THE USE OF FORMS IN VARIATIONAL CALCULATIONS

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Introduction. The purpose of this paper is to present a method of calculating the first and second variation which is suitable for spaces which have a Euclidean connection. I then use this method to calculate the first and second variations along a geodesic in a Finsler space in terms of differential invariants of the Finsler metric. In the special case of Riemannian geometry, this calculation has been carried out by Schoenberg in [**4**].

Indications as to how this calculation should be made are originally due to E. Cartan [**1**]. I wish to thank Prof. S. S. Chern for the privilege of seeing his calculations on this matter for Riemann spaces.

1. Algebraic Preliminaries. Let $I=[0, 1]$ and $0\leq\xi_1, \xi_2\leq 1$. Let M^n be an n-dimensional C^∞ manifold. Assume we have a one parameter family of mappings of I into M^n which we will denote by $f(\xi_1, \xi_2)$, where ξ_2 is taken as the parameter along I and ξ_1 parametrizes the family of mappings. Then we may define a mapping η: $I\times I\rightarrow M^n$ by the equation

$$\eta(\xi_1, \xi_2)=f(\xi_1, \xi_2).$$

We require that η shall also be a C^∞ mapping.

Let η_* denote the mapping induced by η on the tangent space to $I\times I$ into the tangent space to M^n. Let η^* denote the dual mapping induced on the cotangent spaces. Then we define two vector fields X_1 and X_2 over $\eta(I\times I)$ by

$$X_2=\eta_*(\partial/\partial\xi_2) \quad \text{and} \quad X_1=\eta_*(\partial/\partial\xi_1).$$

Then if w is any form in M^n we may write

$$\eta^*(w)=w_\delta d\xi_1+w_d d\xi_2,$$

where w_δ and w_d are defined by the equation.

LEMMA 1.1. *If $\langle X, w\rangle$ denotes the value that X takes on the covector w at each point, then*

$$w_\delta=\langle X_1, w\rangle$$

and

$$w_d=\langle X_2, w\rangle.$$

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Proof. $w_\delta = \langle \partial/\partial\xi_1, \eta^*(w)\rangle = \langle \eta^*(\partial/\partial\xi_1), w\rangle = \langle X_1, w\rangle$.
The proof is analogous for w_d.

Let Ω be any two form and let X_1 and X_2 be any two vector fields. It is well known that $\Lambda^2(V)$ and $\Lambda^2(V^*)$ are dually paired. Let this pairing be denoted by

$$\langle X_1 \wedge X_2, \Omega\rangle .$$

Then if Ω can be decomposed as $w_1 \wedge w_2$, where w_1 and w_2 are one forms, we have that the pairing may be defined by the following expression:

$$\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = \langle X_1, w_1\rangle\langle X_2, w_2\rangle - \langle X_1, w_2\rangle\langle X_2, w_1\rangle .$$

THEOREM 1.1. $\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = w_{1\delta} w_{2d} - w_{1d} w_{2\delta}$.

The proof of this theorem is straightforward.

We define the symbols δw_d and dw_δ by the following equations:

$$\delta w_d = \partial/\partial\xi_1 \langle X_2, w\rangle ,$$
$$dw_\delta = \partial/\partial\xi_2 \langle X_1, w\rangle .$$

If f is any function of ξ_1 and ξ_2, we define

$$d^r \delta^s f = \frac{\partial^t f}{\partial\xi_2^r \partial\xi_1^s} ,$$

where $t = r+s$. Define $\delta^r d^s f$ similarly.

THEOREM 1.2. $\langle X_1 \wedge X_2, dw\rangle = \delta w_d - dw_\delta$.

Proof. Now, in terms of a local coordinate system $(x_1, \cdots, x_n)$,

$$\langle X_1 \wedge X_2, dw\rangle = \sum \left[\frac{\partial}{\partial\xi_1}\left(a_i \frac{\partial x_i}{\partial\xi_2}\right) - \frac{\partial}{\partial\xi_2}\left(a_i \frac{\partial x_i}{\partial\xi_1}\right)\right]$$

since

$$\sum a_i \frac{\partial^2 x_i}{\partial\xi_1 \partial\xi_2} = \sum a_i \frac{\partial^2 x_i}{\partial\xi_2 \partial\xi_1} .$$

This and the definition of δw_d and dw_δ prove the theorem.

2. The First Variation.

Consider the integral

$$I = \int_a^b F(q_1, \cdots, q_n ; q_1', \cdots, q_n' ; t)dt \tag{2.1}$$

in a space M of $2n+1$ dimensions. Then in the cotangent space to the manifold M define the form w by the equation

$$w=\sum \frac{\partial F}{\partial q_i'}dq -\Big(\sum q_i' \frac{\partial F}{\partial q_i'} - F\Big)dt \,. \tag{2.2}$$

Now let C be a curve in M^{2n+1} expressed by the equations

$$q_i = q_i(\xi_2)\,, \qquad q_i' = q_i'(\xi_2)\,, \qquad t=(b-a)\xi_2 + a\,.$$

Assume further that $dq_i/d\xi_2 = q_i'$ for all values of ξ_2. Let X_2 be the image of $\partial/\partial\xi_2$ under the mapping described above. Then

$$X_2 = \sum q_i' \frac{\partial}{\partial q_i} + \sum \frac{\partial q_i}{\partial \xi_2}\frac{\partial}{\partial q_i'} + (b-a)\frac{\partial}{\partial t}\,, \tag{2.3}$$

and

$$w_a d\xi_2 = F(q, q', t)\frac{dt}{(b-a)}\,.$$

Hence

$$I=\int_0^1 w_a d\xi_2 = \int_a^b F\big(q_1(t),\, \cdots,\, q_n(t)\,;\, q_1'(t),\, \cdots,\, q_n'(t)\,;\, t\big)dt\,. \tag{2.4}$$

Now consider a one parameter family of curves $f(\xi_1, \xi_2)$ each with the property described above. For each curve in the family we get a vector field which we will denote by $X_2(\xi_1)$. We may consider the variational problem for this family of curves. The crucial fact is that the requirement that $f(\xi_1, \xi_2)$ is a mapping of a *fixed* interval for each fixed value of ξ_1 enables us to treat the problem of variable end point without the necessity of differentiating limits of integration. We consider

$$I(\xi_1) = \int_0^1 \langle X_2(\xi_1),\, w\rangle d\xi_2$$

and

$$\delta I = \frac{\partial I(\xi_1)}{\partial \xi_1} = \int_0^1 \delta w_a d\xi_2\,. \tag{2.5}$$

If we add and subtract dw_δ under the integral sign we get

$$\delta I = [w_\delta]_0^1 + \int_0^1 (\delta w_a - dw_\delta)d\xi_2 \tag{2.6}$$

$$= [w_\delta]_0^1 + \int_0^1 w'(\delta, d)d\xi_2\,, \tag{2.7}$$

where

$$w'(\delta, d) = \langle X_1 \wedge X_2,\, dw\rangle\,, \tag{2.8}$$

and

$$w'(d, \delta)=\langle X_2\wedge X_1, dw\rangle .$$

It may be noted that $w'(\delta, d)=-w'(d, \delta)$. The term $[w_\delta]_0^1$ is called the transversality term.

THEOREM 2.1. *Assume* $[w_\delta]_0^1=0$. *Then a necessary and sufficient condition for* $\delta I=0$ *for all variations is that* $dw=0$ *along* C.

Proof. The condition is clearly sufficient. An equivalent form of the hypothesis is that

$$\int_0^1\langle X_1\wedge X_2, dw\rangle d\xi_2=0$$

for all vector fields X_1 along C. Assume dw does not equal zero along C. Then there exists an X_1 such that $\langle X_1\wedge X_2, dw\rangle>0$ for some open interval $a<\xi_2<b$. Then we may choose a new vector field X_1 such that:

$$\begin{aligned} &\overline{X}_1=X_1 \quad \text{for} \quad a<\xi_2<b \\ &\overline{X}_1=0 \quad \text{for} \quad 0\leq\xi_2\leq a-\varepsilon \quad \text{or} \quad b+\varepsilon\leq\xi_2\leq 1 , \end{aligned}$$

where ε may be chosen arbitrarily small. Then

$$\int_0^1\langle\overline{X}_1\wedge X_2, dw\rangle d\xi_2=\int_a^b\langle X_1\wedge X_2, dw\rangle d\xi_2+\varepsilon',$$

where ε' depends on ε and $\lim_{\varepsilon\to 0}\varepsilon'=0$. Hence we may choose ε in such a way that

$$\int_0^1\langle\overline{X}_1\wedge X_2, dw\rangle d\xi_2>0 .$$

This contradiction proves the theorem.

Remark: This is essentially the usual argument for the derivation of Euler's equation.

3. **Application to Finsler Geometry.** If we assume that our integral is of the Finsler type then we may proceed to calculate the second variation. For treating this special case we assume that the reader has a familiarity with Euclidean connections and we will use the Euclidean connection for a Finsler space as calculated by E. Cartan in [2] and Chern [3].

Let M be an n-dimensional differentiable manifold and let G be the principal bundle over M with fiber and group the n-dimensional orthogonal groups, $O_{(n)}$. Then in G, we have forms w_i, w_{ij}, where $w_{ij}+w_{ji}=0$ and $i, j=1, \cdots, n$. The equations of structure are

$$(3.1) \qquad dw_i = w_j \wedge w_{ji} + \gamma_{ji\alpha} w_j \wedge w_{\alpha n}$$

$$(3.2) \qquad dw_{ij} = w_{ik} \wedge w_{kj} + \Omega_{ij} ,$$

where $\alpha=1, \cdots, n-1$. (Henceforth we will assume that Greek indices run from 1 to $n-1$ and Latin indices run from 1 to n.) The $\gamma_{ij\alpha}$ are symmetric in all indices and zero if any index is n. Also

$$(3.3) \qquad \Omega_{ij} = \frac{1}{2} \sum_{\alpha,\beta} Q_{ij\alpha\beta} w_{\alpha n} \wedge w_{\beta n} + \sum_{l,\alpha} P_{ijl\alpha} w_l \wedge w_{\alpha n} + \frac{1}{2} \sum_{l,k} R_{ijlk} w_l \wedge w_k .$$

Let C be any path in M^n. Choose any path in G with the property that if $e_1, \cdots, e_n$ represents a righthanded frame, that is, an element of $O_{(n)}$, then e_n is in the tangent direction to C. Then arc length along a path C is

$$I = \int_0^1 (w_n)_d d\xi_2 .$$

This follows from equation (2.4) and the definition of w_n (see [3]).

Now $X_2 = e_n$ and $X_1 = \sum k_i e_i$. Therefore $(w_n)_\delta = \langle X_1, w_n \rangle = k_n$. Hence if X_1 is perpendicular to the curve C, then the transversality term is zero. From equation (3.1), we have

$$dw_n = \sum w_\alpha \wedge w_{\alpha n} .$$

Hence

$$(3.4) \qquad \delta I = [\delta(w_n)]_0^1 + \int_0^1 \sum \{(w_\alpha)_\delta (w_{\alpha n})_d - (w_\alpha)_d (w_{\alpha n})_\delta\} d\xi_2 ,$$

where $\qquad (w_\alpha)_d = \langle w_\alpha, e_n \rangle = 0 .$

It is clear from the last equation that the symbols δ and d and our indices make the notation awkward. Hence a w_d will be written as w and a w_δ will be written as ϕ. In this notation equation (3.4) becomes

$$(3.5) \qquad I = [\phi_n]_0^1 + \int_0^1 \sum \phi_\alpha w_{\alpha n} d\xi_2 ,$$

since $w_\alpha = 0$ along the path C.

From Theorem 2.1 we have the following theorem.

THEOREM 3.1. *The differential equations of a geodesic in Finsler geometry are*

$$w_\alpha = 0 , \qquad w_{\alpha n} = 0 , \qquad \alpha = 1, \cdots, n-1.$$

We will now compute the second variation along a geodesic. We have

$$\delta I = \int_0^1 \delta w_n d\xi_2 ,$$

and $\delta^2 I$ is the second variation. Hence we have to compute $\delta^2(w_n)$ along a geodesic. Now

$$\delta^2(w_n) = \delta d(\phi_n) + \phi_\alpha \delta(w_{\alpha n}) \tag{3.6}$$

since $w_{\alpha n}=0$ along the geodesic. We have

$$\delta(w_{\alpha n}) - d(\phi_{\alpha n}) = \langle X_1 \wedge X_2,\ dw_{\alpha n}\rangle . \tag{3.7}$$

From equation (3.2) we obtain

$$\langle X_1 \wedge X_2,\ dw_{\alpha n}\rangle = \langle X_1 \wedge X_2,\ w_{\alpha\beta} \wedge w_{\beta n}\rangle + \langle X_1 \wedge X_2,\ \Omega_{\alpha n}\rangle .$$

By Theorem 1.1 and since C is a geodesic, we have

$$\delta w_{\alpha n} = d\phi_{\alpha n} - w_{\alpha\beta}\phi_{\beta n} + \langle \Omega_{\alpha n},\ X_1 \wedge X_2\rangle . \tag{3.8}$$

Now by equation (3.2) and the facts that

$$R_{ijkl} = -R_{jikl}\ , \qquad R_{ij,kl} = R_{kl,ij}$$

we have

$$\langle X_1 \wedge X_2,\ \Omega_{\alpha n}\rangle = \sum P_{n\alpha n\beta} w_n \phi_{\beta n} + \sum R_{n\alpha n\beta}\phi_\beta w_n . \tag{3.9}$$

Therefore, from equations (3.6), (3.8) and (3.9), we obtain

$$\delta^2(w_n) = \delta d\phi_n + \sum \phi_\alpha [d\phi_{\alpha n} - \phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta} w_n \phi_{\beta n} + R_{n\alpha n\beta}\phi_\beta w_n] . \tag{3.10}$$

Now,

$$\delta d\phi_n = d\delta\phi_n \quad \text{and} \quad d(\phi_\alpha \phi_{\alpha n}) = \phi_{\alpha n}(d\phi_\alpha) + \phi_\alpha(d\phi_{\alpha n}) .$$

Hence

$$\begin{aligned} \delta^2(w_n) = {} & d[\delta\phi_n + \phi_\alpha\phi_{\alpha n}] - \phi_{\alpha n} d\phi_\alpha \\ & + [-\phi_\alpha\phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta] w_n . \end{aligned} \tag{3.11}$$

But from equation (3.1) we have

$$d\phi_\alpha = \delta w_\alpha + w_j \phi_{j\alpha} - \phi_j w_{j\alpha} \tag{3.12}$$

since

$$\gamma_{j\alpha\beta}[\phi_j w_{\beta n} - w_j \phi_{\beta n}] = 0$$

along the geodesic. Also $\delta w_\alpha = 0$ along the geodesic, since $w_\alpha \geq 0$ and equals zero along the geodesic and hence w_α must attain a minimum along a geodesic.

Hence

$$\delta^2 w_n = d[\delta\phi_n + \sum \phi_\alpha\phi_{\alpha n}] + \sum(\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n\,. \tag{3.13}$$

Hence the integral form of the second variation becomes

$$\delta^2 I = [\delta\phi_n + \sum \phi_\alpha\alpha_{\alpha n}]_0^1 + \int_0^1 \sum (\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n d\xi_2\,.$$

For Riemannian geometry we have $P_{ijkl}=0$ and $\sum \phi_\alpha\phi_{\alpha n}$ represents the second fundamental form of the geodesic surface perpendicular to the geodesic at the point.

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