

AN ALGEBRA ASSOCIATED WITH A COMPACT GROUP

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1. Introduction. This paper deals with a variation on a familiar theme; namely, a proof that a space is determined, in some appropriate sense, by certain properties of a collection of functions on that space. Here, the space in question is a compact abelian group G , and the collection of functions is the set of all continuous functions from G into a commutative, complex Banach algebra R . The relevant properties of the collection of functions make it into a Banach algebra $R(G)$, with addition as well as multiplication by scalars defined in the usual way, that is, pointwise, norm defined by

$$(1) \quad \|x\| = \sup_{g \in G} \|x(g)\| \quad x \in R(G),$$

and multiplication of elements in $R(G)$ defined as the convolution

$$(2) \quad (xy)(g) = \int_G x(gh^{-1})y(h)dh.$$

The integral, like all integrals appearing in this paper, is taken with respect to Haar measure in G , normalized so that the measure of G is 1. The integrand takes on values in the Banach algebra R , and the integral is of the type described in [3]. An alternate approach to this integral is obtained by observing that, as continuous functions on a compact group, the functions with which we deal are almost periodic in the sense of [2], and the integral is the invariant mean whose existence and uniqueness are proved in [2].

We will let \mathcal{H} denote the class of theorems of the type described in the first sentence of the preceding paragraph. Many theorems of this class may be found in the literature; the ones most intimately related to the present investigation appear in [6] and the papers quoted there. I feel, therefore, that some justification is needed for the publication of still another one. Furthermore, there is probably no limit to the number of different kinds of spaces and different sets of functions which might be combined to yield a theorem in \mathcal{H} . The choice of the particular set-up that is being studied here was motivated by an attempt to solve a problem in topology proposed by Fox [4].

If X , Y , and Z are topological spaces, and if the cartesian product $X \times Y$ is homeomorphic with $X \times Z$, then it is known that Y and Z need not be homeomorphic. In the simplest example of this phenomenon [4], Y and Z are compact subsets of the plane that are not at all

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pathological, and X is a closed interval. The non-homogeneity of X at the end points seems to be what makes this example work, and Fox raises the question whether homeomorphism of the products will imply homeomorphism of Y and Z if X is a circle. Now, let R_1 and R_2 be the Banach algebras of all continuous complex (real would work just as well) valued functions on the compact spaces Y and Z , respectively, with multiplication as well as addition defined pointwise. Then a well known theorem (in \mathcal{H}) asserts that R_1 and R_2 are isomorphic, in symbols¹ $R_1 \approx R_2$, if, and only if, Y and Z are homeomorphic. A function from X into R_1 may be identified in an obvious way with a complex function on $X \times Y$. If one could prove that isomorphism of the algebra of all continuous functions from the circle X into R_1 with the space of functions from X into R_2 implies $R_1 \approx R_2$, and if, moreover, the former isomorphism is a consequence of the homeomorphism of $X \times Y$ with $X \times Z$, then Fox's problem would be solved.

It would be pleasant to be able to report that this has been achieved, especially because the theorems in \mathcal{H} have had no noteworthy applications to problems in topology. Unfortunately, although Theorem 2 does say that $R_1(G) \approx R_2(G)$ implies $R_1 \approx R_2$ for any compact abelian group G , in particular for a circle, the algebraic structure which has been placed on $R(G)$ is of such a nature that I cannot prove that homeomorphism of $G \times Y$ with $G \times Z$ implies $R_1(G) \approx R_2(G)$. Section 4 is devoted to a discussion of some of the reasons for the failure of this approach.

Theorem 2 is probably true without the hypothesis that G is abelian, but I have not been able to prove it. This hypothesis does not influence the applicability of the theorem to Fox's problem. The requirement that R be a commutative Banach algebra whose only idempotent is its unit, is equivalent, if R is the algebra of continuous functions on Y with pointwise multiplication, to the assumption that Y is connected.

2. Complex valued functions. In this section, we assume that R is the field of complex numbers, and then we no longer need to require that G be abelian.

THEOREM 1. *If D and A are the Banach algebras of continuous complex functions on the compact groups G and Γ , respectively (with multiplication defined by (2)), and $D \approx A$, then G and Γ are isomorphic.*

This theorem can probably be proved by the technique of [6], but

¹ This symbol will be reserved for isomorphism (including preservation of norms) of Banach algebras.

we choose to base our proof on the theory of Banach spaces of continuous functions. If $f : \mathcal{A} \rightarrow D$ is the isomorphism, then there exists a one-to-one mapping φ of G onto Γ which is a homeomorphism of the underlying topological spaces, and $x_0 \in D$ with the property $|x_0(g)| \equiv 1$, such that²

$$(3) \quad f\xi(g) = x_0(g) \cdot \xi(\varphi g)$$

for all $\xi \in \mathcal{A}$ and $g \in G$. We propose to show that φ is an isomorphism of the groups.

LEMMA 1. $x_0(g'g'') = x_0(g')x_0(g'')$ for all $g', g'' \in G$.

Proof. If $\xi_0 \in \mathcal{A}$ is defined as $\xi_0(\gamma) \equiv 1, \gamma \in \Gamma$, then $f\xi_0 = x_0$. Consequently,

$$x_0 \cdot x_0 = f\xi_0 \cdot f\xi_0 = f(\xi_0 \cdot \xi_0) = f(\xi_0) = x_0;$$

that is, x_0 is an idempotent in D . Hence,

$$x_0(g) = x_0 x_0(g) = \int x_0(gh^{-1})x_0(h)dh.$$

Since $|x_0(g)| \equiv 1$, we have

$$1 = x_0(g)x_0(g) = \int \overline{x_0(g)}x_0(gh^{-1})x_0(h)dh.$$

But the absolute value of the integrand is 1 for all g and h , and the measure of G is also 1, so that

$$\overline{x_0(g)}x_0(gh^{-1})x_0(h) = 1$$

for all g and h (more precisely, for almost all h , but the function is continuous). Setting $g = g'g''$ and $h = g''$, and remembering that $|x_0(g)| = 1$, we obtain $x_0(g'g'') = x_0(g') \cdot x_0(g'')$.

Proof of Theorem 1. It is required only to prove that

$$\varphi(gg') = (\varphi g)(\varphi g') \quad \text{for all } g, g' \in G.$$

Let Ω be a neighborhood of the identity in Γ , and let ω be a continuous function on Γ which vanishes outside of Ω and such that

$$\int_{\Gamma} \omega(\eta^{-1})\omega(\eta)d\eta = 1.$$

² A proof of this assertion for a Banach space of real functions may be found in a number of different places, including [1, p. 172]. A generalization which includes the case of complex functions appears in [5, Theorem 6.2].

Set

$$\omega_1(\gamma) = \omega((\varphi g)^{-1}\gamma) \quad \text{and} \quad \omega_2(\gamma) = \omega(\gamma \cdot (\varphi g')^{-1});$$

then a straightforward computation yields

$$(4) \quad \omega_1 \omega_2((\varphi g)(\varphi g')) = 1.$$

Using (2), (3), and Lemma 1 in the relation $f(\omega_1 \omega_2) = (f\omega_1) \cdot (f\omega_2)$, we find that for any $a \in G$,

$$\begin{aligned} f(\omega_1 \omega_2)(a) &= \int_G x_0(ah^{-1}) \cdot \omega_1(\varphi(ah^{-1})) \cdot x_0(h) \cdot \omega_2(\varphi h) dh \\ &= x_0(a) \int_G \omega((\varphi g)^{-1}\varphi(ah^{-1})) \cdot \omega((\varphi h)(\varphi g')^{-1}) dh. \end{aligned}$$

Since ω vanishes outside of Ω , this implies

$$(5) \quad f(\omega_1 \omega_2)(a) = 0 \quad \text{for } a \notin V$$

where

$$V = \varphi^{-1}[(\varphi g)\Omega] \cdot \varphi^{-1}[\Omega(\varphi g')].$$

$(\varphi g)\Omega$ is a neighborhood of φg in I' , so that $\varphi^{-1}[(\varphi g)\Omega]$ is a neighborhood of g in G . Similarly, $\varphi^{-1}[\Omega(\varphi g')]$ is a neighborhood of g' , and V is a neighborhood of gg' .

Let a be the (unique) element of G such that $\varphi a = (\varphi g) \cdot (\varphi g')$, and suppose $a \neq gg'$. Since no previous restrictions have been placed upon Ω , we may now choose Ω so that $a \notin V$, that is, so that $f(\omega_1 \omega_2)(a) = 0$. But,

$$f(\omega_1 \omega_2)(a) = x_0(a) \cdot \omega_1 \omega_2(\varphi a) = x_0(a) \cdot \omega_1 \omega_2(\varphi g \cdot \varphi g') = x_0(a) \neq 0.$$

This contradicts the assumption that $a \neq gg'$, and therefore $\phi(gg') = (\varphi g)(\varphi g')$.

3. The isomorphism theorem.

THEOREM 2. *Let G_1 and G_2 be compact abelian groups, and R_1 and R_2 , commutative Banach algebras whose only idempotents are their respective units. Then $R_1(G_1) \approx R_2(G_2)$ if, and only if, $R_1 \approx R_2$ and G_1 is isomorphic to G_2 .*

Proof. In one direction, the implication is trivial. To prove the non-trivial half of the theorem, we consider a group G , an algebra R with unit e , and show how R and G may be recovered from $R(G)$, using only the structure of $R(G)$ as a Banach algebra.

The first step is to find D (in the notation of Theorem 1) in $R(G)$.

Specifically, we want to characterize the set De of elements in $R(G)$ of the form $\lambda(g) \cdot e$, where $\lambda(g)$ is a complex function on G .

LEMMA 2. *De is the smallest closed linear subspace containing all of the idempotents of $R(G)$.*

Proof. We review some essential facts concerning Fourier analysis in $R(G)$; the proofs may be found in [2]. Let $\{\chi_\alpha\}$ be the set of all continuous characters of G , that is,

$$|\chi_\alpha(g)|=1 \text{ and } \chi_\alpha(gg')=\chi_\alpha(g) \cdot \chi_\alpha(g')$$

for all $g, g' \in G$. For $x \in R(G)$, define

$$r_\alpha = \int_G \overline{\chi_\alpha(g)} x(g) dg.$$

This is an element of R . The formal series $\sum_\alpha \chi_\alpha(g) r_\alpha$ represents $x(g)$ in exactly the same way that classical Fourier series represent continuous functions. We write $x \sim \sum \chi_\alpha r_\alpha$. If $x' \sim \sum \chi_\alpha r'_\alpha$ then $xx' \sim \sum \chi_\alpha r_\alpha r'_\alpha$. (This is not proved in [2], but can be done, as in the classical case, simply by evaluating the α th coefficient of xx' .)

Since the formal series representation is unique, x is an idempotent if, and only if, $r_\alpha=0$ or e for all α . Thus, every idempotent of $R(G)$ is in De , and, in fact, is an idempotent of D multiplied by e . Since the idempotents of D span D , the idempotents of $R(G)$ span De . It is obvious, that De is a closed linear subspace of $R(G)$.

Lemma 2 asserts that De is determined by $R(G)$. Since $De \approx D$ (assuming $\|e\|=1$), it follows from Theorem 1 that G is determined by $R(G)$. It remains only to prove that R is determined by $R(G)$, and this will be achieved essentially by fishing the constant functions out of $R(G)$. Specifically, we will find all of the constant functions multiplied by some character of G . It is impossible to distinguish between characters using only their algebraic properties in $R(G)$.

LEMMA 3. *Let x be any irreducible idempotent of $R(G)$, that is, any idempotent which is not the sum of other non-zero idempotents. The principal ideal generated by x is isomorphic with the Banach algebra R .*

Proof. From the discussion of idempotents given earlier, it is clear that $x = \chi e$ for some character χ of G . If $y \in R(G)$, then

$$\begin{aligned} yx(g) &= \int y(gh^{-1}) \chi(h) \cdot e dh = \int y(k) \chi(k^{-1}g) \cdot e dk \\ &= \left[\int y(k) \overline{\chi(k)} dk \right] \chi(g) \cdot e = \chi(g) \cdot r \end{aligned}$$

where r is the "Fourier coefficient" of y with respect to the character χ . Similarly $xy(g) = \chi(g)r$. Consequently, the set of functions in $R(G)$ of the form $\chi(g)r$, $r \in R$, is a two-sided ideal. The correspondence $r \leftrightarrow \chi r$ is the desired isomorphism.

4. Fox's problem. It was remarked earlier that the class of theorems \mathcal{H} has been disappointing as a source of solutions of problems in topology, problems that do not involve the function space directly. The comments which will be made here refer only to the failure to solve Fox's problem, but it seems to me that they lie close to the heart of the difficulties in general.

It is unlikely that Theorem 2 can be used to prove Fox's conjecture because the conclusion of the theorem is so strong. What is needed is a theorem with the statement " G_1 isomorphic to G_2 " in the hypothesis rather than in the conclusion. That so much could be proved from the hypothesis $R_1(G_1) \approx R_2(G_2)$ implies that it is a very strong condition and one that will be difficult to verify. Thus, in the application to Fox's problem, we would take R_1 and R_2 as the algebras of continuous functions on Y and Z , respectively, (pointwise multiplication) and we would have to prove that if $G \times Y$ is homeomorphic with $G \times Z$, then $R_1(G) \approx R_2(G)$. One may observe, incidentally, that in the correspondence between these two algebras induced by the homeomorphism of the product spaces, norms are preserved, but the norms do not enter in an essential way into the proof of $R_1 \approx R_2$.

Apparently, then, the source of the difficulty is the peculiar definition of multiplication in $R(G)$. I believe, however, that the trouble goes deeper. A theorem in \mathcal{H} generally has a hypothesis which is so strong that to verify it is tantamount to exhibiting a homeomorphism of the topological spaces on which the functions are defined. One manifestation of this is the fact that the hypothesis implies not only a homeomorphism but also an intimate relationship between the homeomorphism of the conclusion and the isomorphism of the hypothesis, as given by formula (3). The presence of such a formula is implicit in all of the techniques for proving theorems in \mathcal{H} . It is what requires the strong hypothesis, which, in turn, limits the applicability of the theorem.

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