ADDITIONAL NOTE ON SOME TAUBERIAN THEOREMS OF O. SZÁSZ

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1. An additional theorem. In the note [3] to which this is an addition, Theorem II is exhibited as a generalization of Theorem I and an appeal is made to Szász [6] to indicate the transition from Theorem II to the final result stated as Corollary III′. However, in view of the formal simplicity of Corollary III′ and the wide generality (reflected in its apparent complexity) of Theorem II, it seems worth while to adopt the opposite point of view and record a method, based on the following result, of deducing Theorem II and all related theorems (which cover Szász's) from Corollary III′ [3, p. 384].

THEOREM IV. *If a (real) series* $\sum_{n=1}^{\infty} a_n$ *is* (Φ, λ)*-summable to* s*, where* λ *denotes the strictly positive increasing divergent sequence* $\{\lambda_n\}$ *subject to the additional condition* $\lambda_{n+1}/\lambda_n \to 1$*, and if the series satisfies the Tauberian condition*:

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{\nu=n+1}^{m} \lambda_\nu a_\nu \geq 0\,, \qquad m>n,\ \frac{\lambda_m}{\lambda_n}\to 1\,, \tag{1}$$

then $\sum_{n=1}^{\infty} a_n$ *is convergent to* s. (Amnon Jakimovski [1, Theorem 1] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

Proof. We have, by Abel's partial-summation lemma,

$$\sum_{\nu=n+1}^{m} a_\nu = \sum_{\nu=n+1}^{m} \frac{\lambda_\nu a_\nu}{\lambda_\nu} \geq \frac{\lambda_n}{\lambda_{n+1}} \cdot \frac{1}{\lambda_n} \min_{n+1\leq k\leq m} \sum_{\nu=n+1}^{k} \lambda_\nu a_\nu\,.$$

Hence, by (1),

$$\liminf_{n\to\infty} \sum_{\nu=n+1}^{m} a_\nu \geq 0, \qquad m>n, \quad \frac{\lambda_m}{\lambda_n}\to 1.$$

It is well-known [2, p. 33] that the above Schmidt condition is equivalent to the second alternative of hypothesis (12) of Corollary III′ [3, p. 384]. Therefore this corollary establishes that $\sum_{n=1}^{\infty} a_n = s$.

2. Deductions from Theorem IV.

COROLLARY IV.1. *In Theorem* IV, (1) *is implied by, and so can be replaced by,* ONE *of the following conditions*:

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$$\left.\begin{array}{ll}(2) & \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{\nu=n+1}^{m}\lambda_\nu(|a_\nu|-a_\nu)=0\,, \\ (3) & \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{\nu=n+1}^{m}\lambda_\nu|a_\nu|=0\,,\end{array}\right\} \qquad m>n,\quad \frac{\lambda_m}{\lambda_n}\to 1.$$

(Szász [6, Theorem 3] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

COROLLARY IV.2. *In Corollary* IV.1, (2) *can be replaced by the condition*:

$$(4)\qquad \begin{array}{ll} U_n\equiv\sum_{\nu=1}^{n}\lambda_\nu(|a_\nu|-a_\nu)=O(\lambda_n), & n\to\infty\,, \\ \lim_{n\to\infty}\left(\frac{U_m}{\lambda_m}-\frac{U_n}{\lambda_n}\right)=0\,. & m>n,\ \frac{\lambda_m}{\lambda_n}\to 1\,.\end{array}$$

(Szász [6, Theorem 2] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

The above corollary is the same as Theorem II of my note [3]. We can deduce it from the preceding corollary merely by noting that (4) implies (2)[1] as a result of letting $n\to\infty$, $\lambda_m/\lambda_n\to 1$ in the identity:

$$\frac{U_m-U_n}{\lambda_n}=\left(\frac{U_m}{\lambda_m}-\frac{U_n}{\lambda_n}\right)\frac{\lambda_m}{\lambda_n}+\frac{U_n}{\lambda_n}\left(\frac{\lambda_m}{\lambda_n}-1\right), \qquad m>n\,.$$

COROLLARY IV.3. *In Corollary* IV.1, (3) *can be replaced by the hypothesis*:

$$(5)\qquad \begin{array}{ll} V_n\equiv\sum_{\nu=1}^{n}\lambda_\nu|a_\nu|=O(\lambda_n)\,, & n\to\infty\,, \\ \lim_{n\to\infty}\left(\frac{V_m}{\lambda_m}-\frac{V_n}{\lambda_n}\right)=0\,, & m>n,\ \frac{\lambda_m}{\lambda_n}\to 1\,,\end{array}$$

which implies (3) *exactly as* (4) *implies* (2).

Plainly the last hypothesis (5) *can assume the special form*:

$$\lim_{n\to\infty}\frac{V_n}{\lambda_n}=l \qquad l<\infty\,.$$

(Szász and Rényi [6, Theorems 1 and B] give the particular case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

3. A second additional theorem. Theorem IV is a deduction from Corollary III′ [3] and so ultimately from Theorem A [3, p. 378]. The following is another deduction from Theorem A deserving of mention.

[1] In fact (4) is equivalent to (2) as (2) implies (4) by an argument exactly like Szász's in the case $\lambda_n=n$ [**6**, Lemma 2].

THEOREM B. *Let $\phi(u)$ fulfill the conditions* C(i)–(v) *of the Introduction* [3, p. 377].[2] *Suppose that $A(u)$ is a (real) function of bounded variation in every finite interval of $(0, \infty)$, $A(0)=0$. If*

$$(6)\qquad \frac{1}{u}\int_0^u x\, d\{A(x)\}$$

is slowly decreasing, that is,

$$\liminf_{u\to\infty}\left(\frac{1}{v}\int_0^v x\, d\{A(x)\} - \frac{1}{u}\int_0^u x\, d\{A(x)\}\right)\geq 0\,, \qquad v>u\,,\ \frac{v}{u}\to 1\,,$$

and if $A(u)$ is Φ-summable to s, that is, if

$$(7)\qquad \Phi(t)=\int_0^\infty \phi(ut)\, d\{A(u)\}$$

exists for $t>0$ and tends to s as $t\to+0$, then $A(u)\to s$ as $u\to\infty$.

Proof. We write as before [3, pp. 377–378]:

$$A_1(u)=\int_0^u A(x)dx\,, \qquad \phi(u)=\int_u^\infty \psi(x)dx\,.$$

Then (7) gives successively [4, pp. 346–347], as $t\to+0$,

$$\Phi(t)=t\int_0^\infty \psi(ut)A(u)du \to s\,, \qquad \Phi_1(t)=t\int_0^\infty \psi(ut)\frac{A_1(u)}{u}\, du \to s\,.$$

$$\Phi(t)-\Phi_1(t)=t\int_0^\infty \psi(ut)\{A(u)-u^{-1}A_1(u)\}\, du \to 0\,.$$

Thus $A(u)-u^{-1}A_1(u)$ is Φ-summable to 0 and satisfies the Tauberian condition in (6). Hence, by a known result [4, Corollary 2.2] following from Theorem A [3], $A(u)-u^{-1}A_1(u)$ tends to 0 as $u\to\infty$. Consequently, by Theorem A [3], $u^{-1}A_1(u)$, and hence also $A(u)$, tends to s as $u\to\infty$.

4. Remarks. (i) Amnon Jakimovski [1, Theorem 1] has dealt with the case of Theorem B in which $\phi(u)=e^{-u}$ and

$$A(u)=\begin{cases} a_1+a_2+\cdots+a_n & \text{for } n\leq u<n+1,\ n\geq 1\,,\\ 0 & \text{for } 0\leq u<1\,,\end{cases}$$

showing, by a modification of the method used above to prove Theorem B, that we may in this case replace (6) by

$$(6^*)\qquad \liminf_{n\to\infty}\left(\frac{U_m^*}{m}-\frac{U_n^*}{n}\right)\geq 0\,, \qquad m>n,\ \frac{m}{n}\to 1\,,$$

[2] These conditions can be slightly relaxed (for example, [5, Theorem A]).

where $U_n^* = \sum_{\nu=1}^{n} \nu a_\nu$, leaving the statement of Theorem B otherwise unaltered. He also observes that (6*) includes (or generalizes) the second half of (4) with $\lambda_n = n$, implying that, in Szász's result cited under Corollary IV.2, the first half of (4) is superfluous. This observation is, however, incorrect as shown by the following example.

EXAMPLE 1. Let a_n be defined so that

$$\left.\begin{array}{ll} na_n = \nu & \text{for} \quad 4^\nu \leq n < 2\cdot 4^\nu , \\ na_n = -n^{-2} & \text{for} \quad 2\cdot 4^\nu \leq n < 4^{\nu+1} , \end{array}\right\} \nu = 0, 1, 2, \cdots .$$

Then it is easily verified that (4) with $\lambda_n = n$ holds because

$$\sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu) = o(n) , \qquad n \to \infty ,$$

but that (6*) does not hold since

$$\text{if } n = 2\cdot 4^\nu,\ \nu \to \infty,\ \text{then } \frac{U_n^*}{n} = \frac{\sum_{k=0}^{\nu} k\cdot 4^k + O(1)}{2\cdot 4^\nu} \sim \frac{2\nu}{3} ,$$

$$\text{if } m = \text{the integral part of } 2\cdot 4^\nu \frac{\nu}{\nu - \sqrt{\nu}} ,\ \text{then } \frac{U_m^*}{m} = \frac{U_n^* + o(1)}{n} \frac{n}{m}$$

where $(n/m - 1) \sim -\nu^{-1/2}$, so that

$$\liminf_{n\to\infty} \left(\frac{U_m^*}{m} - \frac{U_n^*}{n} \right) = -\infty , \qquad m > n,\ \frac{m}{n} \to 1 .$$

While the above example shows that (4) *with* $\lambda_n = n$ *does not in general imply* (6*), the one which follows makes it clear that *neither does* (6*) *necessarily imply* (4) *with* $\lambda_n = n$.

EXAMPLE 2. Let a_n be defined so that

$$\left.\begin{array}{rl} (-1)^n na_n = \nu & \text{for} \quad 4^\nu \leq n < 2\cdot 4^\nu , \\ a_n = 0 & \text{for} \quad 2\cdot 4^\nu \leq n < 4^{\nu+1} , \end{array}\right\} \nu = 0, 1, 2, \cdots .$$

Then (6*) holds since $U_n^*/n \to 0$ as $n \to \infty$. However, (4) with $\lambda_n = n$ does not hold since now

$$U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu)$$

and we have:

$$\text{if } n = 2\cdot 4^\nu,\ \nu \to \infty,\ \text{then } \frac{U_n}{2n} = \frac{\sum_{k=1}^{\nu} k\cdot 4^k/2}{2.4^\nu} \sim \frac{\nu}{3} ,$$

if m=the integral part of $2\cdot 4^{\nu}\dfrac{\nu}{\nu-\sqrt{\nu}}$, then $\dfrac{U_m}{2m}=\dfrac{U_n}{2n}\dfrac{n}{m}$

where $(n/m-1)\sim-\nu^{-1/2}$, with the result that

$$\liminf_{n\to\infty}\left(\frac{U_m}{m}-\frac{U_n}{n}\right)=-\infty\ , \qquad m>n,\ \frac{m}{n}\to 1\,.$$

(ii) In the definition of Φ-summability of $A(u)$, set forth in (7) and assumed in both Theorem A [3] and Theorem B, the integral $\Phi(t)$ is to be interpreted as a Lebesgue-Stieltjes integral (absolutely) convergent for $t>0$ unless further considerations, as in the case $\phi(u)=e^{-u}$, permit us to view it as a (non-absolutely) convergent Riemann-Stieltjes integral (cf. [5, p. 103, Note]).

(iii) In Theorem III [3, p. 383] the condition $\lambda_{n+1}/\lambda_n\to\infty$ of hypothesis (11) is a misprint for $\lambda_{n+1}/\lambda_n\to 1$.

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