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SOME THEOREMS ON GENERALIZED DEDEKIND–RADEMACHER SUMS

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Radamacher has defined a generalized Dedekind sum

$$s(h,k;x,y)=\sum_{a\,(\mathrm{mod}\,k)}\left(\left(h\frac{a+y}{k}+x\right)\right)\left(\left(\frac{a+y}{k}\right)\right)$$

and proved a reciprocity theorem for this sum that generalizes the well known result for $s(h,k)$. In the present paper we define

$$\phi_{r,s}(h,k;x,y)=\sum_{a\,(\mathrm{mod}\,k)}\bar{B}_r\left(h\left(\frac{a+y}{k}\right)+x\right)\bar{B}_s\left(\frac{a+y}{k}\right),$$

$$\psi_{r,s}(h,k;x,y)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\phi_{j,r+s-j}(h,k;x,y),$$

where $\bar{B}_n(x)$ is the Bernoulli function, and show that

$$\begin{aligned}&(s+1)k^s\psi_{r+1,s}(h,k;x,y)-(r+1)h^r\psi_{s+1,r}(k,h;y,x)\\&\quad=(s+1)k\bar{B}_{r+1}(x)\bar{B}_s(y)-(r+1)h\bar{B}_r(x)\bar{B}_{s+1}(y)\qquad((h,k)=1).\end{aligned}$$

We also prove the polynomial reciprocity theorem

$$(1-v)\sum_{a=0}^{k-1}u^{h-[(ha+z)/k]}v^a-(1-u)\sum_{b=0}^{h-1}v^{k-[(kb+z)/h]}u^b=u^h-v^k$$
$$((h,k)=1)$$

as well as some related results.

1. Introduction.

For real x put

$$((x))=\begin{cases}x-[x]-\frac{1}{2} & (x\neq\text{integer})\\ 0 & (x=\text{integer}),\end{cases}$$

where $[x]$ denotes the greatest integer $\leqq x$. The Dedekind sum $s(h,k)$ is defined by

$$s(h,k)=\sum_{a\,(\mathrm{mod}\,k)}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ha}{k}\right)\right). \tag{1.1}$$

The most striking property of $s(h, k)$ is the reciprocity theorem

$$12hk\{s(h,k)+s(k,h)\}=h^2-3hk+k^2+1 \qquad ((h,k)=1). \tag{1.2}$$

For an excellent introduction and many references to Dedekind sums see **[9]**.

The Bernoulli function $\bar{B}_n(x)$ is defined by

$$\bar{B}_n(x)=B_n(x-[x]),$$

where $B_n(x)$ is the Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}. \tag{1.3}$$

Note that, for $x \neq$ integer, $\bar{B}_1(x)=((x))$.

Apostol **[1]**, **[2]** defined the generalized sum

$$s_n(h,k)=\sum_{r\,(\mathrm{mod}\,k)}\bar{B}_1\left(\frac{r}{k}\right)\bar{B}_n\left(\frac{hr}{k}\right) \tag{1.4}$$

and proved the reciprocity theorem

$$(n+1)\{hk^n s_n(h,k)+kh^n s_n(k,h)\}=(Bk+Bh)^{n+1}+nB_{n+1} \qquad ((h,k)=1). \tag{1.5}$$

This result is indeed valid for all $n \geqq 0$. For a simple proof see [**4**, §3].

A further generalization of (1.4) is furnished by

$$\phi_{r,s}(h,k)=\sum_{a\,(\mathrm{mod}\,k)}\bar{B}_r\left(\frac{a}{k}\right)\bar{B}_s\left(\frac{ha}{k}\right), \tag{1.6}$$

where r, s are arbitrary nonnegative integers. Put

$$\psi_{r,s}(h,k)=\sum_{t=0}^{r}(-1)^t\binom{r}{t}h^t\phi_{r-t,s+t}(h,k). \tag{1.7}$$

The writer **[3]**, **[7]** has proved the following reciprocity theorem which includes (1.5) as a special case.

$$(1.8)\qquad (s+1)k^s\psi_{r+1,s}(h,k)-(s+1)kB_{r+1}B_s=(r+1)h^r\psi_{s+1,r}(k,h) -(r+1)hB_{s+1}B_r \qquad ((h,k)=1).$$

He has also proved the following polynomial reciprocity:

$$(1.9)\qquad (u-1)\sum_{r=1}^{k-1}u^{k-r-1}v^{[hr/k]}-(v-1)\sum_{r=1}^{h-1}v^{h-r-1}u^{[kr/h]} = u^{k-1}-v^{h-1} \qquad ((h,k)=1),$$

where u, v are indeterminates.

Rademacher **[10]** has generalized $s(h,k)$ in the following way:

$$(1.10)\qquad s(h,k;x,y)=\sum_{a \pmod k}\left(\left(h\frac{a+y}{k}+x\right)\right)\left(\left(\frac{a+y}{k}\right)\right),$$

where x, y are arbitrary real numbers. He proved that

$$(1.11)\qquad \begin{aligned} s(h,k;x,y)+s(k,h;y,x) &= -\tfrac{1}{4}\delta(x)\delta(y)+((x))((y)) \\ &\quad +\frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y)+\frac{1}{hk}\bar{B}_2(hy+kx)+\frac{k}{h}\bar{B}_2(x)\right\}, \end{aligned}$$

where $(h,k)=1$ and

$$\delta(x)=\begin{cases}1 & (x=\text{integer})\\ 0 & (x\neq\text{integer}).\end{cases}$$

For a simplified version of the proof see **[5]**.

In the present paper we define

$$(1.12)\qquad \phi_{r,s}(h,k;x,y)=\sum_{a \pmod k}\bar{B}_r\left(h\frac{a+y}{k}+x\right)\bar{B}_s\left(\frac{a+y}{k}\right)$$

and

$$(1.13)\qquad \psi_{r,s}(h,k;x,y)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\phi_{j,r+s-j}(h,k;x,y),$$

corresponding to (1.6) and (1.7), respectively. We prove the reciprocity theorem

$$(s+1)k^s\psi_{r+1,s}(h,k;x,y)-(r+1)h^r\psi_{s+1,r}(k,h;y,x)=(s+1)k\bar{B}_{r+1}(x)\bar{B}_s(y)$$

$$-(r+1)h\bar{B}_r(x)\bar{B}_{s+1}(y) \qquad ((h,k)=1). \tag{1.14}$$

It should be observed that there is no loss in generality in assuming that

$$0 \leqq x < 1, \qquad 0 \leqq y < 1. \tag{1.15}$$

We show also, assuming (1.15), that

$$(1-v)\sum_{a=0}^{k-1} u^{h-[(ha+z)/k]}v^a-(1-u)\sum_{b=0}^{h-1} v^{k-[(kb+z)/h]}u^b = u^h - v^k \tag{1.16}$$

$$((h,k)=1),$$

where $z = kx + hy$. For $x = y = 0$, (1.16) reduces to (1.9) after a little manipulation. Clearly (1.16) holds for all z such that $0 \leqq z < h + k$.

For some additional results see §4 below, in particular (4.1), (4.2), (4.3), (4.4), (4.5), (4.6).

2. Proof of (1.14). We recall that [**8**, Ch. 2]

$$\bar{B}_n(hx) = h^{n-1}\sum_{b \,(\mathrm{mod}\, h)} \bar{B}_n\left(x+\frac{b}{h}\right). \tag{2.1}$$

Thus (1.12) becomes

$$\phi_{r,s}(h,k;x,y) = h^{r-1}\sum_{\substack{a \,(\mathrm{mod}\, k)\\ b\,(\mathrm{mod}\, h)}} \bar{B}_r\left(\frac{a+y}{k}+\frac{b+x}{h}\right)\bar{B}_s\left(\frac{a+y}{h}\right).$$

We shall write this in the abbreviated form

$$\phi_{r,s}(h,k;x,y) = h^{r-1}\sum_{a,b} \bar{B}_r(\alpha+\beta)\bar{B}_s(\alpha), \tag{2.2}$$

where

$$\alpha = \frac{a+y}{k}, \qquad \beta = \frac{b+x}{h} \tag{2.3}$$

and the summation on the right of (2.2) is over complete residue systems (mod k) and (mod h), respectively.

Substituting from (2.2) in (1.13), we get

$$\psi_{r,s}(h, k; x, y) = h^{r-1} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sum_{a,b} \bar{B}_j(\alpha + \beta) \bar{B}_{r+s-j}(\alpha). \tag{2.4}$$

Now consider

$$\begin{aligned} &\Phi(h, k; x, y; u, v) \\ &\quad = \sum_{r,s=0}^{\infty} sk^{s-1} \psi_{r,s-1}(h, k; x, y) \frac{u^r v^s}{r!s!} \\ &\quad = \sum_{r,s} \frac{h^{r-1} k^{s-1} u^r v^s}{r!(s-1)!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \bar{B}_j(\alpha + \beta) \bar{B}_{r+s-j-1}(\alpha). \end{aligned} \tag{2.5}$$

We assume in what follows that

$$0 \leqq x < 1, \qquad 0 \leqq y < 1, \tag{2.6}$$

which implies

$$0 \leqq \alpha < 1, \qquad 0 \leqq \beta < 1. \tag{2.7}$$

Thus

$$\bar{B}_{r+s-j-1}(\alpha) = B_{r+s-j-1}(\alpha).$$

Taking $m = r + s - j - 1$, (2.5) becomes

$$\begin{aligned} &\Phi(h, k; x, y; u, v) \\ &\quad = h^{-1} v \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \sum_{j=0}^{\infty} \frac{(hu)^j}{j!} \bar{B}_j(\alpha + \beta) \sum_{m=0}^{\infty} \frac{1}{m!} B_m(\alpha) \\ &\qquad\qquad \sum_{r=j}^{m+j} (-1)^{r-j} \frac{m!}{(r-j)!(m-r+j)!} (hu)^{r-j} (kv)^{m-r+j} \\ &\quad = h^{-1} v \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \sum_{j=0}^{\infty} \frac{(hu)^j}{j!} \bar{B}_j(\alpha + \beta) \sum_{m=0}^{\infty} \frac{(-hu + kv)^m}{m!} B_m(\alpha) \\ &\quad = h^{-1} v \sum_{j,m=0}^{\infty} \frac{(hu)^j (-hu + kv)^m}{j!m!} \sum_{a,b} \bar{B}_j(\alpha + \beta) B_m(\alpha). \end{aligned} \tag{2.8}$$

Since

$$B_j(x + 1) - B_j(x) = jx^{j-1},$$

the inner sum

$$\sum_{a=0}^{k-1}\sum_{b=0}^{h-1}\bar{B}_j(\alpha+\beta)B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta-[\alpha+\beta])B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta)B_m(\alpha) - \sum_{\substack{a,b\\ \alpha+\beta\geqq 1}} (B_j(\alpha+\beta)-B_j(\alpha+\beta-1))B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta)B_m(\alpha) - j\sum_{\substack{a,b\\ \alpha+\beta\geqq 1}} (\alpha+\beta-1)^{j-1}B_m(\alpha).$$

Thus (2.8) becomes

(2.9) $\Phi(h,k;x,y;u,v) = \Phi_1(h,k;x,y;u,v) - \Phi_2(h,k;x,y;u,v),$

where

$$\Phi_1(h,k;x,y;u,v) = h^{-1}v\sum_{j,m=0}^{\infty}\frac{(hu)^j(-hu+kv)^m}{j!m!} \cdot \sum_{a,b} B_j(\alpha+\beta)B_m(\alpha),$$

$$\Phi_2(h,k;x,y;u,v) = h^{-1}v\sum_{j,m=0}^{\infty}\frac{(hu)^j(-hu+kv)^m}{j!m!} \cdot \sum_{\substack{a,b\\ \alpha+\beta\geqq 1}} (\alpha+\beta-1)^{j-1}B_m(\alpha).$$

Clearly, by (1.3) and (2.3),

$$\Phi_1(h,k;x,y;u,v) = h^{-1}v\frac{hu}{e^{hu}-1}\frac{-hu+kv}{e^{-hu+kv}-1} \cdot \sum_{a,b} e^{hu(\alpha+\beta)}e^{(-hu+kv)\alpha}$$

(2.10)

$$= \frac{uv}{e^{hu}-1}\frac{-hu+kv}{e^{-hu+kv}-1}\cdot e^{xu+yv}\frac{e^{hu}-1}{e^{u}-1}\frac{e^{kv}-1}{e^{v}-1}$$

$$= \frac{uv}{e^{u}-1}\frac{e^{kv}-1}{e^{v}-1}\frac{hu-kv}{e^{hu}-e^{kv}}e^{hu+xu+yv},$$

$$\begin{aligned}\Phi_2(h,k;x,y;u,v) &= uv\frac{-hu+kv}{e^{-hu+kv}-1}\cdot\sum_{\substack{a,b\\ \alpha+\beta\geqq 1}} e^{hu(\alpha+\beta-1)}e^{(-hu+kv)\alpha}\\ &= uv\frac{hu-kv}{e^{hu}-e^{kv}}e^{xu+yv}\sum_{\substack{a,b\\ \alpha+\beta\geqq 1}} e^{av+bu}.\end{aligned} \tag{2.11}$$

It follows from (2.10) that

$$\begin{aligned}&\Phi_1(h,k;x,y;u,v)-\Phi_1(k,h;y,x;v,u)\\ &=\frac{u}{e^u-1}\frac{v}{e^v-1}\frac{hu-kv}{e^{hu}-e^{kv}}e^{xu+yv}\{e^{hu}(e^{kv}-1)-e^{kv}(e^{hu}-1)\}.\\ &=(-hu+kv)\frac{ue^{xu}}{e^u-1}\frac{ve^{yv}}{e^v-1},\end{aligned} \tag{2.12}$$

while

$$\Phi_2(h,k;x,y;u,v)-\Phi_2(k,h;y,x;v,u)=0. \tag{2.13}$$

Therefore, by (2.9), (2.12) and (2.13),

$$\Phi(h,k;x,y;u,v)-\Phi(k,h;y,x;v,u)=(-hu+kv)\frac{ue^{xu}}{e^u-1}\frac{ve^{yv}}{e^v-1}. \tag{2.14}$$

By (2.5), the left hand side of (2.14) is equal to

$$\sum_{r,s=0}^{\infty}\{sk^{s-1}\psi_{r,s-1}(h,k;x,y)-rh^{r-1}\psi_{s,r-1}(k,h;y,x)\}\frac{u^rv^s}{r!s!}.$$

By (1.3), the right hand side of (2.14) is equal to

$$\begin{aligned}&(-hu+kv)\sum_{r,s=0}^{\infty}B_r(x)B_s(y)\frac{u^rv^s}{r!s!}\\ &=\sum_{r,s=0}^{\infty}\{skB_r(x)B_{s-1}(y)-rhB_{r-1}(x)B_s(y)\}\frac{u^rv^s}{r!s!}.\end{aligned}$$

Hence, equating coefficients of $u^rv^s/r!s!$, we get

$$sk^{s-1}\psi_{r,s-1}(h,k;x,y)-rh^{r-1}\psi_{s,r-1}(k,h;y,x)=skB_r(x)B_{s-1}(y)-rhB_{r-1}(x)B_s(y).$$

Finally, dropping the restriction (2.6), we have

$$sk^{s-1}\psi_{r,s-1}(h,k;x,y)-rh^{r-1}\psi_{s,r-1}(k,h;y,x)=sk\bar{B}_r(x)\bar{B}_{s-1}(y)-rh\bar{B}_{r-1}(x)\bar{B}_s(y), \tag{2.15}$$

for all nonnegative r, s and all real x, y.

3. *Proof of* (1.16). We again assume that

$$0\leqq x<1,\quad 0\leqq y<1. \tag{3.1}$$

By (1.12) and (1.13) we have

$$\psi_{r,s}(h,k;x,y)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\sum_{a=0}^{k-1}\bar{B}_j\left(h\frac{a+y}{k}+x\right)B_{r+s-j}\left(\frac{a+y}{k}\right).$$

Then, as in the previous proof,

$$\begin{aligned}
\Phi&(h,k;x,y;u,v)\\
&=\sum_{r,s=0}^{\infty}sk^{s-1}\psi_{r,s-1}(h,k;x,y)\frac{u^rv^s}{r!s!}\\
&=v\sum_{r,s=0}^{\infty}\frac{u^r(kv)^s}{r!s!}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\sum_{a=0}^{k-1}\bar{B}_j\left(h\frac{a+y}{k}+x\right)B_{r+s-j}\left(\frac{a+y}{k}\right)\\
&=v\sum_{j,m=0}^{\infty}\frac{u^j(-hu+kv)^m}{j!m!}\sum_{a=0}^{k-1}B_j\left(\frac{ha+z}{k}-\left[\frac{ha+z}{k}\right]\right)B_m\left(\frac{a+y}{k}\right)\\
&=v\frac{u}{e^u-1}\,\frac{-hu+kv}{e^{-hu+kv}-1}\sum_{a=0}^{k-1}\exp\left\{\left(\frac{ha+z}{k}-\left[\frac{ha+z}{k}\right]\right)u\right.\\
&\qquad\qquad\left.+\frac{a+y}{k}(-hu+kv)\right\}\\
&=\frac{uv}{e^u-1}\,\frac{hu-kv}{e^{hu}-e^{kv}}\,e^{xu+yv}\sum_{a=0}^{k-1}\exp\left\{\left(h-\left[\frac{ha+z}{k}\right]\right)u+av\right\},
\end{aligned}$$

where

$$z=kx+hy. \tag{3.2}$$

It follows that

$$\begin{aligned}&\Phi(h,k;x,y;u,v)-\Phi(k,h;y,x;v,u)\\ &= \frac{uv}{e^u-1}\,\frac{hu-kv}{e^{hu}-e^{kv}}\,e^{xu+yv}\sum_{a=0}^{k-1}\exp\left\{\left(h-\left[\frac{ha+z}{k}\right]\right)u+av\right\}\\ &\quad-\frac{uv}{e^v-1}\,\frac{hu-kv}{e^{hu}-e^{kv}}\,e^{xu+yv}\sum_{b=0}^{h-1}\exp\left\{\left(k-\left[\frac{kb+z}{h}\right]\right)v+bu\right\}.\end{aligned} \tag{3.3}$$

Comparing (3.3) with (2.14) and simplifying, we get

$$\begin{aligned}&(e^v-1)\sum_{a=0}^{k-1}\exp\left\{\left(h-\left[\frac{ha+z}{k}\right]\right)u+av\right\}\\ &\quad-(e^u-1)\sum_{b=0}^{h-1}\exp\left\{\left(k-\left[\frac{kb+z}{h}\right]\right)v+bu\right\}=-e^{hu}+e^{kv}.\end{aligned} \tag{3.4}$$

Replacing e^u, e^v by u, v, respectively, this becomes

$$(1-v)\sum_{a=0}^{k-1}u^{h-[(ha+z)/k]}v^a-(1-u)\sum_{b=0}^{h-1}v^{k-[(kb+z)/h]}u^b=u^h-v^k \qquad ((h,k)=1). \tag{3.5}$$

Clearly (3.5) is a polynomial identity in the indeterminates u, v. It is not evident how the restriction (3.1) can be removed.

To show that (3.5) includes (1.9), take $x=y=z=0$ and replace a by $k-a$, b by $k-b$. Thus the left hand side of (3.5) becomes

$$\begin{aligned}&(1-v)u^h-(1-u)v^k\\ &\quad+(1-v)\sum_{a=1}^{k-1}u^{h-[h-(ha/k)]}v^{k-a}-(1-u)\sum_{b=1}^{h-1}v^{k-[k-(kb/h)]}u^{h-b}\\ &=(u^h-v^k)-uv(u^{h-1}-v^{k-1})\\ &\quad+(1-v)\sum_{a=1}^{k-1}u^{[ha/k]+1}v^{k-a}-(1-u)\sum_{b=1}^{h-1}v^{[kb/h]+1}u^{h-b},\end{aligned}$$

since

$$[m-x]=m-1-[x] \qquad (m=\text{integer},\ x\neq\text{integer}).$$

Thus we get

$$(1-v)\sum_{a=1}^{k-1} u^{[ha/k]}v^{k-a-1}-(1-u)\sum_{b=1}^{h-1} v^{[kb/h]}u^{h-b-1}=u^{h-1}-v^{k-1},$$

which is (1.9) in a slightly different notation.

4. Additional results. We have

$$\begin{aligned}(e^v-1)\sum_{a=0}^{k-1}\exp\left\{\left(h-\left[\frac{ha+z}{k}\right]\right)u+av\right\}\\ =\sum_{a=0}^{k-1}\sum_{r=0}^{\infty}\left(h-\left[\frac{ha+z}{k}\right]\right)^r\frac{u^r}{r!}\sum_{s=0}^{\infty}((a+1)^s-a^s)\frac{v^s}{s!}.\end{aligned}$$

Thus the left hand side of (3.4) is equal to

$$\begin{aligned}\sum_{r,s=0}^{\infty}\frac{u^rv^s}{r!s!}\Bigg\{\sum_{a=0}^{k-1}\left(h-\left[\frac{ha+z}{k}\right]\right)^r((a+1)^s-a^s)\\ -\sum_{b=0}^{h-1}\left(k-\left[\frac{kb+z}{h}\right]\right)^s((b+1)^r-b^r)\Bigg\}.\end{aligned}$$

Since the right hand side of (3.4) is equal to

$$-\sum_{r=0}^{\infty}\frac{h^ru^r}{r!}+\sum_{s=0}^{\infty}\frac{k^sv^s}{s!},$$

we get

$$\begin{aligned}\sum_{a=0}^{k-1}\left(h-\left[\frac{ha+z}{k}\right]\right)^r((a+1)^s-a^s)-\sum_{b=0}^{h-1}\left(k-\left[\frac{kb+z}{h}\right]\right)^s((b+1)^r-b^r)\\ =-h^r\delta_{s,0}+k^s\delta_{r,0},\qquad ((h,k)=1)\end{aligned}\tag{4.1}$$

for all nonnegative r, s and all z such that

$$0\leqq z<h+k.$$

Hence, in particular,

$$\begin{aligned}\sum_{a=0}^{k-1}\left(h-\left[\frac{ha+z}{k}\right]\right)^r((a+1)^s-a^s)=\sum_{b=0}^{h-1}\left(k-\left[\frac{kb+z}{h}\right]\right)^s((b+1)^r-b^r)\\ (r>0,s>0;\quad 0\leqq z<h+k).\end{aligned}\tag{4.2}$$

For example, for $r = s = 2$,

$$\sum_{a=0}^{k-1} (2a+1)\left(h - \left[\frac{ha+z}{k}\right]\right)^2 = \sum_{b=0}^{h-1} (2b+1)\left(k - \left[\frac{kb+z}{h}\right]\right)^2$$

$$(0 \leqq z < h+k).$$

For $s = 1$ we get

$$\sum_{a=0}^{k-1} \left(h - \left[\frac{ha+z}{k}\right]\right)^r = \sum_{b=0}^{h-1} \left(k - \left[\frac{kb+z}{h}\right]\right)((b+1)^r - b^r) \tag{4.3}$$

$$(r > 0, \quad 0 \leqq z < h+k).$$

Recall [8, Ch. 2] that

$$nx^{n-1} = B_n(x+1) - B_n(x)$$
$$= \sum_{j=0}^{n} \binom{n}{j} B_{n-j}((x+1)^j - x^j),$$

where $B_n = B_n(0)$ is the nth Bernoulli number. Thus if in (4.1) we replace r, s by i, j, respectively, multiply both sides by

$$\binom{r}{i}\binom{s}{j} B_{r-i} B_{s-j}$$

and sum over i, j, we get

$$s \sum_{a=0}^{k-1} a^{s-1} B_r\left(h - \left[\frac{ha+z}{k}\right]\right) - r \sum_{b=0}^{h-1} b^{r-1} B_s\left(k - \left[\frac{kb+z}{h}\right]\right) \tag{4.4}$$
$$= B_s(k)B_r - B_r(h)B_s \qquad (0 \leqq z < h+k).$$

A more general result is

$$s \sum_{a=0}^{k-1} (a+\eta)^{s-1} B_r\left(h + \xi - \left[\frac{ha+z}{k}\right]\right)$$
$$- r \sum_{b=0}^{h-1} (b+\xi)^{r-1} B_s\left(k + \eta - \left[\frac{kb+z}{h}\right]\right) \tag{4.5}$$
$$= B_r(\xi)B_s(k+\eta) - B_r(h+\xi)B_s(\eta) \qquad (0 \leqq z < h+k),$$

where ξ and η are arbitrary. In particular, for $\xi = 1-h$, $\eta = 1-k$, (4.5) reduces to

$$s\sum_{a=0}^{k-1}(a+1-k)^{s-1}B_r\left(1-\left[\frac{ha+z}{k}\right]\right)\ -r\sum_{b=0}^{h-1}(b+1-h)^{r-1}B_s\left(1-\left[\frac{kb+z}{h}\right]\right)$$

$$= B_r(1-h)B_s(1)-B_r(1)B_s(1-k).$$

Since

$$B_n(1-x)=(-1)^nB_n(x),$$

we get

$$(4.6)\qquad s\sum_{a=0}^{k-1}(k-a-1)^{s-1}B_r\left(\left[\frac{ha+z}{k}\right]\right)-r\sum_{b=0}^{h-1}(h-b-1)^{r-1}B_s\left(\left[\frac{kb+z}{h}\right]\right)$$
$$= -B_r(h)B_s+B_rB_s(k)\qquad (0\leqq z<h+k).$$

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