LOOPS WITH TRANSITIVE AUTOMORPHISM GROUPS

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1. Introduction. Every automorphism of an additive loop[1] L maps 0 upon 0. The automorphism group $A(L)$ of L will be called *transitive* if $A(L)$ is transitive on the nonzero elements of L. It is readily deduced from work of L. J. Paige [4] and P. T. Bateman [3] that, for every cardinal number n, there exists a loop L of cardinal number n with a transitive automorphism group. We shall demonstrate that (aside from the obvious exceptions) such a loop L must be simple, that is, its only normal subloops must be 0 and L, if it satisfies the following ascending chain dition:

(C) If $N_1 \subset N_2 \subset N_3 \subset \cdots$ is an ascending chain of normal subloops of the loop L, there exists an integer i such that $N_i = N_{i+1}$.

2. Theorem. We shall establish the following result.

THEOREM 1. *An additive loop L which satisfies* (C) *and has a transitive automorphism group is either* (i) *a simple loop or* (ii) *a finite abelian p-group of type* $(p, p, \cdots, p)$.

Proof. For each nonzero a of L, denote by $M(a)$ the smallest normal subloop of L which contains a.

(1) *The subloop $M(a)$ has a transitive automorphism group and is a minimal normal subloop of L.* If $b \neq 0$ is in $M(a)$, then there exists $\theta \in A(L)$ such that $a\theta = b$. Since θ maps normal subloops upon normal subloops, we have $M(a)\theta = M(b)$. Since $b \in M(a)$, it follows that $M(b) \subset M(a)$. If $\phi = \theta^{-1}$, then $M(a) = M(b)\phi \subset M(a)\phi$, and, by induction, $M(a) \subset M(a)\phi \subset M(a)\phi^2 \subset \cdots$. In view of (C), we have $M(a)\phi^i = M(a)\phi^{i+1}$ for some integer i. Since ϕ is an automorphism of L, it follows that $M(a) = M(a)\phi^{-1} = M(a)\theta = M(b)$. Hence θ induces an automorphism of $M(a)$. This is enough to prove (1).

[1] Readers unfamiliar with loop theory will get the sense of the paper if they read *group* in place of *loop*. The necessary loop theory will be found in Baer [1,2].

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(2) *If N is a normal subloop of L, then $N \cap M(a) = 0$ or $M(a)$.* This follows from the minimality of $M(a)$.

(3) *The loop L is a direct sum of a finite number r of isomorphic simple subloops M_i; that is, $L = M_1 \oplus M_2 \oplus \cdots \oplus M_r$.* If a, b are nonzero elements of L, there exists $\theta \in A(L)$ such that $a\theta = b$. Then $M(a)\theta = M(b)$, showing that all the minimal normal subloops $M(a)$ are isomorphic. If a_1 is an arbitrary nonzero element of L, define $M_1 = M(a_1)$. Now suppose that $M_i = M(a_i)$ has been defined, for $i = 1, 2, \cdots, s$, such that the (normal) subloop N_s generated by the M_i is the direct sum $N_s = M_1 \oplus \cdots \oplus M_s$. Write $t = s + 1$. If there exists a nonzero element a_t of L which is not in N_s, define $M_t = M(a_t)$. Then $N_s \cap M_t = 0$, by (2), and hence $N_t = N_s \oplus M_t = M_1 \oplus \cdots \oplus M_t$. In view of (C), the strictly increasing chain $N_1 \subset N_2 \subset \cdots$ must be finite. Therefore $L = N_r$ for some integer r. If M' is a normal subloop of M_1, then M' is normal in L, by virtue of the direct decomposition. Hence, by (1), each M_i is simple. This proves (3).

The center $Z(L)$ of a loop L is a characteristic subloop and an abelian group. In view of (1), either $Z(M_i) = 0$ or $Z(M_i) = M_i$. Hence, by (3), either (i′) $Z(L) = 0$ or (ii′) *L is a direct sum of isomorphic simple abelian groups.* Since a simple abelian group is cyclic of prime order p, (ii′) implies (ii) of Theorem 1. (Conversely, every finite abelian p-group of type $(p, p, \cdots, p)$ satisfies the hypotheses of the theorem.) In the case (i′), assume $r > 1$ in (3). Since $Z(L) = 0$, the decomposition (3) is unique. However, the nonzero element $c = a_1 + a_2$ is in $M_1 \oplus M_2$ but not in any of the M_i. Yet the proof of (3) shows that $M(c)$ could be chosen as the first factor in the direct decomposition of L, a contradiction. Therefore $r = 1$, and we have (i). This completes the proof of Theorem 1.

As the following (trivial) theorem shows, simple loops need not have transitive automorphism groups:

THEOREM 2. *A finite simple group $G \neq 0$ with a transitive automorphism group is necessarily cyclic of prime order.*

Proof. Every nonzero element of G has the same order p, necessarily prime. Thus G is a p-group, $Z(G) \neq 0$, $Z(G) = G$, and G is cyclic of order p.

3. Remarks. The author does not know whether finiteness is necessary for the conclusion of Theorem 2.

The following is the nonabelian loop L of lowest order with a transitive automorphism group; it is readily verified that $A(L)$ is the (alternating) group of order 12 generated by (12)(34) and (123):

+	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

REFERENCES

1. Reinhold Baer, *The homomorphism theorems for loops*, Amer. J. Math. 67 (1945), 450-460.

2. ———, *Direct decompositions*, Trans. Amer. Math. Soc. 62 (1947), 62-98.

3. P. T. Bateman, *A remark on infinite groups*, Amer. Math. Monthly 57 (1950), 623-624.

4. L. J. Paige, *Neofields*, Duke Math. J. 16 (1949), 39-60.

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