REMARKS ON THE SPACE H^p

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1. Introduction. The space H^p is the collection of all single-valued complex functions f which are regular on the interior of the unit circle in the complex plane, and for which

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty .$$

In [6] it was shown that H^p, $0 < p < 1$, is a linear topological space in which the metric is $\| f - g \|^p$, where we define

$$\| f \| = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} .$$

It was moreover shown that $(H^p)^*$, the conjugate of H^p, has sufficiently many elements (linear functionals on H^p) so as to distinguish elements in H^p, in the sense that if $f \neq 0$ is in H^p, then there is a $\gamma \in (H^p)^*$ such that $\gamma(f) \neq 0$.

In the present paper it will be shown that *if γ is in $(H^p)^*$, $0 < p < 1$, then there exists a unique function G which is regular in the open unit circle, continuous on the closed circle,*[1] *and such that*

$$\gamma(f) = \lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) G\left(\frac{r}{\rho} e^{-i\theta} \right) d\theta, \qquad r < \rho < 1,$$

for every f in H^p. It is further shown that the following is true of G:

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(a) *if* $0 < p < 1/n$, $n = 2,3,\cdots$, *then* $[d^{n-1} G(z)]/dz^{n-1}$ *is continuous on the closure of the unit circle*;

(b) *if* $0 < p < 1/2n$, $n = 1,2,\cdots$, *then* $G(e^{it})$ *has a continuous nth derivative with respect to t*; *and*

(c) *if* $0 < p < 1/2$, *then the power series for G converges absolutely on the boundary of the unit circle.*

It is moreover shown that *if G is regular on the open unit circle and is such that*

$$\lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) G\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta\,, \qquad r < \rho < 1\,,$$

exists for every f in H^p, *then the functional so defined is in* $(H^p)^*$. Thus $(H^p)^*$ is equivalent to a subspace of the functions which are regular on the open unit circle and continuous on the closed unit circle when $0 < p < 1$; and indeed, as p tends toward zero, the spaces $(H^p)^*$ are equivalent to subspaces of spaces whose members have far stronger properties than merely the property of being continuous on the closure of the unit circle.

A generalization of a theorem by Khintchine and Ostrowski [1, p.157], which is a sort of generalization of Vitali's theorem, will also be presented; namely, it will be shown that *a bounded sequence in* H^p, $0 < p < \infty$, *whose boundary values converge on a set of positive measure, converges uniformly on all compact subsets of the unit circle.* Khintchine and Ostrowski proved this theorem in the case that the sequence consists of uniformly bounded elements.

It is worth remarking that under the present "norm" $\|\cdot\|$, H^p, $0 < p < 1$, is definitely *not* a normed linear space, this being due to the complete failure of Minkowski's inequality for index smaller than unity. As a result, it is conjectured by the author that H^p, $0 < p < 1$, is not a normed linear space at all (and hence contains no bounded convex neighborhood). If this conjecture is true, then H^p, $0 < p < 1$, offers an interesting example of a linear topological space which is not locally convex (since H^p is clearly locally bounded) and whose conjugate space has sufficiently many members so as to distinguish the elements in H^p.

2. Representation of linear functionals on H^p, $0 < p < 1$. In this section we shall suppose always that $0 < p < 1$. We let Δ be the set of all z such that $|z| < 1$, and $\mathfrak{A}$ the class of all single-valued complex functions which are regular on Δ. We shall first make some definitions and prove several lemmas before proving the representation theorem.

For many of the topological terms used in the ensuing, see [3]. By a *complete* linear topological space, we shall mean a space in which $f_n - f_m \longrightarrow 0$ implies $\lim_{n=\infty} f_n$ exists in the space. *Locally bounded linear topological space* and *normed linear space* will be abbreviated LBLTS and NLS respectively. By F^*, where F is a linear topological space, we shall mean the conjugate of F, that is, the space of linear functionals on F.

If F is a LBLTS, it is easy to show that F^* is a complete NLS (Banach space) in which

$$\|\gamma\| = \sup_{f \in U} |\gamma(f)| ,$$

where $\gamma \in F^*$, and U is a fixed bounded neighborhood of the origin. Moreover, the topology so introduced into F^* is independent of U. With respect to H^p, we let U be the unit sphere, so that

$$\|\gamma\| = \sup_{\|f\|=1} |\gamma(f)| .$$

It is then simple to prove the following theorem, merely by modeling the proof exactly after that given in the theory of NLS's.

LEMMA 1. *If F is a complete* LBLTS, *and Γ is a subset of F^* having the property that, for each fixed f in F, $\gamma(f)$ is bounded as γ varies over Γ, then Γ is a bounded set.*

We remind ourselves that H^p is locally bounded, and is moreover complete by [6]. We make the following definitions, where f and g are any elements in $\mathfrak{A}$:

(i) $\gamma_n(f) = f^{(n)}(0)/N\,!$, $\quad n = 0, 1, \cdots$,

(ii) $T_w f : T_w f(z) = f(wz)$, $\quad w \in \Delta$, $z \in \Delta$,

(iii) $u_n : u_n(z) = z^n$, $\quad z \in \Delta$, $n = 0, 1, \cdots$,

(iv) $B(f, g; z) = \sum_{n=0}^{\infty} \gamma_n(f)\, \gamma_n(g)\, z^n$, $\quad z \in \Delta$.

It is easily verified that

$$B(f, g; z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta ,$$

where $z_1 z_2 = z$, and z_1 and z_2 are in Δ. The proof is made by expansion of the integrand above in a Taylor series about the origin and then term-by-term integration. In particular,

$$B(f, g; r) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) g\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta , \qquad r < \rho < 1 .$$

LEMMA 2. *If f is in H^p, then $T_w f$ is in H^p, and moreover*

$$T_w f = \sum_{n=0}^{\infty} \gamma_n(f) w^n u_n .$$

Proof. Let $g = \sum_{n=0}^{\infty} \gamma_n(f) w^n u_n$. We first show that this series converges. Note that $\|u_n\| = 1$, and

$$|\gamma_n(f)| \leq \left(\frac{pn+1}{pn}\right) (pn+1)^{1/p} \cdot \|f\| .$$

The last inequality appears in [6, Theorem 6]. Thus

$$\left\| \sum_{n=l}^{m} \gamma_n(f) w^n u_n \right\|^p \leq \sum_{n=l}^{m} \|\gamma_n(f) w^n u_n\|^p \longrightarrow 0 \quad \text{as} \quad l, m \longrightarrow \infty ,$$

whence $\sum_{n=0}^{\infty} \gamma_n(f) w^n u_n$ converges, by the completeness of H^p. Then, noting [6, Theorem 8], which tells us that a convergent sequence in H^p converges pointwise to its limit, we have

$$g(z) = \sum_{n=0}^{\infty} \gamma_n(f) w^n u_n(z) = \sum_{n=0}^{\infty} \gamma_n(f)(wz)^n$$

But $T_w f(z) = \sum_{n=0}^{\infty} \gamma_n(f)(wz)^n$. This completes the proof.

We note that it was obvious that $T_w f$ was in H^p in the first place, merely from the definition of H^p; but the form for $T_w f$, which was obtained above, will be needed later.

THEOREM 1. *If* $G \in \mathfrak{A}$ *such that* $\lim_{r=1} B(f, G; r) = \gamma(f)$ *(that is, we assume that this limit exists) for all* f *in* H^p*, then* γ *is in* $(H^p)^*$*. Conversely, if* γ *is in* $(H^p)^*$*, then there exists a unique* G *in* $\mathfrak{A}$ *such that* $\gamma(f) = \lim_{r=1} B(f, G; r)$ *for all* f *in* H^p.

Proof. To prove the first part of our theorem, let $\gamma_r(f) = B(f, G; r)$. Clearly $\gamma_r(f)$ is distributive in f. Suppose $\|f\| = 1$ and $r < p < 1$. Then

$$|\gamma_r(f)| \leq \sum_{n=0}^{\infty} |\gamma_n(f)| \cdot |\gamma_n(G)| r^n \leq \sum_{n=0}^{\infty} |\gamma_n(G)| \left(\frac{pn+1}{pn}\right)(pn+1)^{1/p} r^n .$$

Thus, $\gamma_r(f)$ is bounded in f for $\|f\| = 1$, r being fixed. It is then clear that γ_r is in $(H^p)^*$. Since $\lim_{r=1} \gamma_r(f)$ exists, it follows that $\gamma_r(f)$ is continuous on $0 \leq r \leq 1$ for each fixed f in H^p. Thus $\{\gamma_r(f)\}$ is bounded for $0 \leq r < 1$. As a result of Lemma 1, we may conclude that $\{\|\gamma_r\|\}$ is bounded for $0 \leq r < 1$; that is, there exists an M such that $\|\gamma_r\| \leq M$ for $0 \leq r < 1$. Let $\|f\| = 1$. Then $|\gamma_r(f)| \leq M$, whence $|\gamma(f)| \leq M$. Thus γ is necessarily in $(H^p)^*$ since it is bounded on the unit sphere in H^p.

We now prove the second part of Theorem 1. We note that if $\lim_{r=1} B(f, G; r) = \gamma(f)$ for some G and all f, then

$$\gamma(u_n) = \lim_{r=1} B(u_n, G; r) = \lim_{r=1} \gamma_n(G) r^n = \gamma_n(G) ;$$

that is, $\gamma_n(G) = \gamma(u_n)$ for all n, or merely $G(z) = \sum_{n=0}^{\infty} \gamma(u_n) z^n$. We note that $\sum_{n=0}^{\infty} \gamma(u_n) z^n$ converges, for $|\gamma(u_n)| \leq \|\gamma\| \cdot \|u_n\| = \|\gamma\|$. Let us now verify that G, as defined, has the desired property. We see that

$$B(f, G; r) = \sum_{n=0}^{\infty} \gamma_n(f) \gamma(u_n) r^n = \gamma \left\{ \sum_{n=0}^{\infty} \gamma_n(f) r^n u_n \right\} = \gamma(T_r f) .$$

But $\|T_r f - f\| \longrightarrow 0$; see [5] for this result; note that

$$\|T_r f - f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta \right)^{1/p},$$

where $f(e^{i\theta})$ is the boundary function for $f(z)$. Thus $\gamma(T_r f) \longrightarrow \gamma(f)$, or $B(f, G; r) \longrightarrow \gamma(f)$. Our proof is thus complete.

THEOREM 2. *The function G in Theorem* 1 *is continuous on the closure of* Δ.

Proof. We first verify that $f_t(z) = (1 - ze^{it})^{-1}$ is in H^p for every real t. It suffices to show that f_0 is in H^p. We see that

$$|1 - re^{i\theta}|^{-2} = [(1 - re^{i\theta})(1 - re^{-i\theta})]^{-1} = (1 - 2r \cos \theta + r^2)^{-1},$$

whence

$$|1 - re^{i\theta}|^{-p} = (1 - 2r \cos \theta + r^2)^{-p/2} .$$

From the character of $(1 - 2r \cos \theta + r^2)$, we see that it suffices to show that $\int_0^\delta (1 - 2r \cos \pi + r^2)^{-p/2}\, d\theta$ is bounded in r, where δ is any positive number. We note that the following is true for $0 \leq \theta \leq \delta$ (where δ is some sufficiently small positive number) and for all r such that $1/2 \leq r < 1$:

$$1 - 2r \cos \theta + r^2 \geq 1 - 2r\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) + r^2 = (1 - 2r + r^2) + r\theta^2\left(1 - \frac{\theta^2}{12}\right)$$

$$= (1 - r)^2 + r\theta^2\left(1 - \frac{\theta^2}{12}\right)$$

$$\geq \frac{r\theta^2}{2} \geq \frac{\theta^2}{4} .$$

Thus, $(1 - 2r \cos \theta + r^2)^{-p/2} \leq 4^{p/2}\ \theta^{-p}$. Since θ^{-p} is integrable on $[0, \delta]$, our statement is proved.

We remind ourselves that we are trying to show that G is continuous on the assumption that

$$\gamma(f) = \lim_{r=1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta})\, G\left(\frac{r}{\rho} e^{-i\theta}\right) d\theta , \qquad r < \rho < 1 ,$$

exists for each f in H^p. Let γ_r be defined as in the proof of Theorem 1. Then

$$\gamma_r(f_t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-\rho e^{i(\theta+t)}} \cdot G\left(\frac{r}{\rho}\, e^{-i\theta}\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{G\left(\frac{r}{\rho}\, e^{i\theta}\right)}{1-\rho e^{i(t-\theta)}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{G\left(\frac{r}{\rho}\, e^{i\theta}\right)\frac{r}{\rho}\, e^{i\theta}}{\frac{r}{\rho}\, e^{i\theta} - re^{it}}\, d\theta$$

$$= G(re^{it})\,, \qquad r < \rho < 1\,.$$

The last equality is true by virtue of Cauchy's integral formula. We then have shown that $G(re^{it}) = \gamma_r(f_t)$. Consequently, since $\lim_{r=1} \gamma_r(f_t)$ exists by hypothesis, $\lim_{r=1} G(re^{it})$ exists for all t, and in fact

$$\gamma(f_t) = G(e^{it})\,,$$

where we define $G(e^{it})$ to be the boundary function $\lim_{r=1} G(re^{it})$.

We now show that $\lim_{t=t_0} f_t = f_{t_0}$ in the topology of H^p. Now, for *any* g in H^p, letting $g(e^{i\theta})$ be its boundary function, we know that

$$\sup_{0\le r<1} \int_0^{2\pi} |g(re^{i\theta})|^p\, d\theta = \int_0^{2\pi} |g(e^{i\theta})|^p\, d\theta\,.$$

It is easily verified that (see, for example, [**4**, Theorem 7, p. 29])

$$\lim_{t=t_0} \int_0^{2\pi} |g(e^{i(\theta+t)}) - g(e^{i(\theta+t_0)})|^p\, d\theta = 0\,.$$

Clearly $f_t(e^{i\theta}) = (1 - e^{i(\theta+t)})^{-1}$, whence $f_t(e^{i\theta}) = f_0(e^{i(\theta+t)})$. Thus $\lim_{t=t_0} f_t = f_{t_0}$, in the topology of H^p.

Now, by Theorem 1, γ is continuous, whence $\lim_{t=t_0}\gamma(f_t) = \gamma(f_{t_0})$; hence $\lim_{t=t_0} G(e^{it}) = G(e^{it_0})$. We have now shown that $G(e^{it})$ is continuous.

We remember that, in the course of proving Theorem 1, we showed that $\{\gamma_r\}$ is bounded in r as a subset of $(H^p)^*$. Obviously $\{f_t\}$ is a bounded subset of H^p, all of the elements having the same norm. Thus $\gamma_r(f_t)$ is bounded in both r and t. In other words, $G(re^{it})$ is bounded in r and t, or equivalently G is uniformly bounded on Δ. We then know that

$$G(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} G(e^{it})\, P_r(\theta - t)\, d\theta,$$

where $P_r(\theta)$ is the Poisson kernel. But, since $G(e^{it})$ is continuous, the right side above is necessarily a continuous function on the closed unit circle. Our proof is now complete.

It will now be shown that even more can be said of G when $0 < p < 1/2$.

THEOREM 3. *If $0 < p < 1/2$, then $G(e^{it})$ satisfies the Lipschitz condition of order one.*

Proof. It suffices to show that

$$\|f_{t+h} - f_t\| = \|f_h - f_0\| \leq A \cdot |1 - e^{ih}|$$

for some fixed constant A. We have

$$\|f_h - f_0\| = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{1 - e^{i(\theta+h)}} - \frac{1}{1 - e^{i\theta}} \right|^p d\theta \right)^{1/p}$$

$$= \frac{|1 - e^{ih}|}{(2\pi)^{1/p}} \left\{ \int_0^{2\pi} |(1 - e^{i(\theta+h)})(1 - e^{i\theta})|^{-p} d\theta \right\}^{1/p},$$

The proof will then be complete after we have shown that

$$\int_0^{2\pi} |(1 - e^{i(\theta+h)})(1 - e^{i\theta})|^{-p} d\theta$$

is bounded for all sufficiently small h. It is evident that

$$|(1 - e^{i\theta})(1 - e^{i(\theta+h)})|^2 = 4(1 - \cos\theta)[1 - \cos(\theta + h)],$$

and hence

$$|(1-e^{i\theta})(1-e^{i(\theta+h)}|^p = 4^{p/2}(1-\cos\theta)^{p/2}(1-\cos(\theta+h))^{p/2}.$$

We now must show that

$$\int_0^{2\pi}(1-\cos\theta)^{-p/2}(1-\cos(\theta+h))^{-p/2}\,d\theta$$

is bounded in h for all sufficiently small h. We note that the following is true for all sufficiently small θ and h:

$$1-\cos\theta \geq \frac{\theta^2}{2}\left(1-\frac{\theta^2}{12}\right) \geq \frac{\theta^2}{4},$$

$$1-\cos(\theta+h) \geq \frac{(\theta+h)^2}{4}.$$

Thus we have

$$(1-\cos\theta)^{-p/2}[1-\cos(\theta+h)]^{-p/2} \leq 4^p\theta^{-p}(\theta+h)^{-p}$$

for all sufficiently small θ and h. Since θ^{-2p} is integrable on the interval $[0,2\pi]$, it is then rather easy to show that

$$\int_0^{2\pi}(1-\cos\theta)^{-p/2}[1-\cos(\theta+h)]^{-p/2}\,d\theta$$

is bounded in h for all sufficiently small h.

We now have the rather interesting result:

COROLLARY. *If* $0<p<1/2$, *then* $\sum_{n=0}^{\infty}|\gamma_n(G)|<\infty$.

Proof. Since $G(e^{it})$ is of bounded variation, it follows that $G(z)$ is a power series of bounded variation according to [7, §7.5]. Hence the conclusion is obtained by [7, (i), p. 158].

We shall now show that even more may be said of G when $0<p<1/2$.

THEOREM 4. *If* $0<p<1/2$, *then* $(d/dz)G(z)$ *is continuous on the closure of*

Δ, *and moreover* $(d/dt)G(e^{it})$ *is continuous on* $[0, 2\pi]$.

Proof. By Cauchy's integral formulas (where $(d/dz)G(z) = G'(z)$):

$$G'(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{G\left(\frac{r}{\rho}e^{i\theta}\right)\frac{r}{\rho}e^{i\theta}}{\left(\frac{r}{\rho}e^{i\theta} - re^{it}\right)^2}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{G\left(\frac{r}{\rho}e^{-i\theta}\right)\frac{\rho}{r}e^{i\theta}}{(1-\rho e^{i(\theta+t)})^2}\,d\theta$$

$$= \frac{1}{2\pi r}\int_0^{2\pi}[f_t^2(\rho e^{i\theta})\cdot \rho e^{i\theta}]\cdot G\left(\frac{r}{\rho}e^{-i\theta}\right)d\theta,\quad r < p < 1.$$

Thus $G'(re^{it}) = (1/r)\gamma_r(f_t^2 u_1)$. We note that since $0 < p < 1/2$, we have $f_t^2 \in H^p$, whence $f_t^2 \cdot u_1 \in H^p$, since u_1 is bounded. Thus we show exactly as in Theorem 2 that

$$\gamma(f_t^2 u_1) = G'(e^{it}),$$

$G'(e^{it})$ is continuous in t,

$G'(z)$ is uniformly bounded on Δ,

$$G'(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} G'(e^{i\theta})P_r(\theta - t)\,d\theta,$$

where we define $G'(e^{it})$ to be the boundary value of $G'(z)$. Let us now consider

$$F(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi}\left[-ie^{-i\theta}\frac{d}{d\theta}G(e^{i\theta})\right]P_r(\theta - t)\,d\theta.$$

We note that $G(e^{i\theta})$ is absolutely continuous by virtue of Theorem 2, whence $(d/d\theta)G(e^{i\theta})$ is integrable. We also note that

$$G(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\theta})P_r(\theta - t)\,d\theta = \sum_{n=0}^{\infty} C_n r^n e^{int},$$

where

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\theta}) e^{-in\theta} d\theta .$$

Moreover, it is not at all difficult to verify that the real and imaginary parts of $-ie^{i\theta}(d/d\theta) G(e^{i\theta})$ are conjugate, whence

$$F(re^{it}) = \sum_{n=0}^{\infty} d_n r^n e^{int} ,$$

where

$$d_n = \frac{1}{2\pi} \int_0^{2\pi} \left[-ie^{-i\theta} \frac{d}{d\theta} G(e^{i\theta}) \right] e^{-in\theta} d\theta .$$

Integration by parts readily yields

$$d_n = (n+1) C_n ;$$

that is,

$$F(re^{it}) = \sum_{n=0}^{\infty} (n+1) C_{n+1} r^n e^{int} ,$$

and hence $F(z) = G'(z)$. Thus, we necessarily have

$$G'(e^{i\theta}) = -ie^{-i\theta} \frac{d}{d\theta} G(e^{i\theta})$$

almost everywhere. Since $G'(e^{i\theta})$ is continuous, it follows that $G(e^{i\theta})$ necessarily has a continuous derivative, and in fact

$$\frac{d}{d\theta} G(e^{i\theta}) = ie^{i\theta} G'(e^{i\theta}) .$$

This completes the proof of the theorem.

We sum up by presenting the following theorem, which is readily proved by induction, the proof being modeled after that given for Theorem 4.

THEOREM 5. *If* $0 < p < 1/n, n = 2,3,\cdots$, *then* $(d^{n-1}/dz^{n-1})\, G(z)$ *is continuous on the closure of* Δ. *Moreover, if* $0 < p < 1/2n, n = 1,2,\cdots$, *then* $G(e^{it})$ *has a continuous nth derivative with respect to t.*

3. Generalization of Vitali's Theorem. In this section we assume merely that p is any positive real number. We here need the following:

LEMMA 3. *If* $\{f_n\}$ *is a bounded sequence in* H^p, *and if* $\lim_{n=\infty} f_n(z)$ *exists on a set having at least one limit point in* Δ, *then* $\lim_{n=\infty} f_n(z)$ *exists uniformly on all compact subsets of* Δ.

Proof. The proof is a simple consequence of the following inequalities:

$$|f(z)| \leq \frac{\|f\|}{(1-|z|)^{1/p}}, \qquad \text{when } 0 < p \leq 1,$$

and

$$|f(z)| \leq \frac{\|f\|}{1-|z|} \qquad \text{when } 1 \leq p < \infty .$$

The first of the above inequalities appears in [6, Theorem 2]. The second is easily obtained as follows. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i\theta})\, \rho e^{i\theta}}{\rho e^{i\theta} - z}\, d\theta ,$$

and hence, by Hölder's inequality,

$$|f(z)| \leq \frac{\rho}{\rho - |z|} \quad \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|\, d\theta$$
$$\leq \frac{\rho}{\rho - |z|} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta \right)^{1/p},$$

whence

$$|f(z)| \leq \frac{\|f\|}{1-|z|}.$$

Let

$$N(r)=\begin{cases}\dfrac{1}{(1-r)^{1/p}}\,, & 0<p\le 1\,.\\[2ex] \dfrac{1}{1-r}\,, & 1\le p<\infty\,.\end{cases}$$

It is then clear that $|f_n(z)| \le N(r)\cdot M$ when $|z| \le r < 1$, where $\|f_n\| \le M$ for all n. We choose r so large that the set $|z| < r$ includes a set having a limit point in $|z| < r$ and such that $\lim_{n=\infty} f_n(z)$ exists on this set. Then, by Vitali's theorem, $\lim_{n=\infty} f_n(z)$ exists uniformly on all compact subsets of $|z| < r$, and hence on all compact subsets of Δ. This completes the proof.

THEOREM 6. *Suppose $\{f_n\}$ is a bounded sequence in H^p. Further, suppose $\lim_{n=\infty} f_n(e^{i\theta})$ exists on a set of positive measure in the interval $[0, 2\pi]$. Then $\lim_{n=\infty} f_n(z)$ exists uniformly on all compact subsets of Δ.*

Proof. It suffices, by the preceding lemma, to show that $\lim_{n=\infty} f_n(z)$ exists on some neighborhood of the origin. Thus, we shall show that this is the case whenever $|z| < 1/9$. Let $|z_0| < 1/9$, and suppose $\lim_{n=\infty} f_n(z_0)$ does *not* exist. Then we may find a positive number α and subsequences $\{f_{n_k}\}$ and $\{f_{m_k}\}$ of $\{f_n\}$ which have the property that $|f_{n_k}(z_0) - f_{m_k}(z_0)| > \alpha$ for all k. We then define $q_k = f_{n_k} - f_{m_k}$. It is clear that $\{q_k\}$ is a bounded sequence in H^p. We then write $q_k = g_k \cdot h_k$, by virtue of F. Riesz's decomposition theorem [5], where g_k and h_k are such that

(i) $g_k \in H^p$ and $g_k(z) \neq 0$ for all z in Δ,

(ii) $|h_k(z)| \le 1$ on Δ and $|h_k(e^{i\theta})| = 1$ almost everywhere,

(iii) $\|g_k\| = \|q_k\|$.

We note that $l_k(z) = [g_k(z)]^{p/2}$ is in H^2, and in fact $\{l_k\}$ is a bounded sequence in H^2. Since $\lim_{k=\infty} [f_{n_k}(e^{i\theta}) - f_{m_k}(e^{i\theta})] = 0$ on a set of positive measure, it follows that $\lim_{k=\infty} l_k(e^{i\theta}) = 0$ on a set E of measure $\mu > 0$. We next shall show that $\lim_{k=\infty} l_k(z_0) = 0$, which will in turn imply that $\lim_{k=\infty} g_k(z_0) = 0$, and hence imply $\lim_{k=\infty} q_k(z_0) = 0$, a contradiction to $|q_k(z_0)| > \alpha$ for all k.

Let $A > 0$, and define

$$\phi(\theta) = \begin{cases} A/\mu & \text{on } E \\ A/(\mu - 2\pi) & \text{on } CE \end{cases},$$

where CE is the set $[0, 2\pi] - E$. There is no loss in supposing that $\mu \leq \pi$. Define

$$u_0(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)\, P_r(\theta - t)\, dt\ ,$$

where $P_r(\theta)$ is the Poisson kernel. Then u_0 is harmonic in Δ and $\lim_{r=1} u_0(re^{i\theta}) = \phi(\theta)$ a.e., by virtue of Fatou's theorem; see [7, § 3.442]. Let

$$u(re^{i\theta}) = u_0(re^{i\theta}) - u_0(z_0)\ .$$

We note that

$$u_0(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\ ,$$

where $\{a_n, b_n\}$ are the Fourier coefficients of $\phi(\theta)$. Since $u_0(0) = 0$, this being due to the fact that $\int_0^{2\pi} \phi(t)\, dt = 0$ and $P_0(\theta - t) = 1$, we then have a_0 equal to zero, or

$$u_0(re^{i\theta}) = \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\ .$$

We note that $|a_n| \leq 2A/\pi$ as well as $|b_n| \leq 2A/\pi$, whence

$$|u_0(re^{i\theta})| \leq \frac{4A}{\pi}\sum_{n=1}^{\infty} r^n = \frac{4A}{\pi}\,\frac{r}{1-r} < \frac{A}{2\pi} \leq \frac{A}{2(2\pi - \mu)}$$

provided $0 \leq r < 1/9$.

Let $v(z)$ be the harmonic conjugate of $u(z)$ which vanishes at z_0, and define $g(z) = e^{u(z)+iv(z)}$. Then $g \in \mathfrak{A}$, and $g(z_0) = 1$. Moreover, since $|g(z)| = e^{u(z)}$, we have $\lim_{r=1} |g(re^{i\theta})| = e^{\phi(\theta) - u_0(z_0)}$. By Cauchy's integral formula we have

$$l_k(z_0) = \frac{1}{2\pi} \int_0^{2\pi} l_k(\rho e^{i\theta})\ g(\rho e^{i\theta})\ \frac{\rho e^{i\theta}}{\rho e^{i\theta} - z_0}\, d\theta\,, \quad |z_0| < \rho < 1\,.$$

This is true since $l_k(z_0) = l_k(z_0)\, g(z_0)$. We note that $u(z)$ is bounded in Δ, and hence so is $g(z)$. Since

$$\lim_{\rho=1} \int_0^{2\pi} |\, l_k(\rho e^{i\theta}) - l_k(e^{i\theta})|^2\, d\theta = 0\,,$$

and since $g(z)$ is bounded on Δ, it is then evident that

$$\lim_{\rho=1} \int_0^{2\pi} |\, l_k(\rho e^{i\theta})\ g(\rho e^{i\theta})\,|\, d\theta = \int_0^{2\pi} |\, l_k(e^{i\theta})\ g(e^{i\theta})\,|\, d\theta\,.$$

Hence

$$|\, l_k(z_0)| \le \frac{1}{2\pi}\ \frac{1}{1 - |z_0|} \int_0^{2\pi} |\, l_k(e^{i\theta})|\ e^{\phi(\theta) - u_0(z_0)}\, d\theta\,.$$

Consequently

$$|\, l_k(z_0)| \le e^{A/\mu - u_0(z_0)}\ \frac{1}{2\pi} \left(\frac{1}{1 - |z_0|}\right) \int_E |\, l_k(e^{i\theta})|\, d\theta$$

$$+\ e^{A/(\mu - 2\pi) - u_0(z_0)} \left(\frac{1}{2\pi}\right)\left(\frac{1}{1 - |z_0|}\right) \int_{CE} |\, l_k(e^{i\theta})|\, d\theta\,.$$

Since

$$\frac{1}{2\pi} \int_{CE} |\, l_k(e^{i\theta})|\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} |\, l_k(e^{i\theta})|\, d\theta \le \left(\frac{1}{2\pi} \int_0^{2\pi} |\, l_k(e^{i\theta})|^2 d\theta\right)^{1/2}$$

and since $\{l_k\}$ is a bounded subset of H^2, we see that

$$\frac{1}{2\pi} \int_{CE} |l_k(e^{i\theta})|\, d\theta$$

is bounded with respect to k. Moreover

$$\frac{A}{\mu-2\pi} - u_0(z_0) \leq \frac{A}{\mu-2\pi} + |u_0(z_0)| \leq \frac{A}{\mu-2\pi} + \frac{4A}{\pi}\left(\frac{|z_0|}{1-|z_0|}\right)$$

$$\leq \frac{A}{\mu-2\pi} + \frac{A}{2(2\pi-\mu)} = \frac{A}{2(\mu-2\pi)} .$$

By virtue of Schwarz's inequality, where ξ is an arbitrary measurable subset of $[0, 2\pi]$, we have

$$\int_{\xi} |l_k(e^{i\theta})|\, d\theta \leq [m(\xi)]^{1/2} \left(\int_{\xi} |l_k(e^{i\theta})|\, d\theta\right)^{1/2} .$$

Hence, by a convergence theorem of Lebesque (see [2, p. 190], we have

$$\lim_{k=\infty} \int_{E} |l_k(e^{i\theta})|\, d\theta = 0,$$

since $\lim_{k=\infty} l_k(e^{i\theta}) = 0$ on E. Now, for arbitrary $\epsilon > 0$, we choose A so large that

$$e^{A/[2(\mu-2\pi)]} \cdot \left(\frac{1}{2\pi}\right)\left(\frac{1}{1-|z_0|}\right) \int_{CE} |l_k(e^{i\theta})|\, d\theta < \epsilon/2 .$$

and hence we obtain, from the foregoing,

$$e^{A/[\mu-2\pi] - u_0(z_0)}\left(\frac{1}{2\pi}\right)\left(\frac{1}{1-|z_0|}\right) \int_{CE} |l_k(e^{i\theta})|\, d\theta < \epsilon/2 .$$

Having so chosen A, choose K so large that $k > K$ implies

$$e^{A/\mu - u_0(z_0)}\left(\frac{1}{2\pi}\right)\left(\frac{1}{1-|z_0|}\right) \int_{E} |l_k(e^{i\theta})|\, d\theta < \epsilon/2 .$$

Hence, $k > K$ implies $|l_k(z_0)| < \epsilon/2 + \epsilon/2 = \epsilon$. This completes the proof of the theorem.

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