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TENSOR AND TORSION PRODUCTS OF SEMIGROUPS

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This paper is concerned with the study of tensor and torsion functors on the category of abelian semigroups. We show that such functors exist, that they satisfy the universal diagram properties required of them in other branches of algebra, and that many of the theorems obtained for tensor and torsion products of modules may also be obtained in this setting. In particular the tensor functor $\otimes_0$ is exact relative to the category of identity preserving homomorphisms. We determine certain structural characteristics of $\otimes$. If E and F are maximal semilattice homomorphic images of abelian semigroups S and T respectively, then $E \otimes F$ is the maximal semilattice homomorphic image of $S \otimes T$. If G and H are maximal subgroups of S and T then $G \otimes H$ may be identified as a subgroup of $S \otimes T$ and if G and H are the groups of units of S and T respectively, then $G \otimes H$ is the group of units of $S \otimes T$. Moreover, the tensor product of abelian inverse semigroups is an abelian inverse semigroup. Similar results are obtained for the torsion functor.

1. Basic properties of tensor. Throughout this paper A and B will denote arbitrary abelian semigroups unless stated otherwise. $\mathscr{F}$ will denote the semigroup of all functions of finite support from $A \times B$ into the additive semigroup N of nonnegative integers under the operation of pointwise addition. Thus $\mathscr{F}$ is the free abelian semigroup on $A \times B$ with an identity adjoined and will be referred to as the free abelian semigroup on $A \times B$. For $(a, b) \in A \times B$, $\langle a, b\rangle$ will denote the element of $\mathscr{F}$ having value 1 at (a, b) and having value 0 elsewhere. Let σ denote the relation on $\mathscr{F}$ such that $(x, y) \in \sigma$ if and only if either $x = y$ or one of the ordered pairs (x, y) or (y, x) is of the form

(1) $(\langle a + b, c\rangle, \langle a, c\rangle + \langle b, c\rangle)$ or

(2) $(\langle a, c + d\rangle, \langle a, c\rangle + \langle a, d\rangle)$

for $a, b \in A$ and $c, d \in B$. The set of all ordered pairs $(x + t, y + t)$ for $(x, y) \in \sigma$ and $t \in \mathscr{F}$ will be denoted by ν and ρ will denote the transitive closure of ν. Thus ρ is the smallest congruence on $\mathscr{F}$ containing pairs of the form (1) and (2). We denote the semigroup $\mathscr{F}/\rho$ by $A \otimes B$ and we say that $A \otimes B$ is the tensor product of A and B. Let ω denote the function from $A \times B$ into $A \otimes B$ defined by $\omega(a, b) = \langle a, b\rangle\rho$. For $(a, b) \in A \times B$, $a \otimes b$ will denote $\omega(a, b)$ and will be called the tensor product of a and b. Note that if $a_0 \in A$, then the function defined by $b \to \omega(a_0, b)$ is a homomorphism from A into $A \otimes B$. Thus ω is a homomorphism in its first argument. It is also a homomorphism in its second argument. Any such function will be called a bihomomorphism. If a bihomomorphism is an identity preserving homomorphism in each of its arguments, then it is called an identity preserving bihomomorphism.

If each of A and B contains an identity (denoted 0 in each semigroup), then σ_0 will denote the relation on $\mathscr{F}$ which contains σ and which also contains all ordered pairs of the form:

$$(\langle a, 0\rangle, 0),\ (0, \langle a, 0\rangle),\ (\langle 0, c\rangle, 0),\ (\langle c, 0\rangle, 0)$$

for $a \in A$ and $c \in B$. Define ν_0 and ρ_0 analogously. Then $A \otimes_0 B$ will denote $\mathscr{F}/\rho_0$, ω_0 will denote the function defined by $\omega_0(a, b) = \langle a, b\rangle\rho_0$, and $a \otimes_0 b$ will denote the element $\omega_0(a, b)$ of $A \otimes_0 B$. Note that ω_0 is an identity preserving bihomomorphism.

PROPOSITION 1. *If S is any abelian semigroup and φ is a bihomomorphism from $A \times B$ into S, then there is a unique homomorphism φ^* from $A \otimes B$ into S such that the diagram*

$$\begin{tikzcd} A \times B \arrow[rr, "\omega"] \arrow[dr, "\varphi"'] & & A \otimes B \arrow[dl, "\varphi^*"] \\ & S & \end{tikzcd}$$

is commutative. If one assumes that A, B, and S have identities and that φ is an identity preserving bihomomorphism, then there is an identity preserving homomorphism φ^* from $A \otimes_0 B$ into S such that $\varphi = \varphi^* \circ \omega_0$.

The proof is straightforward and is omitted.

Let $\mathscr{S}$ denote the category whose "objects" are abelian semigroups and whose "morphisms" are semigroup homomorphisms. Let $\mathscr{S}_0$ denote the category whose "objects" are abelian semigroups each having an identity and whose "morphisms" are identity preserving semigroup homomorphisms.

Suppose A, B, A', B' are in the category $\mathscr{S}$ and that $\varphi: A \to A'$ and $\theta: B \to B'$ are morphisms of $\mathscr{S}$. The function α from $A \times B$ into $A' \otimes B'$ defined by $(a, b) \mapsto \varphi(a) \otimes \theta(b)$ is a bihomomorphism (which is identity preserving if φ and θ are) and thus there is a unique homomorphism, denoted $\varphi \otimes \theta$, such that the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\omega"] \arrow[dr, "\alpha"'] & A \otimes B \arrow[d, "\varphi \otimes \theta"] \\ & A' \otimes B' \end{tikzcd}$$

is commutative. In case φ and θ are identity preserving one obtains a similar diagram for $\otimes_0$. From these remarks it is easy to see that the following proposition is true.

PROPOSITION 2. *$\otimes$ is a functor from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}$ which is covariant in each of its arguments. Moreover, if φ and θ are morphisms of $\mathscr{S}$ which are onto, then so is $\varphi \otimes \theta$. A similar statement for $\otimes_0$ holds relative to the category $\mathscr{S}_0$.*

Assume that for each λ in some set Λ, A_λ is an object in $\mathscr{S}_0$. The *direct sum* of the family $\{A_\lambda\}_{\lambda \in \Lambda}$, denoted $\sum_\lambda A_\lambda$, is the subsemigroup of the direct product of $\{A_\lambda\}_{\lambda \in \Lambda}$ consisting of those members of the product of the form $\{a_\lambda\}_{\lambda \in \Lambda}$ where the set of $\lambda \in \Lambda$ such that a_λ is not the identity of A_λ is finite.

The following proposition has a proof similar to the proof of the corresponding theorem for abelian groups and is omitted.

PROPOSITION 3. *If $\{A_\lambda\}_{\lambda \in \Lambda}$ and $\{B_\mu\}_{\mu \in \Omega}$ are families of semigroups each member of which is abelian and has an identity, then*

$$(\textstyle\sum_\lambda A_\lambda) \otimes_0 (\sum_\mu B_\mu) \cong \sum_\lambda \sum_\mu (A_\lambda \otimes_0 B_\mu) .$$

In other investigations where the notion of a tensor product plays an important role one also has the notion of an exact sequence and a corresponding theorem which yields a relationship between the two ideas. We wow present a definition of "exact sequence" which preserves that relationship for the cotegory $\mathscr{S}_0$.

If S is an abelian semigroup and φ is a homomorphism with domain S, then the kernel of φ, denoted $\ker \varphi$, is the relation on S defined by $(x, y) \in \ker \varphi$ if and only if $\varphi(x) = \varphi(y)$. If φ is a homomorphism from some abelian semigroup T into S, then the image of φ, denoted im φ, is the relation on S defined by $(x, y) \in \operatorname{im} \varphi$ if and only if there exists $t_1 \in T$, $t_2 \in T$ such that $x + \varphi(t_1) = y + \varphi(t_2)$. Note that $\ker \varphi$ and im φ are always congruences. If A, B, and C are abelian semigroups, φ is a homomorphism from A into B, and θ is a homomorphism from B into C, then we say that $A \xrightarrow{\varphi} B \xrightarrow{\theta} C$ is an *exact sequence* if and only if $\ker \theta = \operatorname{im} \varphi$. Unlike the situation for abelian groups one may have $A \xrightarrow{\varphi} B \xrightarrow{\theta} C$ exact and C trivial (C

has only one element) and yet not have φ onto. If we wish to indicate that φ is onto B we write $A \xrightarrow{\varphi} > B$. Similarly, if we write $\varphi: A > \rightarrow B$, then we mean that φ is one-to-one.

PROPOSITION 3. *The functor* $\otimes_0$ *is right-exact on* $\mathscr{S}_0 \times \mathscr{S}_0$. *More generally, if* A, B, C, *and* D *are abelian semigroups each having an identity and if* φ *and* θ *are identity preserving homomorphisms such that*

$$A \xrightarrow{\varphi} B \xrightarrow{\theta} > C$$

is an exact sequence, then the sequences

(1) $A \otimes_0 D \xrightarrow{\varphi^*} B \otimes_0 D \xrightarrow{\theta^*} > C \otimes_0 D$, *and*

(2) $D \otimes_0 A \xrightarrow{\bar{\varphi}} D \otimes_0 B \xrightarrow{\bar{\theta}} > D \otimes_0 C$,

are exact where $\varphi^*, \theta^*, \bar{\varphi}$ *and* $\bar{\theta}$ *are the natural maps induced from* φ *and* θ *via the tensor functor* $\otimes_0$.

Proof. It is sufficient to prove that (1) is exact since the proof of (2) is analogous.

First assume $(x, y) \in \operatorname{im} \varphi^*$. We show that $(x, y) \in \ker \theta^*$. Since $(x, y) \in \operatorname{im} \varphi^*$, there exists $p, q \in A \otimes_0 D$ such that $x + \varphi^*(p) = y + \varphi^*(q)$ and consequently

$$\theta^*(x) = \theta^*(x) + \theta^*(\varphi^*(p)) = \theta^*(y) + \theta^*(\varphi^*(q)) = \theta^*(y) .$$

Thus $(x, y) \in \ker \theta^*$.

We now show that $\ker \theta^* \subseteqq \operatorname{im} \varphi^*$. First we show that there is a function α from $C \times D$ into $(B \otimes_0 D)/\operatorname{im} \varphi^*$ such that for $(c, d) \in C \times D$

$$\alpha(c, d) = (\theta^{-1}(c) \otimes_0 d) \operatorname{im} \varphi^*$$

where $\theta^{-1}(c)$ denotes any element of B such that $\theta(\theta^{-1}(c)) = c$. To see that α is well-defined, assume $b, b' \in B$ such that $\theta(b) = c = \theta(b')$. Then there exists $a, a' \in A$ such that $b + \varphi(a) = b' + \varphi(a')$. If $d \in D$, then

$$\begin{aligned}(b \otimes_0 d) + \varphi^*(a \otimes_0 d) &= [b + \varphi(a)] \otimes_0 d = [b' + \varphi(a')] \otimes_0 d \\ &= (b' \otimes_0 d) + \varphi^*(a' \otimes_0 d)\end{aligned}$$

and $(b \otimes_0 d, b' \otimes_0 d) \in \operatorname{im} \varphi^*$. Thus α is well-defined and is clearly an identity preserving bihomomorphism. Let α^* denote the induced homomorphism from $C \otimes_0 D$ into $(B \otimes_0 D)/\operatorname{im} \varphi^*$. A straightforward computation now shows that if $(x, y) \in \ker \theta^*$, then

$$x \operatorname{im} \varphi^* = \alpha^*(\theta^*(x)) = \alpha^*(\theta^*(y)) = y \operatorname{im} \varphi^*$$

and $(x, y) \in \operatorname{im} \varphi^*$. The proposition now follows.

2. Semigroup properties of $A \otimes B$. In this section we investigate the subgroup and semilattice structure of the tensor product.

PROPOSITION 4. *Assume that each of A and B is an abelian semigroup and that η and ξ are the natural maps onto their respective maximal semilattice homomorphic images E and F. Then $E \otimes F$ is the maximal semilattice homomorphic image of $A \otimes B$ and $\eta \otimes \xi$ is the natural mapping of $A \otimes B$ onto its maximal semilattice homomorphic image. A similar statement holds for $\otimes_0$.*

Proof. Assume G is a semilattice and that τ is a homomorphism from $A \otimes B$ onto G. We define a map τ^* such that the diagram

$$\begin{tikzcd} A \otimes B \arrow[rr, "\eta\ \xi"] \arrow[dr, "\tau"'] & & E \otimes F \arrow[dl, "\tau^*"] \\ & G & \end{tikzcd}$$

is commutative. Let α denote the function from $E \times F$ into G defined by $\alpha((e, f)) = \tau(a \otimes b)$ where $a \in A$ and $b \in B$ such that $\eta(a) = e$ and $\xi(b) = f$. It follows from Theorem 4.12 of [1] that if $\eta(a') = e$ and $\xi(b') = f$, then there exists positive integers n, n, p, q and $s_1, s_2 \in A$ and $t_1, t_2 \in B$ such that

$$\begin{aligned} na &= a' + s_1 \qquad & pb &= b' + t_1 \\ ma' &= a + s_2 \qquad & qb' &= b + t_2 \,. \end{aligned}$$

Thus

$$\begin{aligned} \tau(a \otimes b) &= np\tau(a \otimes b) = \tau(na \otimes pb) \\ &= \tau(a' \otimes b') + \tau(a' \otimes t_1 \ + \ s_1 \otimes (b' + t_1)) \end{aligned}$$

and $\tau(a \otimes b) \leqq \tau(a' \otimes b')$. Similarly, $\tau(a' \otimes b') \leqq \tau(a \otimes b)$ and $\tau(a \otimes b) = \tau(a' \otimes b')$. It follows that α is a well-defined map which is evidently a bihomomorphism. Let τ^* denote the unique homomorphism from $E \otimes F$ into G induced by α. It is easy to show that $\tau^* \circ (\eta \otimes \xi) = \tau$. It follows that $\eta \otimes \xi$ is the natural map of $A \otimes B$ onto its maximal semilattice homomorphic image $E \otimes F$ (the kernel of $\eta \otimes \xi$ is the smallest semilattice congruence on $A \otimes B$). The proof of the analogous statement for $\otimes_0$ is similar.

If A and B are abelian semigroups and G and H are subgroups of A and B, respectively, then $\otimes(G, H)$ will denote the set of all elements of $A \otimes B$ of the form

$$\sum_{i=1}^{n} n_i(g_i \otimes h_i)$$

where n is a positive integer and for $1 \leqq i \leqq n$, $n_i \in N$, $g_i \in G$, and $h_i \in H$. A similar definition is supposed for $\bigotimes_0$.

LEMMA 1. *If G and H are subgroups of A and B respectively, then $\bigotimes(G, H)$ is a subgroup of $A \otimes B$. A similar statement holds for $\bigotimes_0$.*

Proof. Observe that if $g \in G$, $h \in H$, and e and f are the respective identities of G and H, then $g \otimes f = e \otimes f = e \otimes h$. The rest follows in a straightforward manner. The following lemma is an immediate consequence of Proposition 1 and Lemma 1.

LEMMA 2. *If G and H are abelian groups, then $G \otimes H$ and $G \otimes_0 H$ both become the "usual" tensor product of G and H as defined in group theory* (see, for example, Fuchs [2]).

PROPOSITION 5. *If G and H are maximal subgroups of A and B respectively, then $G \otimes H \cong (G, H)$ and $G \otimes_0 H \cong \bigotimes_0 (G, H)$.*

Proof. Throughout this proof $\bigotimes$ will denote the tensor operation in $A \otimes B$ and $\bigotimes'$ will denote the tensor operation in $G \otimes H$. Let α denote the function from $G \times H$ into $\bigotimes(G, H)$ defined by $\alpha(g, h) = g \otimes h$ for $(g, h) \in G \times H$. Since α is a bihomomorphism it induces a homomorphism α^* from $G \otimes H$ into $\bigotimes(G, H)$. Clearly α^* is onto. We show that it is one-to-one. Let $\mathscr{F}$, ρ, ν, and σ be defined as at the beginning of this paper. Assume

$$\alpha^*(\sum n_i(g_i \otimes' h_i)) = \alpha^*(\sum n_j^*(g_j^* \otimes' h_j^*))$$

for n_i, $n_j^* \in N$, g_i, $g_j^* \in G$, and h_i, $h_j^* \in H$. Then there exists $x_0, x_1, \cdots, x_{q+1}$ in $\mathscr{F}$ such that $x_0 = \sum n_i\langle g_i, h_i\rangle$, $x_{q+1} = \sum n_j^*\langle g_j^*, h_j^*\rangle$ and for $0 \leqq k \leqq q$, $(x_k, x_{k+1}) \in \nu$. For each k, there exists a positive integer q_k and elements $m_{pk} \in N$, $a_{pk} \in A$, and $b_{pk} \in B$ for $1 \leqq k \leqq q_k$, such that $x_k = \sum_p m_{pk}\langle a_{pk}, b_{pk}\rangle$. Clearly we may choose $a_{p0} = g_p$ and $b_{p0} = h_p$. Now define x_k' to be $\sum_p m_{pk}\langle a'_{pk}, b'_{pk}\rangle$ where $a'_{pk} = a_{pk} + e$ and $b'_{pk} = b_{pk} + f$, and e and f are the identities of G and H, respectively. We show (by induction on k) that for each p and k the "components" a'_{pk} and b'_{pk} of x'_p are in G and H, respectively. The latter statement is clearly true for $k = 0$. Assume $a'_{ph} \in G$ and $b'_{ph} \in H$ for all p and for all $1 \leqq h < k$. Since $x_{k-1}\nu x_k$, there exists $(x, y) \in \sigma$ and $t \in \mathscr{F}$ such $x_{k-1} = x + t$, $x_k = y + t$. We may assume (x, y) is of the form (1) or (2) at the beginning of the paper and since for each p, $a'_{p(k-1)} \in G$ and $b'_{p(k-1)} \in H$, all the "components" of t' are appropriately in G or H. Thus it suffices to show that if one has an ordered pair (x, y) of the form (1) and (2) such that x' has its "components" in the appropriate group G or H, then so does y' (here, as before, x', y', t' denote elements obtained from x, y, and t by adding e or f to the appropriate "components" of x, y, and t). To show the latter statement, one merely needs to show that if $a, b \in A$ such that $a + b + e \in G$, then $a + e$ and $b + e$ are in G (plus a similar statement for H, of course). Clearly $a + e$ and $b + e$ are in the archimedean components of A containing e. But if g is in an archimedean semigroup C which contains an idempotent e, then $g + e$ is in the maximal subgroup of C. It follows that x'_k has its "components" in the appropriate group G or H. For each k, let $\bar{x}_k$ denote the restriction of x'_k to $G \times H$. If $\mathscr{F}'$ is the free abelian group on $G \times H$, and ρ', ν', σ' are the relations on $\mathscr{F}'$ corresponding to ρ, ν, and σ on $\mathscr{F}$, then the fact that $x_k \nu x_{k+1}$ for each k, implies that $\bar{x}_k \nu' \bar{x}_{k+1}$ for each k, and thus $\bar{x}_0 \rho' \bar{x}_{q+1}$. We have

$$\sum n_i(g_i \otimes' h_i) = \bar{x}_0\rho' = \bar{x}_{q+1}\rho' = \sum n_j^*(g_j^* \otimes' h_j^*)$$

and α^* is one-to-one. The proposition follows.

The following proposition is easy and its proof is omitted.

PROPOSITION 6. *If A and B are abelian inverse semigroups, then so are $A \otimes B$ and $A \otimes_0 B$.*

REMARK. If A and B are semilattice unions of abelian groups, $A = \bigcup_{e \in E} A_e$ and $B = \bigcup_{f \in F} B_f$, then $A \otimes B$ is a semilattice union of groups by the last proposition. It is clear that each element x of $A \otimes B$ may be written in the form

$$x = \sum_{i=1}^{n} x_i$$

where, for each i, x_i is an element of $A_e \otimes B_f$ for some $e \in E$ and $f \in F$. At a later point in the exposition it shown that $E \otimes F$ is isomorphic to the direct product of E and F. It therefore follows from Proposition 4 that

$$A \otimes B = \bigcup_{(e,f) \in E \otimes F} (A_e \otimes B_f)$$

is the semilattice decomposition of $A \otimes B$ into a union of disjoint groups.

PROPOSITION 7. *Assume A and B are abelian semigroups and that η and ξ are the natural maps onto their respective maximal semilattice homomorphic images E and F. Also assume e is an* *identity of E, f is an identity of F, and that e_1 and f_1 are idempotents in $\eta^{-1}(e)$ and $\xi^{-1}(f)$, respectively. Then the maximal subgroup of $A \otimes B$ containing $e_1 \otimes f_1$ is $G_{e_1} \otimes H_{f_1}$ where G_{e_1} and H_{f_1} are the respective maximal subgroups of A and B containing e_1 and f_1.*

Proof. It is clear that $G_{e_1} \otimes H_{f_1}$ is a subgroup of the maximal subgroup M of $A \otimes B$ which contains $e_1 \otimes f_1$. Let $x \in M$, then $x = \sum n_i(a_i \otimes b_i)$ for some $n_i \in N$, $a_i \in A$, and $b_i \in B$. Now

$$e \otimes f = (\eta \otimes \xi)(x) = \sum n_i(\eta(a_i) \otimes \xi(b_i))$$

and since $e \otimes f$ is the identity of $E \otimes F$, it follows that $\eta(a_i) \otimes \xi(b_i) = e \otimes f$ for each i. Let $e' = \eta(a_i)$ and $f' = \xi(b_i)$. Then by expanding $(e + e') \otimes (f + f')$ one sees that $e' \otimes f = e \otimes f = e \otimes f'$. Note, however, that the function $\sigma: E \times F \to E$ defined by $(s, f) \mapsto s$ is a bihomomorphism and if $\sigma^*: E \otimes F \to E$ is its induced morphism, then $e = \sigma^*(e \otimes f) = \sigma^*(e' \otimes f) = e'$. Similarly $f = f'$. Thus $\eta(a_i) = e$ and $\xi(b_i) = f$ for each i. Since $a_i \otimes f_1$ and $e_1 \otimes b_i$ are both idempotents in the same archimedean component of $A \otimes B$ it follows that $a_i \otimes f_1 = e_1 \otimes b_i = e_1 \otimes f_1$ and thus

$$(a_i + e_1) \otimes (b_i + f_1) = (a_i \otimes b_i) + (e_1 \otimes f_1) .$$

We have

$$\begin{aligned} x &= \sum n_i(a_i \otimes b_i) = \sum n_i((a_i \otimes b_i) + (e_1 \otimes f_1)) \\ &= \sum n_i((a_i + e_1) \otimes (b_i + f_1)) . \end{aligned}$$

But $a_i + e_1 \in G_{e_1}$ and $b_i + f_1 \in H_{f_1}$ for each i; thus $x \in G_{e_1} \otimes H_{f_1}$. The proposition follows. The following corollaries are immediate.

COROLLARY 8. *Assume A and B are abelian semigroups with respective groups of units G and H. Then $G \otimes H$ is the group of units of $A \otimes B$.*

COROLLARY 9. *Assume A and B are abelian archimedean semigroups each of which contains an idempotent. If G and H are the maximal subgroups of A and B respectively, then $G \otimes H$ is the maximal subgroup of $A \otimes B$.*

3. The torsion functor. We follow MacLane [3]. Throughout this section A and B will denote abelian semigroups with nonvoid sets of idempotents E and F, respectively. Let N^* denote the set of positive integers and let $T(A, B)$ denote the set of all triples $(a, n, b) \in A \times N^* \times B$ such that na and nb are idempotent. For each (a, n, b) in $T(A, B)$, let $\langle a, n, b \rangle$ denote the corresponding element of the free abelian semigroup $\mathscr{T}$ on $T(A, B)$. Finally, define Tor (A, B) to be the semigroup $\mathscr{T}$ subject to the relations:

$$\begin{aligned} \langle a_1 + a_2, n, b\rangle &= \langle a_1, n, b\rangle + \langle a_2, n, b\rangle \\ \langle a, n, b_1 + b_2\rangle &= \langle a, n, b_1\rangle + \langle a, n, b_2\rangle \\ \langle a, nm, b\rangle &= \langle na, m, b\rangle \\ \langle a, nm, b\rangle &= \langle a, n, mb\rangle \end{aligned}$$

for $a_1, a_2, a \in A$, $b_1, b_2, b \in B$ and $m, n \in N^*$ such that each of the triples $\langle\ ,\ ,\ \rangle$ above is member of $\mathscr{T}$ determined by a member of $T(A, B)$. Whenever $(a, n, b) \in T(A, B)$, $[a, n, b]$ will denote the member of Tor (A, B) which (as an equivalence class of $\mathscr{T}$) contains $\langle a, n, b\rangle$.

Observe that Tor (A, B) has a universal property similar to the one stated for $\otimes$ in Proposition 1. More precisely, if φ: $T(A, B) \rightarrow C$ is a function from $T(A, B)$ into an abelian semigroup C such that φ "preserves" the relations which define Tor (A, B), then there exists a unique semigroup morphism φ^*: Tor $(A, B) \rightarrow C$ such that $\varphi^*([a, n, b]) = \varphi((a, n, b))$ for all $(a, n, b) \in T(A, B)$. This property along with many elementary arguments similar to the ones displayed for tensor above may be used to establish various propositions regarding the torsion functor. We state some of these propositions below without proof as the proofs are not particularly instructive. First we need some terminology. If A is an abelian semigroup (having a nonvoid set of idempotents) and $x \in A$, then there is $n \in N^*$ such that nx is idempotent if and only if there exist distinct r and s in N^* such that $rx = sx$. Each such x is said to be *torsion*. The least $s \in N^*$ such that $sx = rx$ for some $r \in N^*$ such that $r \neq s$ is called the *index* of x. The subsemigroup of A consisting of all the torsion elements of A will be denoted by A_t.

PROPOSITION 10. *Assume A and B are abelian semigroups each of which contains idempotent elements. Then*

(1) Tor $(A, B) =$ Tor (A_t, B_t)

(2) Tor $(A, B) \cong$ Tor (B, A)

(3) Tor *is a covariant bifunctor defined on the category of pairs (A, B) where A and B are objects of $\mathscr{S}$ which contain idempotents.*

PROPOSITION 11. *If, for each $\lambda \in \Lambda$ and each $\mu \in \Omega$, A_λ and B_μ are abelian semigroups with identity, then*

$$\text{Tor}\,(\textstyle\sum_\lambda A_\lambda, \sum_\mu B_\mu) \cong \sum_\lambda \sum_\mu \text{Tor}\,(A_\lambda, B_\mu)\ .$$

PROPOSITION 12. *If G and H are abelian groups* Tor (G, H) *is a group and is isomorphic to the usual torsion product of two groups* (as is defined, for example, in MacLane [3]).

PROPOSITION 13. *If A and B are abelian semigroups and G and H are maximal subgroups of A and B, respectively, then* $\operatorname{Tor}(G, H)$ *may be naturally identified with the set of elements of* $\operatorname{Tor}(A, B)$ *of the form*

$$\textstyle\sum_i n_i[g_i, m_i, h_i]$$

for $n_i \in N^*$, $m_i \in N^*$, $g_i \in G$, and $h_i \in H$ such that $m_i g_i$ and $m_i h_i$ are the identities of G and H, respectively.

COROLLARY 14. *The torsion product of two abelian inverse semigroups is an abelian inverse semigroup.*

PROPOSITION 15. *If E and F are semilattices, then* $\operatorname{Tor}(E, F) \cong E \otimes F \cong E \times F$ *where $E \times F$ denotes the direct product of E and F.*

REMARK. The isomorphism $E \otimes F \cong E \times F$ is obtained in the proof of Proposition 7.

PROPOSITION 16. *Assume that A and B are abelian semigroups each of which contains idempotent elements, that A_t and B_t are their respective torsion subsemigroups and that E_A and E_B are their respective idempotent subsemigroups. Let $\delta_A: A_t \rightarrow E_A$ and $\delta_B: B_t \rightarrow E_B$ denote the functions which associate with each x the idempotent in the cyclic subsemigroup generated by x. Then δ_A and δ_B are the natural maps of A_t and B_t onto their respective maximal semilattice homomorphic images E_A and E_B. Moreover the maximal semilattice homomorphic image of*

$$\operatorname{Tor}(A, B) = \operatorname{Tor}(A_t, B_t) \quad \textit{is} \quad \operatorname{Tor}(E_A, E_B) \cong E_A \otimes E_B \cong E_A \times E_B$$

and the canonical mapping of $\operatorname{Tor}(A_t, B_t)$ *onto its maximal semilattice homomorphic image is precisely* $\operatorname{Tor}(\delta_A, \delta_B)$.

REMARK. As with tensor, note that if $A = \bigcup_{e \in E} A_e$ and $B = \bigcup_{f \in F} B_f$ are semilattice unions of groups, then

$$\operatorname{Tor}(A, B) = \operatorname{Tor}(A_t, B_t) = \bigcup_{(e,f) \in E \times F} \operatorname{Tor}((A_e)_t, (B_f)_t) \,.$$

4. The Grothendieck functor. Recall that if A is an abelian semigroup then the Grothendieck group of A is an abelian group $K(A)$ having the property that there is a homomorphism K_A from A into $K(A)$ such that if G is any abelian group and φ any homomorphism from A into G then there exists a unique homomorphism φ^* from $K(A)$ into G such that the diagram

$$\begin{tikzcd} A \arrow[r, "K_A"] \arrow[dr, "\varphi"'] & K(A) \arrow[d, dashed, "\varphi^*"] \\ & G \end{tikzcd}$$

is commutative. Also recall that $K(A)$ is obtained as follows. Let $\mathscr{F}$ denote the free abelian group on A and H the subgroup of $\mathscr{F}$ generated by all elements of $\mathscr{F}$ of the form

$$(*) \qquad \langle a_1 + a_2\rangle - \langle a_1\rangle - \langle a_2\rangle$$

for a_1 and a_2 in A. Define $K(A) = \mathscr{F}/H$. One may then show that K is actually a functor by using the universal property above. We call this functor the Grothendieck functor.

PROPOSITION 17. *If A and B are abelian semigroups, then*

$$K(A \otimes B) \cong K(A) \otimes K(B) .$$

Proof. Let $\mathscr{F}_A$, $\mathscr{F}_B$, and $\mathscr{F}_{A\otimes B}$ denote the respective free abelian groups on A, B, and $A \otimes B$. Let H_A, H_B, and $H_{A\otimes B}$ denote the subgroups defined by (*) above so that $K(A) = \mathscr{F}_A/H_A$, $K(B) = \mathscr{F}_B/H_B$, and $K(A \otimes B) = \mathscr{F}_{A\otimes B}/H_{A\otimes B}$. Let

$$\eta_A: \mathscr{F}_A \to K(A), \eta_B: \mathscr{F}_B \to K(B), \quad \text{and} \quad \eta_{A\otimes B}: \mathscr{F}_{A\otimes B} \to K(A \otimes B)$$

denote the natural mappings. Using the universal properties of free abelian groups one obtains the existence of a bihomomorphism $\sigma: \mathscr{F}_A \times \mathscr{F}_B \to K(A \otimes B)$ such that $\sigma(\langle a\rangle, \langle b\rangle) = \eta_{A\otimes B}(a \otimes b)$ for $a \in A$ and $b \in B$. Define $\sigma^*: K(A) \times K(B) \to K(A \otimes B)$ by $\sigma^*(\eta_A(x), \eta_B(y)) = \sigma(x, y)$ for $x \in \mathscr{F}_A$ and $y \in \mathscr{F}_B$. We show that σ^* is well-defined. Assume $\eta_A(x) = \eta_A(x')$ and $\eta_B(y) = \eta_B(y')$. Then $x = x' + h$ and $y = y' + k$ for some $h \in H_A$ and $k \in H_B$. Thus

$$\sigma(x, y) = \sigma(x' + h, y' + k) = \sigma(x', y') + \sigma(x', k) + \sigma(h, y' + k) .$$

We show that $\sigma(h, y' + k) = 0$. Since $y' + k \in \mathscr{F}_B$ and $h \in H_A$ there exist integers n_i and m_j elements of A, a_{1j} and a_{2j} and b_i in B such that $y' + k = \sum_i n_i\langle b_i\rangle$ and $h = \sum m_j[\langle a_{1j} + a_{2j}\rangle - \langle a_{1j}\rangle - \langle a_{2j}\rangle]$. Thus

$$\begin{aligned} &\sigma(h, y' + k) \\ &\quad = \sum_i n_i \sum_j m_j[\sigma(\langle a_{1j} + a_{2j}\rangle, \langle b_i\rangle) - \sigma(\langle a_{1j}\rangle, \langle b_i\rangle) - \sigma(\langle a_{2j}\rangle, \langle b_i\rangle)] \end{aligned}$$

which is zero by the definition of σ. Similarly $\sigma(x', k) = 0$ and $\sigma(x, y) =$

$\sigma(x', y')$. Thus σ^* is well-defined. The map σ^* is clearly a bihomomorphism thus there exists a unique homomorphism $\psi: K(A) \otimes K(B) \to K(A \otimes B)$ such that $\psi(x \otimes y) = \sigma^*(x, y)$ for $(x, y) \in K(A) \times K(B)$. We claim that ψ is an isomorphism and we prove that this is so by constructing its inverse. Let $\varphi: A \times B \to K(A) \otimes K(B)$ be defined by $\varphi(a, b) = K_A(a) \otimes K_B(b)$. Let φ^* denote the homomorphism from $A \otimes B$ into $K(A) \otimes K(B)$ induced by φ. Let θ denote the homomorphism for which the diagram

$$\begin{tikzcd} A \otimes B \arrow[rr, "K_{A \otimes B}"] \arrow[dr, "\varphi^*"'] & & K(A \otimes B) \arrow[dl, "\theta"] \\ & K(A) \otimes K(B) & \end{tikzcd}$$

is commutative. It is a tedious computation to show that θ and ψ are inverses of one another but the computation is straightforward and thus is omitted. The proposition follows.

At this point it seems appropriate to mention the work of two others who have done some work on the notion of tensor products of semigroups. T. J. Head has written a series of papers on the subject and has obtained our Proposition 4. Also Pierre Grillet has obtained Proposition 4. There seems to be not a great deal of other overlap among these papers. All three of us obtained our results independently and almost simultaneously.

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