Let $V$ be the standard 4-dimensional module for $Sz(q)$, the Suzuki group based on the field of $q = 2^{2n+1}$ elements. In this paper we determine $H^2(Sz(q), V)$. This is usually $(q^{32})$ of dimension one (otherwise zero) and is generated by a cocycle which is the restriction of a generator of $H^2(Sp_4(q), V)$. In addition, the well known groups $H^2(Sz(q), GF(q))$ and $H^1(Sz(q), V)$ are calculated. The proof involves the use of the Hochschild-Serre spectral sequence to determine the cohomology of the normalizer of a Sylow 2-subgroup acting on the various one-dimensional modules involved.

Let $K = GF(q)$, $q = 2^{2n+1}$, let $Sz(q) (B_2(q))$ be the Suzuki group based on the field $K$ and let $B$ be a normalizer of a Sylow 2-subgroup of $Sz(q)$. In this paper we use the Hochschild-Serre spectral sequence to determine $H^i(B, V)$ $i = 1, 2$, where $V$ is a one dimensional $KB$-module, in terms of the solutions to certain equations in $End(K^*)$. These equations are solved when $V$ is trivial or involved in $K^4$, the standard four dimensional module for $KSz(q)$. Using this information we determine $H^3(Sz(q), K^*)$ as well as the previously known groups $H^3(Sz(q), K)$ and $H^3(Sz(q), K^*)$. These may be viewed as results concerning conjugacy classes in semi-direct products and concerning exact sequences of groups using the well known group-theoretic interpretation of cohomology of degree 1 and 2 [6].

We will assume all cocycles are normalized, i.e. vanish when any one of their arguments is the identity. When $[f] \in H^2(G, V)$, where $G$ is a group and $V$ is a left $G$-module, let $E(f)$ denote the extension of $V$ by $G$ using $f$, that is, $E(f) = \{(v, g) | v \in V, g \in G\}$ with multiplication $(v_1, g_1)(v_2, g_2) = (v_1 + g_1(v_2) + f(g_1, g_2), g_1g_2)$.

We use the explicit description of $Sz(q)$ given in [9]. Let $K_0$ be the prime subfield of $K$, $\Gamma = Gal(K/K_0)$ and $\theta \in \Gamma$ defined by $\theta: x \rightarrow x^{2n}$. For $\alpha, u \in K$ and $t \in K^*$ put

$$\begin{bmatrix}
1 & u^\theta & h & g \\
1 & u & \alpha & \\
1 & u^\theta & \\
1 &
\end{bmatrix},
T(t) =
\begin{bmatrix}
t^\theta \\
t^{1-\theta} \\
t^{\theta-1} \\
t^{-\theta}
\end{bmatrix},
J =
\begin{bmatrix}
1 & \\
1 & 1
\end{bmatrix}$$

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where \( h = h(\alpha, u) = u^{\theta + 1} + \alpha \) and \( g = g(\alpha, u) = u^{2\theta + 1} + u^\theta \alpha + \alpha^2 \). Set \( U = \{(\alpha, u) | \alpha, u \in K\} \), \( T = \{T(t) | t \in K^*\} \), \( B = UT \) so \( Sz(q) = \langle B, J \rangle \subset \text{SL}_4(q) \) (in [9], \( U^J \) is used in place of \( U \)). Then \( K^4 \) (columns) is the standard module on which \( Sz(q) \) acts as multiplication on the left. In fact \( Sz(q) \) is contained in the Symplectic group defined by \( J \).

Since \( U \) is a Sylow 2-subgroup of \( Sz(q) \) which is a T. I. set with normalizer \( B \), the Cartan–Eilenberg stability theorem tells us that if \( V \) is a \( KSz(q) \)-module then the restriction maps \( H^i(Sz(q), V) \rightarrow H^i(B, V) \rightarrow H^i(U, V)^T \) are isomorphisms for \( i > 0 \). Thus (after the case \( q = 2 \)) we shall replace \( Sz(q) \) by \( B \). Furthermore these isomorphisms show that when giving explicit cocycles it is sufficient to give their restrictions to \( U \) and show they are \( T \)-stable.

Assume first \( q = 2 \). Then \( Sz(q) \) is a group of order 20. Its Sylow 5-subgroup is cyclic, normal and a generator acts fixed-point-freely on \( K^4 \). This implies \( H^i(Sz(2), K^*) = 0 \) for \( i > 0 \) [7]. Henceforth we assume \( q \geq 8 \).

Throughout we assume \( \alpha, \beta, u, v \in K \) and \( t \in K^* \). We identify \( T \) with \( K^* \). It is seen that \((\alpha, u)(\beta, v) = (\alpha + \beta + uv^\theta, u + v) \) and \((\alpha, u)^{T(t)} = T(t)(\alpha, u)T(t)^{-1} = (ta, t^{t^\theta}u) \) where \( \theta' = 2 - 2\alpha \). Also \( Z = \{(\alpha, 0)\} \) is the center and derived subgroup of \( U \). Set \( A = U/Z \) and \( X = B/Z \) so \( X \) is the semidirect product \( AT \).

When \( V \) is a \( KT \)-module and \( \nu \in \text{End}(K^*) \) we say \( T \) acts with weight \( \nu \) on \( V \) provided \( T(t)v = t^\nu v \) for all \( t \in K^*, v \in V \). The above formulas show \( Z \) and \( A \) are \( KT \)-modules of weight 1 and \( \theta' \) respectively. Observe \( \text{End}(K^*) = \mathbb{Z}/(q - 1)\mathbb{Z} \) and so is a commutative ring.

When \( V \) and \( W \) are (finite dimensional) \( K \)-modules \( \text{Hom}(W, V) = \bigoplus_{\sigma \in \Gamma} H_\sigma(W, V) \) where \( H_\sigma(W, V) \) are the \( \sigma \)-semilinear maps from \( W \) to \( V \). If additionally \( V \) and \( W \) are \( KT \)-modules of weight \( \nu \) and \( \omega \) then \( H_\sigma(W, V) \) is a \( KT \)-module of weight \( \nu - \omega \sigma \).

Now fix \( V \), a one dimensional \( KB \)-module on which \( U \) acts trivially and \( T \) acts with weight \( \nu \). We shall often identify \( V \) with \( K \). From the (nonsplit) exact sequence of groups \( 1 \rightarrow Z \rightarrow B \rightarrow X \rightarrow 1 \) the Hochschild–Serre spectral sequence gives us the exact sequences of \( K \)-modules

\[
0 \rightarrow H^2(B, V)_0 \rightarrow H^2(B, V) \xrightarrow{\text{Res}} H^2(Z, V)^X
\]

(\( * \))

\[
0 \rightarrow H^1(X, V) \rightarrow H^1(B, V) \rightarrow H^1(Z, V)^X \rightarrow H^2(X, V)
\]

\[
\rightarrow H^2(B, V)_0 \rightarrow H^1(X, H^1(Z, V)) \rightarrow H^2(X, V).
\]

Our aim is to determine \( H^2(B, V) \). In Lemmas 1, 2 and 3 we determine most of the other terms in (\( * \)) and study the maps \( \text{Res} \) and \( \Phi \).
Lemma 1. Let W and V (each identified with K) be one dimensional KT-modules of weight \( \omega \) and \( \nu \) respectively and regard V as a trivial W-module. For \( \sigma, \tau \in \Gamma \) define \( h_\sigma: W \to V \) by \( h_\sigma(w) = w^\sigma \) and \( f_{(\sigma,\tau)}: W \times W \to V \) by \( f_{(\sigma,\tau)}(w_1, w_2) = w_1^\sigma w_2^\tau \).

(a) \( \{[h_\sigma] | \nu = \omega \sigma \} \) \( \sigma \in \Gamma \) is a \( K \)-base for \( H^1(W, V)_T \).

(b) \( \{f_{(\sigma,\tau)} | \nu = \omega(\sigma + \tau)\} \) \( \{\sigma, \tau\} \subseteq \Gamma \) is a \( K \)-base for \( H^2(W, V)_T \).

Proof. (a) This statement is immediate since \( H^1(W, V)_T = \text{Hom}(W, V)_T \) and \( T \) acts on \( H_\sigma(W, V) = K h_\sigma \) with weight \( \nu - \omega \sigma \).

(b) Since \( W \) is abelian and trivial on \( V \) we have an exact sequence of \( K \)-modules \( 0 \to H^2_{ab}(W, V) \to H^2(W, V)_T \to \text{Alt}^2(W, V) \to 0 \) where \( \text{Alt}^2(W, V) \) is the group of alternate 2-forms: \( W \times W \to V \) and \( \Psi[f]: (w_1, w_2) \to f(w_1, w_2) - f(w_2, w_1) \). Furthermore \( H^2_{ab}(W, V) \cong \text{Hom}(W, V) \). See [7] for the proofs of these statements. Taking \( T \)-cohomology of the above sequence gives the exact sequence of \( K \)-modules \( 0 \to \text{Hom}(W, V)_T \to H^1(W, V)_T \to \text{Alt}^2(W, V)_T \to 0 = H^1(T, \text{Hom}(W, V)) \). We have seen \( \dim_K \text{Hom}(W, V)_T = \# \{\sigma \in \Gamma | \nu = \omega \sigma\} \) and it can be seen that when \( \nu = \omega \sigma \) then \( f_{(\mu, \nu)} \) is a corresponding cocycle in \( H^2_{ab}(W, V)_T = \text{Hom}(W, V)_T \).

In [5] it is shown that \( \text{Alt}^2(W, V)_T = \bigoplus K f_{(\sigma, \tau)} \) where we sum over all sets \( \{\sigma, \tau\} \subseteq \Gamma, \sigma \neq \tau \) and \( f_{(\sigma, \tau)}: (w_1, w_2) \to w_1^\sigma w_2^\tau - w_1^\tau w_2^\sigma \). Since \( T \) acts with weight \( \nu - \omega(\sigma + \tau) \) on \( K f_{(\sigma, \tau)} \), we have \( \text{Alt}^2(W, V)_T = \bigoplus K f_{(\sigma, \tau)} \) summed over those \( \{\sigma, \tau\} \) such that \( \nu = \omega(\sigma + \tau) \). For such \( \{\sigma, \tau\} \) it can be seen that \( [f_{(\sigma, \tau)}] \in H^2(W, V)_T \) with \( \Psi[f_{(\sigma, \tau)}] = F_{(\sigma, \tau)} \). Note \( [f_{(\sigma, \tau)}] + [f_{(\sigma, \varphi)}] = 0 \) since \( f_{(\sigma, \tau)} + f_{(\sigma, \varphi)} = \delta g \) where \( g(w) = w^{\sigma + \tau} \). This completes the proof.

Using Lemma 1 and the Cartan–Eilenberg stability theorem we can determine the terms of (\*)\( \). We have \( H^1(X, V) = H^1(A, V)_T = \text{Hom}(A, V)_T \cong \text{Hom}(U, V)_T = H^1(U, V)_T = H^1(B, V) \) has \( K \)-dimension \( \# \{\sigma \in \Gamma | \nu = \theta' \sigma\} \). Also \( H^1(X, H^1(Z, V)) = \bigoplus H_{\nu}(A, H_{\nu}(Z, V)) \) (summed over \( (\sigma, \tau) \in \Gamma \times \Gamma \)) has \( K \)-dimension \( \# \{(\sigma, \tau) \in \Gamma \times \Gamma | \nu = \sigma \theta' + \tau\} \) and \( H^2(X, V) = H^2(A, V)_T \) has \( K \)-dimension \( \# \{\sigma, \tau \subseteq \Gamma | \nu = \theta'(\sigma + \tau)\} \). Since \( A \) acts trivially on \( Z \) and \( V \) we have \( H^1(Z, V)_x = H^1(Z, V)_T \) has \( K \)-dimension \( \# \{\sigma \in \Gamma | \nu = \sigma\} \) when \( i = 1 \), and \( \# \{\sigma, \tau \subseteq \Gamma | \nu = \sigma + \tau\} \) when \( i = 2 \).

Lemma 2. If \( \nu = \sigma + \tau \) for some \( \sigma, \tau \in \Gamma \) assume \( \nu \) is invertible in \( \text{End}(K^*) \). Then \( \text{Res} = 0 \) in (\*).\( \)

Proof. First we claim \( \dim_K H^2(Z, V)_x \leq 1 \). By the previous remarks this is evident if we show \( \sigma + \tau = \varphi + \rho \) in \( \text{End}(K^*) \), where \( \sigma, \tau, \varphi, \rho \in \Gamma \), implies \( \{\sigma, \tau\} = \{\varphi, \rho\} \). For this apply both sides to \( (x + 1), \)
expand, cancel and see the same equality holds in \( \text{End}(K^*) \). The claim follows from Dedekind's lemma.

Thus if \( H^2(Z, V)^k \neq 0 \) it is generated by some \( \bar{f} \) of the form \( \bar{f}((\alpha, 0), (\beta, 0)) = \alpha^* \beta^* \) with \( \nu = \sigma + \tau \). If \( \text{Res} \neq 0 \) we can find \( f \in Z^2(B, V) \) with \( \text{Res} f = \bar{f} \), that is, \( f(\alpha, 0, \beta, 0) = \alpha^* \beta^* \) (we use \( f(\alpha, u, \beta, v) \) for \( f((\alpha, u), (\beta, v)) \)). Let \( E = E(f) \), the extension using \( f \), and let \( \tilde{U} \) be its Suzuki 2-group. We show \( \tilde{U} \) is a Suzuki 2-group of exponent 8 contradicting a theorem of \( G. \) Higman [3]. A Suzuki 2-group is a non-abelian 2-group with more than one involution and an automorphism \( \varphi \) with \( \langle \varphi \rangle \) transitive on the involutions.

Writing \( (a, a, u) \) for \( (a, (a, u)) \in \tilde{U} \) we see \( (0, 0, 0) = (a, a, u)^2 = (f(\alpha, u, a, u), u^{a_1}, 0) \) implies \( u = 0 \). Now \( f(\alpha, u, a, u) = \alpha^* u^* = 0 \) implies \( \alpha = 0 \). Thus \( V^* = \{(a, 0, 0)| a \in K^*\} \) is the set of involutions. There are \( q - 1 > 1 \) of them. It is easily seen that \( (a, a, u) \) is of exponent 8 when \( u \neq 0 \).

Choose \( t \) with \( \langle t \rangle = K^* \). Since \( \nu \) is invertible in \( \text{End}(K^*) \), we have \( (1, 0, 0)^{T(t)} = \{(t^v, 0, 0)| t \in K^*\} = V^* \). Thus \( T(t) \in \text{Aut}(U) \) will serve as the required automorphism showing \( \tilde{U} \) is a Suzuki 2-group. This completes the proof.

**Lemma 3.** In \( (*) \) the map \( \Phi \) is a surjection \( \Leftrightarrow H^1(X, H^1(Z, V))) = 0. \)

**Proof.** First we give the description of \( \Phi \) as found in [7]. Choose a set splitting \( S: X \to B \) with \( \pi_S = 1, S(1) = 1 \). For \( f \in Z^2(B, V) \) define \( \hat{f} \in C^1(X, Z^2(Z, V)) \) by \( \hat{f}(x)(\alpha) = f(S(x), \alpha^*) - f(\alpha, S(x)) \). Now \( \hat{f} \) induces a well defined map \( \Phi \) on the classes (this uses only the fact that \( Z \) is abelian).

Now assume \( \text{Im} \Phi = H^1(X, H^1(Z, V))) \neq 0 \) and choose a nonzero \( [d] \in H^1(X, H^1(Z, V)) = \oplus H_\sigma(A, H_r(Z, V))^T \) of the form \( \Phi[d] = u^* \alpha^r \) where \( u \in A, \alpha \in Z, \sigma, \tau \in \Gamma \). Find \( [f] \in H^2(B, V) \) with \( \Phi[f] = [d] \). We no longer need the action of \( T \) so replace \( f \) by \( f|U \times U \). We use \( S \) defined by \( S(u) = (0, u) \). Since \( B^1(A, H^1(Z, V)) = 0 \) we may assume \( \hat{f} = d \), that is

\[
(1) \quad f(0, u, \alpha, 0) + f(\alpha, 0, 0, u) = u^* \alpha^r.
\]

Let \( E = E(f) = \{(a, \alpha, u)| a, \alpha, u \in K\} \), the extension of \( V \) by \( U \) using \( f \), and let \( \tilde{Z} = \{(a, 0, 0)\} \). Then \( \tilde{Z} \triangleleft E \) and \( \tilde{Z} \) is abelian since \( f|Z \times Z = 0 \). We have an exact sequence of groups \( 1 \to \tilde{Z} \to E \to A \to 1 \). Define \( \rho: A \to E \) by \( \rho(u) = (0, 0, u) \) and let \( g \in Z^2(A, \tilde{Z}) \) be the corresponding cocycle, that is, \( g(u, v) = \rho(u)\rho(v)\rho(u + v)^{-1} \). All multiplication in \( E \) can be performed in terms of \( f \) and it can be computed that \( g = (g_1, g_2, 0) \) where \( g_1(u, v) = f(u^v, u + v, (u + v)^{a_1}, u + v) \) and \( g_2(u, v) = uv^s \).
Similarly it can be computed that \((b, a, 0)p(u) = (b + \frac{1}{u}, a, 0, u, a, 0, u)\) since \(G \subset \mathbb{Z}^2(u, v, w)\) we have \(0 = S/(a, 0, 0, 0) = \omega(a, 0, 0, 0) + \omega(a, 0, 0, u) + \omega(a, 0, 0, u) + \omega(a, 0, 0, u)\).

Using this expression for the action of \(A\) on \(\mathbb{Z}\) the first slot of the equation \(0 = g(u, v, w)\) implies

\[0 = g_i(u, v) + g_i(u + v, w) + g_i(v, w) + u^xg_i(v, w) + g_i(u, v + w).\]

Take \(u = v = w = 1\) and use the fact that \(g_i\) vanishes when either of its arguments is 0 to obtain \(0 = g_i(1, 1) = 1\), a contradiction. This completes the proof.

Let \(\{e_i\}, i = 1, 2, 3, 4\) be the standard base for \(K^4\) (columns) and put \(V_i = \langle e_i, \cdots, e_i\rangle/\langle e_i, \cdots, e_i\rangle\) as \(KB\)-module. Then \(V_i\) is a \(KB\)-module on which \(U\) acts trivially and \(T\) acts with weight \(v\) where \(v_i = \theta, v_2 = 1 - \theta, v_3 = \theta - 1, v_4 = -\theta\). For convenience we set \(v_0 = 0\). In the following lemma we determine the terms occurring in (*) when \(v = v_i, i = 0, 1, 2, 3, 4\) by solving the equations following Lemma 1.

**Lemma 4.** The solutions are as indicated when \(q > 2\) and \(i \in \{0, 1, 2, 3, 4\}\).

(a) \(v_i = \theta'\sigma: (i, q, \sigma) = (2, q, 1/2); (4, 8, 1).\)

(b) \(v_i = \sigma': (i, q, \sigma) = (1, q, \theta); (3, 8, 1).\)

(c) \(v_i = \sigma\theta' + \tau: (i, q, \sigma, \tau) = (0, 8, \sigma, 2\sigma)\) (any \(\sigma \in \Gamma\));
\[= (1, q, \theta/2, 1/2); (2, 8, 1, 1); (3, 8, 4, 2); (4, 8, 2, 2); (4, 32, 2, 8); (4, 32, 1, 2).\]

(d) \(v_i = \theta'(\sigma + \tau): (i, q, \{\sigma, \tau\}) = (1, q, \{1/2, \theta\}); (2, q, \{1/4\}); (3, 8, \{1/2\}); (4, 8, \{1/2\}).\)

(e) \(v_i = \sigma + \tau: (i, q, \{\sigma, \tau\}) = (1, q, \{\theta/2\}); (2, 8, \{2, 3\}); (3, 8, \{4\}); (3, 32, \{2, 1\}); (4, 8, \{1, 4\}).\)

The following will be useful for solving these equations.

**Lemma 5.** Let \(\varphi_i \in \Gamma \mapsto \text{End}(K^*)\) \(i = 1, 2, \cdots, m\). The following is arithmetic in \(\text{End}(K^*)\).

(a) If \(\varphi_1 + \varphi_2 = \varphi_3 + \varphi_4\) then \(\{\varphi_1, \varphi_2\} = \{\varphi_3, \varphi_4\}.\)

(b) If the \(\varphi_i\)'s are distinct then \(\sum_{i=1}^n \varphi_i \not\in \Gamma.\)

(c) If \(\sum_{i=1}^n \varphi_i = 0\) then \(m \geq |\Gamma|\), and \(m = |\Gamma| \iff \{\varphi_i\}_{i=1}^n = \Gamma.\)

**Proof of Lemma 5.** (a) A proof is included in the proof of Lemma 2.

For (b) and (c) write \(\varphi_i: x \mapsto x^{n_i}\) for \(0 \leq n_i < |\Gamma|\).

(b) Here we assume the \(n_i\)'s are distinct. Then \(\sum_{i=1}^n \varphi_i \in \Gamma\) implies
for all $x \in K$ we have $(x^{2^n} + 1) = (x + 1)^{2^n} = \Pi(x^n + 1) = \Sigma x^{|J|}$ where we sum over all $J \subseteq \{1, 2, \cdots, m\}$. Cancelling the terms on the left with the corresponding terms on the right there remains a polynomial of degree less than $2^n$ with $2^n = |K|$ solutions.

(c) Assume $m$ is minimal with $\Sigma \varphi_i = 0$. Then the $\varphi_i$'s are distinct since $\varphi_i + \varphi_i = 2 \varphi_i \in \Gamma$. Then $\Sigma \varphi_i = 0$ implies $(q - 1)|\Sigma 2^n$. Thus $q - 1 = 2^n - 1 = \Sigma_{i=0}^{m} 2^i \leq \Sigma_{i=1}^{m} 2^i$ implying $m = \Gamma$ and $\{\varphi_i\} = \Gamma$.

We now indicate a proof of Lemma 4. Observe first that from their definitions we have $\theta \neq 1$, $2\theta^2 = 1$, $\theta'(\theta + 1) = 1$. Thus $\theta'$, $\theta + 1$, $1 - \theta = \theta'/2$ are invertible in $\text{End}(K^*)$. Using these facts the equations can be manipulated to take advantage of Lemma 5 and reduce the problem to a few case by case investigations. We illustrate with the solution of $\nu_i = \sigma \theta' + \tau$.

$i = 0$: $0 = \sigma \theta' + \tau \Rightarrow \tau \sigma^{-1} = -\theta' = 2\theta - 2 \Rightarrow 2\theta = 2 + \tau \sigma^{-1}$. Now Lemma 5 (b) says $2 = \tau \sigma^{-1}$ so $\theta = 2$, $q = 8$, $\tau = 2\sigma$.

$i = 1$: $\theta = \sigma \theta' + \tau = 2\sigma - 2\sigma \theta + \tau \Rightarrow \theta + 2\sigma \theta = 2\sigma + \tau$ and Lemma 5 (a) implies $\{\theta, 2\sigma \theta\} = \{2\sigma, \tau\}$. Now $\theta \neq 1 \Rightarrow (\sigma, \tau) = (\theta/2, \theta^2) = (\theta/2, 1/2)$.

$i = 2$: Multiplying by $1 + \theta$ we obtain $1/2 = \sigma + \tau \theta + \tau$ and Lemma 5 (b) says $\sigma, \tau \theta, \tau$ are not distinct. $\theta \neq 1 \Rightarrow \tau \theta \neq \tau$. $\sigma = \tau \theta \Rightarrow 1/2 = 2\tau \theta + \tau \Rightarrow 2\theta + 1 \Rightarrow 1 = 2\theta^2 = \theta$, a contradiction. $\sigma = \tau \Rightarrow 1/2 = 2\tau + 2\theta \Rightarrow 2 = \theta$, $q = 8$ and it may be seen $\sigma = \tau = 1$.

$i = 3$: Since $\nu_3 = -\nu_2$ we obtain $0 = 1/2 + \sigma + \tau \theta + \tau$ and Lemma 5 (c) implies $|\Gamma| \leq 4$. Thus $q = 8$, $\sigma = \tau = 1$.

$i = 4$: Since $\nu_4 = -\nu_2$ we obtain $2\sigma \theta = 2\sigma + \theta + \tau$ implying $2\sigma, \theta, \tau$ are not distinct. $2\sigma = \theta \Rightarrow \theta^2 = 2\theta + \tau \Rightarrow 2\theta = \tau \Rightarrow \theta^2 = 4\theta$, $\theta = 4$, $q = 32$, $(\sigma, \tau) = (2, 8)$. $2\sigma = \tau \Rightarrow 2\sigma \theta = 4\sigma + \theta \Rightarrow 4\sigma = \theta$, $\theta = 4$, $q = 32$, $(\sigma, \tau) = (1, 2)$. $\tau = \theta \Rightarrow 2\sigma \theta = 2\sigma + 2\theta \Rightarrow \sigma = \theta$, $\theta = 2$, $q = 8$, $(\sigma, \tau) = (2, 2)$.

**Lemma 6.** When $i \in \{1, 2, 3, 4\}$ we have

$$\dim_k H^1(B, V_i) = \begin{cases} 1 & (i, q) = (2, q); (4, 8) \\ 0 & \text{otherwise} \end{cases}$$

$$\dim_k H^2(B, V_i) = \begin{cases} 1 & (i, q) = (2, q); (4, 8); (4, 32) \\ 0 & \text{otherwise} \end{cases}$$
Proof. The first statement is immediate from Lemma 4 and the remarks following Lemma 1. For the second observe $v_i \in \{ \pm \theta, \pm \theta'/2 \}$ and so $v_i$ is invertible in $\text{End}(K^*)$. Now Lemmas 1, 2, 3 and 4 may be used to determine the relevant terms of sequences (*) when $v = v_t$. These considerations prove the claim except to show $H^2(B, V) \not= 0$ when $q = 32$. In this case it may be seen that $(\alpha, u), (\beta, v) \rightarrow u^2\theta^8 + u\beta^2 + u^2v^8 + u^2v^9$ gives a nonzero class in $H^2(U, V) = H^2(B, V)$.

We are now ready to proceed to the main results of this paper.

**Theorem 1.** Let $K$ be the trivial module for $Sz(q)$, $q^8$. Then $\dim K H^2(Sz(q), K)$ is 0 if $q > 8$, and is 2 if $q = 8$ with generators (on a Sylow 2-subgroup) any two of $f_a: (\alpha, u), (\beta, v) \rightarrow (\alpha^2v + u^2\beta^4)^{\sigma}$, $\sigma \in \Gamma$.

**Proof.** We use $B$ in place of $Sz(q)$ and sequences (*) with $v = 0$ and $V = K$. According to Lemma 4 we have $H^2(Z, V)^x = H^2(X, V) = 0$ and $\dim K H^1(X, H^1(Z, V)) = 0$ if $q > 8$, and $|\Gamma| = 3$ if $q = 8$. Now sequences (*) with Lemma 3 give the upperbound. For the lowerbound it is easily checked that $f_\sigma$ as given is a $T$-stable cocycle and when $\sigma \neq \tau$, $\Phi[f_\sigma]$ and $\Phi[f_\tau]$ are independent in $H^1(A, H^1(Z, V))^T = H^1(X, H^1(Z, V))$.

**Theorem 2.** Assume $q \geq 8$ and $K^4$ is the standard module for $Sz(q)$. Then $H^1(Sz(q), K^4)$ is of dimension one and is generated by the restriction of a generator of $H^1(Sp_4(q), K^4)$.

**Proof.** Define $[d] \in H^1(U, K^4)^T = H^1(B, K^4)$ by $d(\alpha, u) = (\alpha^6, u^{1/2}, 0, 0)^*$ (* denotes transpose). It can be checked explicitly that $d$ is a nontrivial $T$-stable cocycle defined on $U$ giving the claimed lowerbound. Furthermore it can be seen that if $v \in K^4$, $x \in U$, then $v^*x^*J_0d(x) = (v^*J_0d + v^*x^*J_0xv)^{1/2}$ where $J_0$ is the $4 \times 4$ matrix with all entries 0 except $(J_0)_{41} = (J_0)_{32} = 1$. This means $d$ is the restriction of Dickson’s derivation which generates $H^1(Sp_4(q), K^4)$ [8].

For the upperbound we use Lemma 6 to conclude $\dim_K H^1(B, K^4) \leq \Sigma_{i=1} \dim_K H^1(B, V_i) = 1$ if $q > 8$, and 2 if $q = 8$. We are done at $q > 8$ and continue at $q = 8$.

Define $V_{12} = \langle e_1, e_2 \rangle$, $V_{34} = K^4/V_{12}$. We obtain the exact sequence of $K$-modules

$$0 \rightarrow H^1(B, V_{12}) \rightarrow H^1(B, K^4) \xrightarrow{\pi_1} H^1(B, V_{34})$$

(2) $$\rightarrow H^2(B, V_{12}) \rightarrow H^2(B, K^4) \xrightarrow{\pi_2} H^2(B, V_{34}) \rightarrow.$$

The given cocycle shows $\dim_K H^1(B, V_{12}) = 1$ so it suffices to see $(\pi_1)_* =$
Lemma 6 implies \( \dim_K H^1(B, V_{34}) \leq 1 \). It can be seen that \((\alpha, u) \mapsto (\alpha, \alpha + u^3, u)^*\) is a nontrivial \( T \)-fixed cocycle in \( Z^1(U, V_{34})^T \) so its class generates \( H^1(U, V_{34})^T \approx H^1(B, V_{34}) \). If \((\pi_1)_* \neq 0\) we can find \( f \in Z^1(U, K^4) \) of the form \( f(\alpha, u) = (f_1(\alpha, u), f_2(\alpha, u), \alpha + u^3, u)^*\). The \( e_2 \) coordinate of the equation \( \delta f((\alpha, u), (\beta, v)) = 0 \) gives the equation \( f_2(\alpha + \beta + uv^4, u + v) = f_2(\alpha, u) + f_2(\beta, v) + u(\beta + v^3) + \alpha v \). Set \( u = v = 0 \) to obtain \( f_2(\alpha + \beta, 0) = f_2(\alpha, 0) + f_2(\beta, 0) \); and set \((\alpha, u) = (\beta, v)\) to obtain \( f_2(u^3, 0) = u^4 \), that is, \( f_2(u, 0) = u^6 \). This is a contradiction as \( u \to u^6 \) is not an additive function.

**Theorem 3.** Let \( K^4 \) be the standard module for \( Sz(q) \). Then \( H^2(Sz(q), K^4) \) is zero if \( q = 8 \), and is of dimension one if \( q > 8 \) generated by a cocycle which is the restriction of a generator of \( H^2(Sp(q), K^4) \).

**Proof.** Landáuzuri (see [7]) has explicitly constructed (on a Sylow 2-subgroup) a nontrivial cocycle in \( Z^2(Sp_4(2^m)), GF(2^m)^4 \) and further (see [5]) has shown \( H^2(Sp_4(2^m)), (GF(2^m))^4 \) is of dimension one when \( m > 1 \). Restricting his cocycle gives

\[
(\alpha, u, (\beta, v) \mapsto ((\alpha, u) \mapsto (\alpha^6 u^6 v^{1/2} + \alpha^6 \beta^6 + u^6 \beta + u^6 v^{6+1} + u^6 \beta v^{1/2} v^{12}, (uv)^{14}, 0, 0)^*).
\]

We will see \( f \) is a coboundary only at \( q = 8 \). McLaughlin [7] has given a somewhat different argument to see \( Res(Sz(q), Sp_4(q)) \) is nonzero when \( q > 8 \) using the sufficient condition of Griess [2].

Consider now sequence (2). We have seen \((\pi_1)_* = 0\) and \( \dim K H^1(B, V_{34}) = 0 \) if \( q > 8 \), and 1 if \( q = 8 \). Next we show \( \dim K^1(B, V_{34}) = 1 \). The upper bound follows from Lemma 6 and the lower bound follows from the displayed cocycle \( f \). Also from Lemma (6), \( H^1(B, V_{34}) = 0 \) when \( q > 32 \). Using sequence (2) the proof is now complete when \( q > 32 \). Furthermore, the cases \( q = 8, 32 \) follow if we show there is no \( f \in Z^2(B, K^4) \) which has a nontrivial projection onto \( V_4 \).

Assuming we have such an \( f \), a contradiction is obtained by using the following: Let \( L = K^4/V_1 \) as \( KB \)-module.

(a) \( H^2(Z, L)^x \approx K \) generated by \((\alpha, \beta) \mapsto (\alpha, \alpha^2 \beta^4, 0, 0)^*\) when \( q = 8 \) and by \((\alpha, \beta) \mapsto (\alpha, 0, \alpha \beta^2, 0)^*\) when \( q = 32 \).

(b) \( H^2(X, L^x) \approx K \) generated by \((u, v) \mapsto (u, (uv)^{14}, 0, 0)^*\).

(c) \( H^1(X, H^1(Z, L)) = 0 \).

We now assume (a), (b), (c). From the exact sequence of groups 
\( 1 \to Z \to B \to X \to 1 \) the Hochschild–Serre spectral sequence gives the exact sequences

\[
0 \to H^2(B, L)_0 \to H^2(B, L) \xrightarrow{Res} H^2(Z, L)^x \to H^2(X, L^x) \to H^1(X, H^1(Z, L)) \to .
\]
In general when we have a function whose range is \( K^4 \) let the subscript \( i \) denote its projection onto \( V_i \). Thus \( f = (f_1, f_2, f_3, f_4) \). We are assuming \( 0 \neq [f_4] \in H^2(B, V_4) \). Let \( \tilde{f} \) denote the projection of \( f \) onto \( L \). We write this as \( \tilde{f} = (f_1, f_2, f_3, f_4) \). Thus \( \tilde{f} \in Z^2(B, L) \).

Assume first \( \text{Res}[\tilde{f}] = 0 \). Then using (c) and the above sequences \( \tilde{f} \) is cohomologous to the image under the inflation map of a generator of \( H^2(X, L^2) \), i.e. there is a \( g \in C^1(B, L) \) with \( (\tilde{f} - \delta f)((\alpha, u), (\beta, v)) = (\alpha, (uv)^{i4}, \alpha, 0)^* \). Using the fact that \( (\alpha, u) \) is an upper triangular matrix it is easily seen that this equation implies \( f_4 = \delta g \in B^2(B, V_4) \), contradicting present assumptions.

Now we assume \( \text{Res}[\tilde{f}] \neq 0 \). Let \( \tilde{f} = \text{Res}(f) \) so \([\tilde{f}] \in H^2(Z, K^4)^X \). Assume first \( q = 8 \). Now (a) tells us we may assume \( f(\alpha, \beta) = (f_1(\alpha, \beta), \alpha^2 \beta^4, 0, 0)^* \). Let \( u = (0, 1) \in U \). Then \( (u - 1)\tilde{f} = \delta g \) for some \( g \in C^1(Z, K^4) \). Apply both sides to \( (\alpha, \beta) \) and obtain

\[
\begin{bmatrix}
\alpha^2 \beta^4 \\
0 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
g_1(\alpha + \beta) \\
g_2(\alpha + \beta) \\
g_3(\alpha + \beta) \\
g_4(\alpha + \beta)
\end{bmatrix} +
\begin{bmatrix}
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4
\end{bmatrix}
\begin{bmatrix}
g_1(\beta) \\
g_2(\beta) \\
g_3(\beta) \\
g_4(\beta)
\end{bmatrix}
+ \begin{bmatrix}
g_1(\alpha) \\
g_2(\alpha) \\
g_3(\alpha) \\
g_4(\alpha)
\end{bmatrix}.
\]

The third and fourth rows tell us \( g_3 \) and \( g_4 \) are additive; \( \alpha = \beta \) in the second row tells us \( 0 = \alpha g_4(\alpha) \) implying \( g_4 = 0 \); \( \alpha = \beta \) in the first row tells us \( \alpha^5 = g_3(\alpha) \), contradicting the additivity of \( g_3 \).

Assume now \( q = 32 \). Here (a) tells us we may assume \( f(\alpha, \beta) = (f_1(\alpha, \beta), \alpha_2, \alpha^2 \beta^4, 0, 0)^* \). Now, with \( u = (0, 1) \in U \), the equation \( (u - 1)\tilde{f} = \delta g \) implies \( (\alpha \beta^2, \alpha \beta^2, 0, 0)^* = \delta g(\alpha, \beta) \). As before \( g_3 \) and \( g_4 \) are additive. Set \( \alpha = \beta \). The second coordinate implies \( g_4(\alpha) = \alpha^2 \); the first implies \( \alpha^5 = g_3(\alpha) + \alpha^{28 + 1} \); these imply \( \alpha \rightarrow \alpha^2 \) is additive, a contradiction.

We now prove (a), (b), (c). Note that if \( x \) is an involution in some group and \( d \) and \( f \) are 1 and 2-cocycles from that group to some module then \( \delta d(x, x) = 0 \) and \( \delta f(x, x, x) = 0 \) imply \( d(x) = -xd(x) \) and \( f(x, x) = xf(x, x) \). Regard \( L = K^3 \) (columns) = \( \langle e_2, e_3, e_4 \rangle \) on which \( (\alpha, u) \) acts as multiplication by

\[
\begin{bmatrix}
1 & u & \alpha \\
1 & u & \alpha \\
1 & 1 & \alpha
\end{bmatrix}.
\]

(a) Take \([f] \in H^2(Z, L)^X \) and using our convention we have \( f = (f_2, f_3, f_4)^* \). Since \([f_4] \in H^2(Z, V_4)^T \), by Lemma 4 (e) we may assume
\[ f_4(\alpha, \beta) = \alpha \beta^* k_4 \] and \( k_4 = 0 \) when \( q = 32 \). The relation \( f(\alpha, \alpha) = \alpha f(\alpha, \alpha) \) implies \( k_4 = 0 \). Now \( [f_3] \in H^2(Z, V_2) \) and we may assume \( f_3(\alpha, \beta) = \alpha^* \beta^* k_3 \) where \( \{\sigma, \tau\} = \{4\} \) if \( q = 8 \) and \( \{\sigma, \tau\} = \{2, 1\} \) if \( q = 32 \). Set \( u = (0, 1) \in U \). Then \( (u - 1) f = \delta g \) for \( g \in C'(Z, L) \). In the usual way this equation implies \( g_3 \) and \( g_4 \) are additive. Setting \( \alpha = \beta \) we obtain \( \alpha^* \beta^* k_3 = g_4(\alpha) \) implying \( k_3 = 0 \) or \( \alpha \to \alpha^* \beta^* - 1 \) is additive. At \( q = 8 \) the latter is false implying \( k_3 = 0 \).

Since \( k_4 = 0 \) it follows that \( [f_2] \in H^2(Z, V_2) \) and by Lemma 4 (e) we may assume \( f_2(\alpha, \beta) = \alpha^2 \beta^* k_2 \) with \( k_2 = 0 \) when \( q = 32 \). This proves (a).

(b) We see \( L^\ast = \langle e_2, e_3 \rangle \cong K^2 \) (columns) on which \((0, u) \mapsto \frac{1}{u} \) acts as multiplication by \( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \). Take \( [f] \in H^2(X, K^2) \). By Lemma 4 (d) we may assume \( f_3(u, v) = uv^2 k_3 \) with \( k_3 = 0 \) when \( q = 32 \). Now the relation \( uf(\bar{u}, \bar{u}) = f(\bar{u}, \bar{u}) \) implies \( f_3 = 0 \). Thus \( f_2 \in Z^2(X, V_2) \) and (b) follows from Lemma 4 (d).

(c) Take \( f \in Z^1(Z, L) \). Then \( f(\alpha) = \alpha f(\alpha) \) implies the image of \( f \) lies in \( L^\ast = L^\ast = \langle e_2, e_3 \rangle \). Thus \( f_4 = 0 \). Taking \( Z \)-cohomology of the exact sequence \( 0 \to L^\ast \to L \to V_4 \to 0 \) gives the exact sequence of \( KX \)-modules \( 0 \to V_4 \to H^1(Z, L^\ast) \to H^1(Z, L) \to H^1(Z, V_4) \to 0 \). We have just seen \( \pi_* = 0 \). Set \( V_3 = L^\ast \). It is easily seen that \( \text{Im} \delta_* = \text{Hom}_K(Z, V_3) \subset \text{Hom}_K(Z, V_2) \subset \text{Hom}(Z, L^\ast) = H^1(Z, L^\ast) \) showing \( H^1(Z, L) = \bigoplus_{\tau \neq 1} H_\tau(Z, V_3) \oplus H \) where

\[ H = \text{Hom}_K(Z, V_2) / \text{Hom}_K(Z, V_3) \cong \text{Hom}_K(Z, V_3). \]

Now \( \pi_* \) is the composite of the \( \delta_* \) and \( H^1(Z, H\tau(Z, V_2)) \) is the \( \delta_* \). We show \( H^1(X, H\tau(Z, V_2)) = 0 \) when \( \tau \neq 1 \). Take \( [f] \in H^1(A, H\tau(Z, V_2)) \). Taking \( u = v \) in the cocycle condition on \( f \) we see \( 0 = uf_3(u)(\alpha) \) showing \( f_3 = 0 \). Thus

\[ H^1(X, H\tau(Z, V_2)) = H^1(X, H\tau(Z, V_2)) = \bigoplus_{\tau \neq 1} H_\tau(A, H\tau(Z, V_2)) \] since by Lemma 4 (c) there is no \( \sigma \in \Gamma \) with \( \nu_1 = \sigma \theta' + 1 \). Finally, we show \( H^1(X, H\tau(Z, V_2)) = 0 \) when \( \tau \neq 1 \). Take \( [f] \in H^1(A, H\tau(Z, V_2)) \). Taking \( u = v \) in the cocycle condition on \( f \) we see \( 0 = uf_3(u)(\alpha) \) showing \( f_3 = 0 \). Thus

\[ H^1(X, H\tau(Z, V_2)) = H^1(X, H\tau(Z, V_2)) = \bigoplus_{\tau \neq 1} H_\tau(A, H\tau(Z, V_2)) \] since by Lemma 4 (c) there is no \( \sigma \in \Gamma \) with \( \nu_2 = \theta' \sigma + \tau \) when \( \tau \neq 1 \).

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