QUASI-CONVEXITY AND THE LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

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1. Introduction. We are concerned in this paper with integrals of the form

$$I(z,D) = \int_D f\left[x, z^i(x),\ z^i_{x^\alpha}(x)\right] dx, \tag{1.1}$$

where

$$x = (x^1, \cdots, x^\nu),\ z = (z^1, \cdots, z^N),\ p = p^i_\alpha$$

$$(i = 1, \cdots, N;\ \alpha = 1, \cdots, \nu),$$

$f(x, z, p)$ is continuous in its arguments, and D is a bounded domain.

The object of the paper is to discuss necessary and sufficient conditions on the function f for the integral I to be lower semicontinuous with respect to various types of convergence of the vector functions z. Because of the success of the "direct methods" in the Calculus of Variations, many writers have shown that certain integrals are lower semicontinuous. However, the writer knows of no paper in which a *necessary* condition for lower semicontinuity was discussed, although such a condition is very easy to obtain (see Theorem 2.1).

In §2, a general condition called "quasi-convexity" (see Definition 2.2) on the behavior of f as a function of p is obtained which is both necessary and sufficient for the lower semicontinuity of I with respect to the type of convergence given in Definition 2.1. This condition is that any linear function furnish the absolute minimum to $I(z,D)$ among all Lipschitzian (see below) functions which coincide with it on D^*, D being any bounded domain and D^* its boundary; here, of course, we consider f to be a function of p only. Section 3 discusses cases involving more general types of convergence and gives an existence theorem. In §4, it is shown that if $f(p)$ is continuous and quasi-convex, then it satisfies a certain generalized Weierstrass condition which reduces to the ordinary one (for the case at hand) when f is of class C'; this is, in turn, seen to be equivalent to the Legendre-Hadamard condition (see (4.8)) (quasi-regularity in its general form) when f is of class C . In §5, a general sufficient

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condition for quasi-convexity is proved and the necessary condition of §4 is seen to be sufficient when f is either a quadratic form in the p_α^i or is the integrand of a parametric problem with $N = \nu + 1$. The view of Terpstra's negative result [5] that even the strong Legendre-Hadamard condition (> 0) does not necessarily imply the existence of an alternating form $C_{ij}^{\alpha\beta}\, p_\alpha^i\, p_\beta^j$ ($C_{ij}^{\beta\alpha} = -C_{ij}^{\alpha\beta}$, and so on) such that

$$f(p) + C_{ij}^{\alpha\beta}\, p_\alpha^i\, p_\beta^j \equiv \left(a_{ij}^{\alpha\beta} + C_{ij}^{\alpha\beta}\right) p_\alpha^i\, p_\beta^j \tag{1.2}$$

is positive definite when $\nu > 2$, it would seem that there is still a wide gap in the general case between the necessary and sufficient conditions for quasi-convexity which the writer has obtained. In fact, after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves f and only a finite number of its derivatives, and which is both necessary and sufficient for quasi-convexity in the general case.

In (1.2), we have used the usual tensor summation convention, and will continue to use it throughout the paper; unless otherwise specified, the Greek letters will run from 1 to ν and the Latin letters from 1 to N.

We shall denote the sum and difference of vectors of the various sorts (x, z, p, and so on) in the usual way. We shall define

$$|x| = (x^\alpha x^\alpha)^{1/2}, \quad |z| = \left(z^i z^i\right)^{1/2}, \quad |p| = \left(p_\alpha^i\, p_\alpha^i\right)^{1/2}.$$

If $\zeta(x)$ is a vector function with derivatives, $\pi(x)$ will denote the vector function $\pi_\alpha^i(x) = \zeta_{x\alpha}^i(x)$; similar notations involving other letters will be introduced as the occasiòn demands.

All integrals are Lebesgue integrals, frequently of vector functions. It is sometimes desirable to consider the behavior of a function $z(x)$ with respect to a particular variable x^α or to the $\nu - 1$ variables $(x^1, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^\nu)$. In such a case, we write x_α' for $(x^1, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^\nu)$, (x_α', x^α) for x and so on. It is also convenient to write the boundary integrals

$$\int_{D^*} A_\alpha \;(x)\; dx_\alpha',$$

where each $A_\alpha(x)$ may be a vector $A_\alpha^i(x)$ and the boundary D^* of the domain is sufficiently regular; such an integral is to be regarded as a Lebesgue-Stieltjes integral with respect to the set function $x_\alpha'(e)$ on D^* chosen so that Green's theorem

$$\int_{D^*} \zeta^i\; dx_\alpha' = \int_D \zeta_{x\alpha}^i\; dx$$

holds. The closure of a set E will be denoted by $\bar{E}$.

Ordinary functions *of class* $\mathfrak{P}_s$, $\mathfrak{P}_s'$, $\mathfrak{P}_s''$, and so on, $s \geqq 1$, have been discussed at length in the papers [1] and [2]; the extension to vector functions is trivial. We define the integrals $\bar{D}_s(z, G)$ and $D_s(z, G)$ by

$$\bar{D}_s(z, G) = \int_G |z(x)|^s\, dx + D_s(z, G), \quad D_s(z, G) = \int_G \left[z^i_{x^\alpha}(x)\, z^i_{x^\alpha}(x)\right]^{s/2} dx.$$

Each function z of class $\mathfrak{P}_s$ is equivalent to a function $\bar{z}$ defined uniquely almost everywhere as that number such that the Lebesgue derivative of the set function

$$\int_e |z(x) - \bar{z}(x_0)|^s\, dx$$

is zero at x_0; $\bar{z}$ is supposed to be defined at *every* point x_0 where such a number exists; $\bar{z}$ is of class $\mathfrak{P}_s'$ (see [1] and [2]) and is also of class $\mathfrak{P}_s'$ in any coordinate system related to the original by a regular Lipschitzian transformation (cf. [2], Theorem 6.3; the $\bar{z}$ there used has a slightly different definition from the present one but the present theorem has been proved for vectors z with values in a Riemannian manifold in [4], Lemma 2.3 and Theorem 2.5).

A function z is said to *satisfy a* (uniform) *Lipschitz condition with coefficient* M on a set S if and only if

$$|z(x_1) - z(x_2)| \leqq M|x_1 - x_2|, \quad x_1 \in S, \quad x_2 \in S.$$

A function is *Lipschitzian* if it satisfies a Lipschitz condition.

If $g(y)$, $y = (y^1, \cdots, y^n)$, is summable on a domain D, we define the h-average function g_h by

$$g_h(y) = (2h)^{-n} \int_{y-h}^{y+h} g(\eta)\, d\eta, \qquad h > 0;$$

if g is summable then g_h is continuous where defined; if g is continuous on D then g_h is of class C' and g_h tends uniformly to g on each bounded closed set interior to D; if g is of class $\mathfrak{P}_s$ on D then g_h tends strongly in $\mathfrak{P}_s$ to g on each domain G with $\bar{G} \subset D$ (see [1], Lemma 5.1).

A form

$$C^{\alpha_1, \cdots, \alpha_\mu}_{i_1, \cdots, i_\mu}\, \pi^{i_1}_{\alpha_1} \cdots \pi^{i_\mu}_{\alpha_\mu} \quad \left(\mu \leqq \nu,\ 1 \leqq \alpha_\gamma \leqq \nu,\ 1 \leqq i_\gamma \leqq N,\ \gamma = 1, \cdots, \mu\right),$$

is called *alternating* if and only if the C's satisfy the obvious symmetry requirements and also the antisymmetry condition that

$$C^{\beta_1, \cdots, \beta_\mu}_{i_1, \cdots, i_\mu} = \pm C^{\alpha_1, \cdots, \alpha_\mu}_{i_1, \cdots, i_\mu}$$

according as $(\beta_1, \cdots, \beta_\mu)$ is an even or odd permutation of the indices $(\alpha_1, \cdots, \alpha_\mu)$; if $\zeta(x)$ is a vector function, then

$$\mu!\, C^{\alpha_1, \cdots, \alpha_\mu}_{i_1, \cdots, i_\mu} \pi^{i_1}_{\alpha_1}(x) \cdots \pi^{i_\mu}_{\alpha_\mu}(x) = C^{\alpha_1, \cdots, \alpha_\mu}_{i_1, \cdots, i_\mu} \frac{\partial(\zeta^{i_1}, \cdots, \zeta^{i_\mu})}{\partial(x^{\alpha_1}, \cdots, x^{\alpha_\mu})},$$

the fractions on the right denoting Jacobians.

2. A necessary and sufficient condition for lower-semicontinuity. We begin with some definitions.

DEFINITION 2.1. For the purposes of this section, we say that the vector functions z_n tend to the vector function z on the domain D if and only if the z_n and z all satisfy a uniform Lipschitz condition on D, independent of n, and the z_n tend uniformly to z on D. We shall write $z_n \longrightarrow z$ to denote this type of convergence.

DEFINITION 2.2. A function $f(p^i_\alpha)$ is said to be *quasi-convex* if and only if

$$\int_D f[p + \pi(x)]\, dx \geqq f(p) \cdot m(D), \qquad \pi^i_\alpha(x) = \zeta^i_{x^\alpha}(x),$$

for each constant p, each domain D, and each vector function ζ which satisfies a uniform Lipschitz condition on D and vanishes on D^*.

We shall show in this section that the integral $I(z,D)$ is lower semicontinuous with respect to the type of convergence specified in Definition 2.1 on each bounded domain D if and only if $f(x,z,p)$ is quasi-convex in p for each fixed (x,z).

THEOREM 2.1. *Suppose $I(z,D)$ is lower semicontinuous with respect to the type of convergence indicated on every region D. Then f is quasi-convex in p for each fixed (x, z).*

Proof. Let x_0 be any point, R be the cell $x_0 \leqq x^i \leqq x^i_0 + h$, Q be the cell $0 \leqq x^i \leqq 1$, and ζ be any function of class C' and periodic in each x^i with period 1. Let z_0 be any function of class C' on R.

For each n, define $\zeta_n(x)$ on R by

$$\zeta_n^i(x) = n^{-1} h\, \zeta^i\, [nh^{-1}(x - x_0)].$$

Then

$$\zeta_{nx^\alpha}^i(x) = \zeta_{x^\alpha}^i\, [nh^{-1}(x - x_0)]$$

and

$$I(z_0 + \zeta_n, R) = \int_R f\left\{x, z_0^i(x) + \zeta_n^i(x), p_{0\gamma}^i(x) + \pi_\gamma^i\, [nh^{-1}(x - x_0)]\right\} dx$$

$$= \sum_\alpha \int_{R_\alpha}\Big(f\left\{x, z_0^i(x) + \zeta_n^i(x), p_{0\gamma}^i(x) + \pi_\gamma^i\, [nh^{-1}(x - x_0)]\right\}$$

$$- f\left\{x_\alpha, z_0^i(x_\alpha), p_{0\gamma}^i(x_\alpha) + \pi_\gamma^i\, [nh^{-1}(x - x_0)]\right\}\Big) dx$$

$$+ \sum_\alpha (n^{-1} h)^\nu \int_Q f\left\{x_\alpha, z_0^i(x_\alpha), p_{0\gamma}^i(x_\alpha) + \pi_\gamma^i(\xi)\right\} d\xi,$$

where

$$\alpha = (\alpha_1, \cdots, \alpha_\nu),\ R_\alpha = R_{\alpha_1, \cdots, \alpha_\nu},\ n^{-1}(\alpha_\beta - 1) \leqq x^\beta \leqq n^{-1}\alpha_\beta,$$

$$x_\alpha = \left(x_{\alpha_1, \cdots, \alpha_\nu}^\beta\right),\ x_{\alpha_1, \cdots, \alpha_\nu}^\beta = n^{-1}(\alpha_\beta - 1),\ \beta = 1, \cdots, \nu.$$

As $n \to \infty$, we see, since f is uniformly continuous on any bounded part of space, $\zeta_n(x)$ tends uniformly to zero, and the π_γ^i are bounded, that

$$\lim_{n \to \infty} I(z_0 + \zeta_n, R) = \int_R \left[\int_Q f[x, z_0(x), p_0(x) + \pi(\xi)]\, d\xi\right] dx.$$

From the lower semicontinuity of I, we must have

$$\int_R \left\{\int_Q f[x, z_0(x), p_0(x) + \pi(\xi)]\, d\xi\right\} dx \geqq \int_R f[x, z_0(x), p_0(x)]\, dx.$$

Now, let x_0, z_0, and p_0 be any constant vectors. By letting

$$z_0(x) = z_0 + p_{0\alpha}(x^\alpha - x_0^\alpha),$$

dividing by h^ν and letting $h \to 0$, we obtain

$$\int_Q f[x_0, z_0, p_0 + \pi(\xi)]\, d\xi \geqq f(x_0, z_0, p_0).$$

By approximations, we can extend this to all ζ which satisfy a uniform Lipschitz condition over the whole space and are periodic of period 1 in each x^α.

Now, let D be a bounded domain and suppose ζ satisfies a uniform Lipschitz condition on D and vanishes on D^*. Let R be a hypercube of edge h, with edges parallel to the axes which contains D. Extend ζ to the whole space by first defining it to be zero on $\overline{R}-D$ and then extending it to be periodic of period h in each variable. Then a simple change of function and variable reduces R to $\mathcal{Q}$ and establishes the result.

LEMMA 2.1. *Suppose R is a cell with edges $(2h^1), \cdots, (2h^\nu)$ and center x_0. Let h be the smallest h^α. Suppose also that $0 < k < h$, that ζ^* satisfies a uniform Lipschitz condition with coefficient $M \geqq 1$ on R^*, and suppose*

$$|\zeta^*(x)| \leqq k, \; x \in R^*.$$

Then there is a function ζ on $\overline{R}$ which satisfies a Lipschitz condition with coefficient M on $\overline{R}$, coincides with ζ^* on R^*, and is zero except on a set of measure at most

$$m(R) \cdot [1 - (1 - h^{-1}k)^\nu].$$

Proof. Let R_1 be the cell with center at x_0 and edges $2(h^\alpha - k)$, $\alpha = 1, \cdots, \nu$. Then, since $h = \min h^\alpha$, we have

$$m(R_1) \geqq m(R) \cdot (1 - h^{-1}k)^\nu.$$

Define $\zeta_1 = 0$ on $\overline{R_1}$ and equal to ζ^* on R^*. Then

$$|\zeta_1(x_1) - \zeta_1(x_2)| \leqq |x_1 - x_2| \qquad \text{if } x_1 \in \overline{R}_1, \quad x_2 \in R^*.$$

Thus ζ_1 satisfies a uniform Lipschitz condition with coefficient M on $\overline{R}_1 \cup R^*$. By a well known theorem, there exists an extension of ζ_1 to $\overline{R}$ (the whole space in fact) which satisfies the same Lipschitz condition.

LEMMA 2.2. *Suppose the vectors $\zeta_n \longrightarrow 0$ (in our sense) on $\overline{R}$ and suppose f is quasi-convex in p. Then if p_0 is a constant vector we have*

$$m(R)\, f(p_0) \leqq \liminf_{n \to \infty} \int_R f[p_0 + \pi_n(x)]\, dx.$$

Proof. For all sufficiently large n, we have $k_n < h$, and $k_n \longrightarrow 0$, k_n being the maximum of $|\zeta_n(x)|$ for $x \in R^*$. For each n for which $k_n < h$, let η_n be the function of the preceding lemma which coincides on R^* with ζ_n, and let $\omega_n = \zeta_n - \eta_n$. Then if each ζ_n satisfies a uniform Lipschitz condition with coefficient $M \geqq 1$ on $\bar{R}$, then each η_n and ω_n satisfies one with coefficient M and $2M$, respectively. Moreover, each derivative $\eta^i_{nx\alpha}$ is uniformly bounded and $\eta^i_{nx\alpha} \longrightarrow 0$ almost everywhere. Since f is uniformly continuous on any bounded portion of p-space, we see that

$$\lim_{n\to\infty} \int_R |f\left(p^i_{0\alpha} + \eta^i_{nx\alpha} + \omega^i_{nx\alpha}\right) - f\left(p^i_{0\alpha} + \omega^i_{nx\alpha}\right)| \, dx = 0.$$

But the result then follows, since, for each n, we have

$$\int_R f\left(p^i_{0\alpha} + \omega^i_{nx\alpha}\right) \, dx \geqq m(R) f(p_0),$$

because of the quasi-convexity of the function f.

THEOREM 2.2. *Suppose f is continuous in (x,z,p) for all (x,z,p) and is quasi-convex in p for each (x,z). Suppose also that $z_n \longrightarrow z_0$ on the bounded domain D. Then*

$$I(z_0,D) \leqq \liminf_{n\to\infty} I(z_n,D).$$

Proof. Let ϵ be any positive number. For each positive integer k, let D_k consist of all the hypercubes of edge 2^{-k} whose faces lie along hyperplanes $x^\alpha = 2^{-k} i^\alpha$ (each i^α an integer) which lie in D. Since all the points $[x, z_0(x), p_0(x)]$ and $[x, z_n(x), p_n(x)]$ for $x \in D$ lie in a bounded portion of (x, z, p) space, we may choose k_1 so large that

$$\int_{D-D_{k_1}} |f(x,z_n,p_n)| \, dx < \epsilon/5, \quad \int_{D-D_{k_1}} |f(x,z_0,p_0)| \, dx < \epsilon/5 \tag{2.1}$$

for all n.

Let the hypercubes of D_{k_1} be $R_1, \cdots, R_N$. For each $k \geqq k_1$, let R_{ki}, $i = 1, \cdots, N \cdot 2^{\nu(k-k_1)}$, be all the hypercubes of side 2^{-k} described above which lie in D_{k_1}. For each such k, define $x^*_k(x)$, $z^*_k(x)$, $p^*_k(x)$ on D_{k_1} by

$$x^*_k(x) = [m(R_{ki})]^{-1} \int_{R_{ki}} x \, dx, \quad z^*_k(x) = [m(R_{ki})]^{-1} \int_{R_{ki}} z_0(x) \, dx,$$

$$p^*_k(x) = [m(R_{ki})]^{-1} \int_{R_{ki}} p_0(x) \, dx \tag{2.2}$$

$$r_k(x) = \left\{ |x^*_k(x) - x|^2 + |z^*_k(x) - z_0(x)|^2 + |p^*_k(x) - p_0(x)|^2 \right\}^{1/2},$$

where $x \in R_{ki}$. Let $\zeta_n(x) = z_n(x) - z_0(x)$, $\pi_n(x) = p_n(x) - p_0(x)$. Then, on D_{k_1},

$$
\begin{aligned}
& f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_0(x)] \\
&= \left\{ f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)] \right\} \\
&+ \left\{ f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] \right\} \qquad (2.3) \\
&- \left\{ f[x, z_0(x), p_0(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)] \right\} \\
&+ \left\{ f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] - f[x_k^*(x), p_k^*(x), p_k^*(x)] \right\} .
\end{aligned}
$$

Now, all the arguments of f occurring in (2.3) for $x \in D_{k_1}$ lie in a bounded closed cell in (x, z, p)-space over which f is uniformly continuous. Let

$$
\epsilon(\rho) = \max |f(x', z', p') - f(x'', z'', p'')|, \qquad \rho \geqq 0
$$

for all (x', z', p') and (x'', z'', p'') in this cell with

$$
|x' - x''|^2 + |z' - z''|^2 + |p' - p''|^2 \leqq \rho^2 .
$$

then $\epsilon(\rho)$ is continuous for $\rho \geqq 0$ with $\epsilon(0) = 0$. Then, for each n and each $k \geqq k_1$, we have

$$
|f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]| \leqq \epsilon(|z_n(x) - z_0(x)|),
$$

$$
|f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)]| \leqq \epsilon[r_k(x)],
$$

$$
|f[x, z_0(x), p_0(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)]| \leqq \epsilon[r_k(x)].
$$

Now, the $r_k(x)$ are uniformly bounded on D_{k_1} and tend to zero almost everywhere on D_{k_1}. Hence we may choose a $k \geqq k_1$ so large that

$$
\begin{aligned}
& \int_{D_{k_1}} |f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)]| \, dx < \epsilon/5, \\
(2.4) \quad & \\
& \int_{D_{k_1}} |f[x, z_0(x), p_0(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)]| \, dx < \epsilon/5,
\end{aligned}
$$

for all n. Since z_n converges uniformly to z_0, there is an n_1 such that

$$
(2.5) \qquad \int_{D_{k_1}} |f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]| \, dx < \epsilon/5, \quad n > n_1 .
$$

Finally, since $x_k^*(x)$, and so on, are constant on each R_{ki}, and f is quasi-convex, we conclude from the previous lemma that

$$\liminf_{n \to \infty} \int_{D_{k_1}} \left\{ f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)] \right\} dx \geqq 0.$$

Using (2.3)-(2.5) and the above inequality, we see that

$$\liminf_{n \to \infty} I(z_n, D) \geqq I(z_0, D) - \epsilon.$$

Since ϵ is any positive number, the result follows.

3. Lower semicontinuity and weak convergence in $\mathfrak{P}_s$ ($s \geqq 1$). In this section, we discuss additional conditions which with the quasi-convexity of f in p are sufficient to guarantee the lower semicontinuity of $I(z, D)$ with respect to weak convergence in $\mathfrak{P}_s$ on D.

DEFINITION 3.1. Suppose ζ is of class $\mathfrak{P}_s$ on the bounded domain D and suppose R is a cell with $\overline{R} \subset D$. Then ζ is said to be *strongly of class* $\mathfrak{P}_s$ *on* R^* if and only if $\overline{\zeta}$ is of class $\mathfrak{P}_s$ in x_α' on each face $x^\alpha =$ const. of R^* and there is a sequence ζ_n of class C' on $\overline{R}$ such that

$$\overline{D}_s(\zeta_n - \zeta, R) \longrightarrow 0, \quad \overline{D}_s(\zeta_n - \overline{\zeta}, R^*) \longrightarrow 0.$$

LEMMA 3.1. *Suppose ζ is of class $\mathfrak{P}_s$ ($s \geqq 1$) on the bounded domain D. For each α, $1 \leqq \alpha \leqq \nu$, let (a^α, b^α) be the open interval projection of D on the x^α axis. Then there exist sets Z^α of measure zero such that if $R: c^\alpha \leqq x^\alpha \leqq d^\alpha$ ($\alpha = 1, \cdots, \nu$) is any closed cell in D with*

$$c^\alpha \in (a^\alpha, b^\alpha) - Z^\alpha, \quad d^\alpha \in (a^\alpha, b^\alpha) - Z^\alpha \qquad (\alpha = 1, \cdots, \nu),$$

then ζ is strongly of class $\mathfrak{P}_s$ on R^*.

Proof. Let R' be any rational cell in D (that is, $R = [C, D]$ with C^α, D^α rational). In [1], Lemma 5.1, we have seen that if ζ is of class $\mathfrak{P}_s$ on D, then

$$\lim_{h \to 0} \overline{D}_s(\zeta_h - \zeta, R) = 0. \tag{3.1}$$

For each α, define

$$\phi_h^\alpha(x^\alpha, R') = \int_{C_\alpha'}^{D_\alpha'} \left\{ |\zeta_h - \overline{\zeta}|^s + \left[\sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\nu} |\zeta_{hx^\beta} - \overline{\zeta}_{x^\beta}|^2 \right]^{s/2} \right\} dx_\alpha'.$$

Since $\overline{\zeta}$ is obviously of class $\mathfrak{P}_s'$ in x_α' for almost all x^α on $[C^\alpha, D^\alpha]$, $\phi_h^\alpha(x^\alpha, R')$ is defined for almost all x^α and

$$\lim_{h \to 0} \int_{C^\alpha}^{D^\alpha} |\phi_h^\alpha(x^\alpha, R')| \, dx^\alpha = 0.$$

By arranging the rational cells R' in some order and choosing successive subsequences, we may choose (on account of (3.1)) a final sequence $h_n \longrightarrow 0$ such that $\phi_{h_n}^\alpha(x^\alpha, R') \longrightarrow 0$ and $\overline{\zeta}$ is of class $\mathfrak{P}_s'$ in x_α' on $[C_\alpha', D_\alpha']$ for each x^α not in a set $Z^\alpha(R')$ of measure zero $(\alpha = 1, \cdots, \nu)$. Now let

$$Z^\alpha = \bigcup Z^\alpha(R');$$

then

$$m(Z^\alpha) = 0 \qquad (\alpha = 1, \cdots, \nu).$$

Now suppose R is one of the cells described in the lemma. Then it lies in some rational cell R' and we may take $\zeta_n = \zeta_{h_n}$.

LEMMA 3.2. *Suppose R is a cell with edges $(2h^1), \cdots, (2h^\nu)$ and center x_0. Let*

$$h = \min_{1 \leq \alpha \leq \nu} h^\alpha, \quad K = h^{-1}(h^\alpha h^\alpha)^{1/2}.$$

Suppose also that $0 < k < h$, that ζ^* is of class $\mathfrak{P}_s$ on an open domain containing $\overline{R}$ in its interior, and that ζ^* is strongly of class $\mathfrak{P}_s$ on R^* with

$$\int_{R^*} |\zeta^*|^s \, dS \leq k^s, \quad D_s(\zeta^*, R^*) \leq M^s \qquad (s \geq 1).$$

Then there is a function ζ of class $\mathfrak{P}_s$ on R which coincides with ζ^* on R^*, is zero except on a set of measure

$$m(R) \cdot [1 - (1 - h^{-1}k)^\nu],$$

and satisfies

$$D_s(\zeta, R) \leq \tau_s \, h^{-1} k (1 + K^s M^s), \quad \tau_s = \begin{cases} 2^{s/2} & (s \leq 2), \\ 2^{s-1} & (s \geq 2). \end{cases}$$

Proof. For each $x \in \overline{R}$, $x \neq x_0$, let $x^*(x)$ be the intersection of the ray $\overrightarrow{x_0 x}$ with R^*, and for each $x \in \overline{R}$ define

$$r(x) = \begin{cases} 0 & (x = x_0), \\ |x^*(x) - x_0|^{-1} \cdot |x - x_0| & (x \neq x_0). \end{cases}$$

Let $\Pi_\alpha^\pm$ be the pyramid in $\overline{R}$ with vertex x_0 and base the face $F_\alpha^\pm$ where

$$x^\alpha = x_0^\alpha \pm h^\alpha.$$

On the pyramid Π_ν^+, introduce coordinates $\xi^1, \cdots, \xi^{\nu-1}$, r by

$$x^\nu = x_0^\nu + rh^\nu,\ x^\gamma = x_0^\gamma + r\xi^\gamma \quad (0 \leqq r \leqq 1,\ \gamma = 1, \cdots, \nu - 1).$$

Then, if r and ξ^γ are considered as functions of x, we have

$$r(x) = r,\ x^*(x) = [\xi^1(x) + x_0^1, \cdots, \xi^{\nu-1}(x) + x_0^{\nu-1}, h^\nu + x_0^\nu].$$

Similar coordinate systems may be set up on each of the other $\Pi_\alpha^\pm$.

Define

$$\phi(r) = \begin{cases} 0 & (0 \leqq r \leqq 1 - kh^{-1}), \\ hk^{-1}(r - 1 + kh^{-1}) & (1 - kh^{-1} \leqq r \leqq 1). \end{cases}$$

Choose a sequence ζ_n^* satisfying the conditions of Definition 3.1; and for each n, define

$$\zeta_n(x) = \phi[r(x)] \cdot \zeta_n^*[x^*(x)].$$

Then each $\zeta_n(x)$ is of class D' on $\overline{R}$.

We now compute the derivatives of ζ_n on each pyramid $\Pi_\alpha^\pm$ taking Π_ν^+ as an example. Then

$$\zeta_{nx^\gamma} = r^{-1}\phi(r)\zeta^*_{n\xi^\gamma} \qquad (1 \leqq \gamma \leqq \nu - 1),$$

$$\zeta_{nx^\nu} = (h^\nu)^{-1}\phi'(r)\zeta_n^* - (h^\nu)^{-1} r\phi(r)\xi^\gamma\zeta^*_{n\xi^\gamma} \ (\gamma \text{ summed from 1 to } \nu - 1).$$

Then, since $r^{-1}\phi(r) \leqq 1$ and $\phi'(r) = k^{-1}h$ for $1 - hk^{-1} \leqq r \leqq 1$,

$$|\pi_n(x)|^2 \leqq \left(\zeta^{*i}_{n\xi^\gamma}\zeta^{*i}_{n\xi^\gamma}\right) + 2k^{-2}|\zeta_n^*|^2 + 2(h^\nu)^{-2}(\xi^\gamma\xi^\gamma)\left(\zeta^{*i}_{n\xi^\gamma}\zeta^{*i}_{n\xi^\gamma}\right)$$

$$\leqq 2\left[k^{-2}|\zeta_n^*|^2 + K^2\left(\zeta^{*i}_{n\xi^\gamma}\zeta^{*i}_{n\xi^\gamma}\right)\right] \qquad (n \text{ not summed}).$$

Using the inequality

$$(a^2 + b^2)^{s/2} \leqq \sigma_s(|a|^s + |b|^s), \quad \sigma_s \leqq \begin{cases} 1 & (s \leqq 2) \\ 2^{(s-2)/2} & (s \geqq 2), \end{cases}$$

we obtain

$$D_s(\zeta_n, \Pi_\nu^+) \leqq \tau_s \int_{1-kh^{-1}}^1 r^{\nu-1}\, dr \int_{F_\nu^+} \Big[k^{-s} |\zeta_n^*|^s + K^s \left(\zeta_{n\xi^\gamma}^{*i} \zeta_{n\xi^\gamma}^{*i}\right)^{s/2}\Big] ds$$

$$\leqq \tau_s h^{-1} k \left[k^{-s} \int_{F_\nu^+} |\zeta_n^*|^s\, ds + K^s D_s(\zeta_n^*, F_\nu^+)\right].$$

Also

$$\int_{\Pi_\nu^+} |\zeta_n|^s\, dx = \int_{1-kh^{-1}}^1 r^{\nu-1} \phi^s(r)\, dr \int_{F_\nu^+} |\zeta_n^*|^s\, dS$$

$$\leqq h^{-1} k \int_{F_\nu^+} |\zeta_n^*|^s\, dS.$$

Adding these results for all the $\Pi_\alpha^\pm$, we obtain the result for each n; and also $\overline{D}_s(\zeta_n, R)$ is uniformly bounded. Thus, we may extract a subsequence which tends weakly in $\mathfrak{P}_s$ to some function ζ of class $\mathfrak{P}_s$ on R. Since each $\zeta_n = \zeta_n^*$ on R^*, ζ_n^* tends strongly in L_s to $\bar{\zeta}^*$ on R^*, we see from [2], Theorem 8.5, that $\zeta = \bar{\zeta}^*$ on R^*. From the lower semicontinuity of D_s (see [2], Theorem 8.2), the result follows.

LEMMA 3.3. *Suppose f is quasi-convex and of class C' for all p, and suppose for all p that*

$$\sum_{i,\alpha} \left(f_{p_\alpha^i}\right)^2 \leqq K^2 (|p|^{s-1} + 1)^2 \qquad (s \geqq 1).$$

If p_0 is any constant vector, D is any bounded domain, and ζ is of class $\mathfrak{P}_s$ on D and vanishes on D^*, then $f[p_0 + \pi(x)]$ is summable over D and

$$\int_D f[p_0 + \pi(x)]\, dx \geqq m(D) \cdot f(p_0).$$

Proof. There exists a sequence of functions ζ_n, each of class C' on D and vanishing on and near D^*, such that $\overline{D}_s(\zeta_n - \zeta, D) \to 0$ (see [2], Definition 9.1). For each n and almost all x on D, we have

$$|f[p_0 + \pi_n(x)] - f[p_0 + \pi(x)]| =$$

$$\left| [\pi_{n\alpha}^i(x) - \pi_\alpha^i(x)] \int_0^1 f_{p_\alpha^i}[p_0 + (1-t)\pi(x) + t\pi_n(x)]\, dt\right|$$

$$\leqq |\pi_n(x) - \pi(x)| \cdot K \cdot \int_0^1 \left\{ |(1-t)p_0 + \pi(x) + tp_0 + \pi_n(x)| \quad ^{-} + 1 \right\} dt$$

(3.2)

$$\leqq K\, |\pi_n(x) - \pi(x)| \left\{ h_s\, |p_0 + \pi(x)|^{s-1} + h_s\, |p_0 + \pi_n(x)|^{s-1} + 1 \right\},$$

where

$$h_s = \begin{cases} s^{-1} & (1 \leqq s \leqq 2) \\ s^{-1}\, 2^{s-2} & (s \geqq 2). \end{cases}$$

Using the Hölder inequality, and so on, and the strong convergence in $\mathfrak{P}_s$, we see that

$$\lim_{n \to \infty} \int_D f[p_0 + \pi_n(x)]\, dx = \int_D f[p_0 + \pi(x)]\, dx.$$

Since f is quasi-convex, the result follows.

LEMMA 3.4. *Suppose that f satisfies the hypotheses of Lemma 3.3. Suppose also that each ζ_n is of class $\mathfrak{P}_s$ on a domain D and is strongly of class $\mathfrak{P}_s$ on R^*, $\overline{R} \subset D$, with*

$$\lim_{n \to \infty} \int_{R^*} |\zeta_n|^s\, dS = 0,\ D_s(\zeta_n, R^*) \leqq M^s,\ D_s(\zeta_n, R) \leqq M^s \qquad (n = 1, 2 \cdots).$$

Then for each p_0, $f[p_0 + \pi_n(x)]$ is summable for all sufficiently large n, and

$$\liminf_{n \to \infty} \int_R f[p_0 + \pi_n(x)]\, dx \geqq m(R) \cdot f(p_0),\ \pi^i_{n\alpha}(x) = \zeta^i_{nx^\alpha}(x).$$

Proof. For each n, let

$$k_n = \left[\int_{R^*} |\zeta_n|^s\, dS \right]^{1/s},$$

and let K and h be the quantities of Lemma 3.2 for R. Since $k_n \longrightarrow 0$, we have $k_n < h$ for all $n >$ some n_1. For each such n, let η_n be the function of Lemma 3.2 which coincides on R^* with ζ_n, and let

$$\chi_n = \zeta_n - \eta_n,\ \kappa^i_{n\alpha} = \eta^i_{nx^\alpha},\ \omega^i_{n\alpha} = \chi^i_{nx^\alpha}.$$

Then, since $\chi_n = 0$ on R^*, we have

$$\int_R f[\pi_0 + \omega_n(x)]\, dx \geqq m(R)\, f(p_0).$$

As in (3.2), we see that, for each n, and almost all x on D,

$$|f[p_0 + \omega_n(x) + \kappa_n(x)] - f[p_0 + \omega_n(x)]|$$

$$\leqq K \cdot |\kappa_n(x)| \cdot (h_s |p_0 + \omega_n(x) + \kappa_n(x)|^{s-1} + h_s |p_0 + \omega_n(x)|^{s-1} + 1)$$

$$\leqq K \cdot |\kappa_n(x)| \cdot [(1 + sh_s) h_s |p_0 + \pi_n(x)|^{s-1} + sh_s^2 |\kappa_n(x)|^{s-1} + 1].$$

Using the Hölder inequality, and so on, we see that

$$\lim_{n \to \infty} \int_R |f[p_0 + \pi_n(x)] - f[p_0 + \omega_n(x)]| \, dx = 0,$$

from which the result follows.

THEOREM 3.1. *Suppose f is of class C' in (x, z, p) and quasi-convex in p. Suppose also that there are numbers k and K, $K > 0$, such that*

(i) $f(x,z,p) \geqq k$, (iii) $f_{x^\alpha} f_{x^\alpha} \leqq K^2 (|p|^s + 1)^2$

(ii) $f_{p^i_\alpha} f_{p^i_\alpha} \leqq K^2 (|p|^{s-1} + 1)^2$, (iv) $f_{z^i} f_{z^i} \leqq K^2 (|p|^s + 1)^2$.

for all (x,z,p).

Suppose also that $z_n \longrightarrow z_0$ weakly in $\mathfrak{P}_s$ on the bounded domain D and that either

(a) *each z_n and z_0 are continuous on D and z_n converges uniformly to z_0 on each closed set interior to D, or*

(b) *the set functions $D_s(z_n, e)$ are uniformly absolutely continuous on each closed set interior to D.*

Then

$$I(z_0, D) \leqq \liminf_{n \to \infty} I(z_n, D).$$

REMARK. If $s = 1$, weak convergence in $\mathfrak{P}_s$ implies the hypothesis (b).

Proof. We note first that hypothesis (ii) implies

$$|f(x,z,p) - f(x,z,0)| = |p^i_\alpha \int_0^1 f_{p^i_\alpha}(x,z,tp^i_\alpha)\, dt| \tag{3.3}$$

$$\leqq |p| \cdot \int_0^1 [f_{p^i_\alpha} f_{p^i_\alpha}(x,z,tp^i_\alpha)]^{1/2}\, dt$$

$$\leq |p| \cdot \int_0^1 K(t^{s-1} p^{s-1} + 1)\, dt \leq K(s^{-1} |p|^s + |p|).$$

Also, hypotheses (iii) and (iv) similarly imply

$$(3.4) \qquad |f(x,z,0) - f(0,0,0)| \leq K(|x| + |z|).$$

Thus, for all (x, z, p), we have

$$(3.5) \qquad |f(x,z,p)| \leq |f(0,0,0)| + K(|x| + |z| + s^{-1} |p|^s + |p|).$$

Therefore $I(z_0, D)$ and the $I(z_n, D)$ are uniformly bounded.

For each α ($1 \leq \alpha \leq \nu$), let (a^α, b^α) be the open interval projection of D on the x^α axis and let Z_0^α and Z_n^α be the sets of Lemma 3.1 for z_0 and z_n. Also for each α, n, k, let $E_{n,k}^\alpha$ be the set of x^α in $(a^\alpha, b^\alpha) - Z_n^\alpha$, where

$$\overline{D}_s(\overline{z}_n, D_{x^\alpha}) \leq k,$$

D_{x^α} being the set of x'_α such that $(x'_\alpha, x^\alpha) \in D$. Suppose that $\overline{D}_s(z_n, D) \leq M$, some uniform bound existing because of the weak convergence. Let

$$Z_{n,k}^\alpha = (a^\alpha, b^\alpha) - E_{n,k}^\alpha .$$

Then

$$m(Z_{n,k}^\alpha) \leq Mk^{-1}, \quad m(E_{n,k}^\alpha) \geq (b^\alpha - a^\alpha) - Mk^{-1} .$$

For each α, let

$$E^\alpha = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_{n,k}^\alpha, \quad \tilde{Z}_0^\alpha = (a^\alpha, b^\alpha) - E^\alpha \cup Z_0^\alpha \cup \bigcup_{n=1}^{\infty} Z_n^\alpha .$$

Then $m(\tilde{Z}_0^\alpha) = 0$. For each α, each natural number n, and each integer i, define $Z_{n,i}^\alpha$ as the set of all x^α such that $x^\alpha - i \cdot 2^{-n} \in \tilde{Z}_0^\alpha$, and define

$$Z^\alpha = \bigcup_{n,i} Z_{n,i}^\alpha .$$

Then $m(Z^\alpha) = 0$.

Now, choose a point x_0 such that x_0^α is not in Z^α ($\alpha = 1, \cdots, \nu$). For each natural number k, let $\mathfrak{G}_k$ be the totality of hypercubes of side 2^{-k} bounded by hyperplanes of the form $x^\alpha = x_0^\alpha + i \cdot 2^{-k}$. None of the numbers $x_0^\alpha + i \cdot 2^{-k}$ is in Z^α and, moreover, $\overline{z}_0$ and each $\overline{z}_n$ is strongly of class $\mathfrak{P}_s$ on R^* with

$\overline{D}_s(\overline{z}_n, R)$ uniformly bounded for infinitely many values of n, R being any hypercube of any $\mathcal{G}_k$. Since the totality of these hypercubes is countable, we may choose a subsequence, still called z_n, such that $I(z_n, D)$ tends to the former lim inf, $\overline{z}_n \longrightarrow \overline{z}_0$ almost everywhere on D, and $\overline{D}_s(z_n, R^*)$ is uniformly bounded for R of any $\mathcal{G}_k$ in D. Since $z_n \longrightarrow z$ in $\mathfrak{P}_s$, we also have

$$\lim_{n \to \infty} \int_{R^*} |z_n - z_0|^s \, dS = 0$$

for each such R.

Now, we first consider the alternative (a). Let ϵ be any positive number. For each k, let D_k be the union of all the cells of $\mathcal{G}_k$ which are interior to D. Since f is bounded below and $I(z_0, D)$ is finite, we first choose k_1 so large that

$$\text{(3.6)} \qquad \begin{aligned} &I(z_n, D - D_{k_1}) > -\epsilon/5 \qquad (n = 1, 2, \cdots). \\ &I(z_0, D_{k_1}) > I(z_0, D) - \epsilon/5. \end{aligned}$$

For this k_1, let $R_1, \cdots, R_q$ be the cells of D_{k_1} and for each $k \geqq k_1$, let

$$R_{ki} \qquad \left(i = 1, \cdots, q \cdot 2^{\nu(k - k_1)}\right)$$

be the cells of $\mathcal{G}_k$ in D_{k_1} For each k, define $x_k^*(x)$, $z_k^*(x)$, and $p_k^*(x)$ on D_{k_1} by (2.9). Then, from (ii), (iii), and (iv), it follows that

$$\text{(3.7)} \qquad \begin{aligned} &|f[x, z_0(x), p_0(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)]| \\ &\leqq K(|p_0(x)|^s + 1) \cdot (|x - x_k^*(x)| + |z_0(x) - z_k^*(x)|) \\ &\quad + K(h_s |p_0(x)|^{s-1} + h_s |p_k^*(x)|^{s-1} + 1) \cdot |p_0(x) - p_k^*(x)|, \end{aligned}$$

where

$$h_s = \begin{cases} s^{-1} & (1 \leqq s \leqq 2), \\ s^{-1} \cdot 2^{s-2} & (s \geqq 2); \end{cases}$$

the method of proof is similar to that of (3.3). If we let

$$\zeta_n = z_n - z_0, \quad \pi_n = p_n - p_0,$$

we see similarly that

$$|f[x, z_0(x),\ p_0(x) + \pi_n(x)] - f[x_k^*(x),\ z_k^*(x),\ p_k^*(x) + \pi_n(x)]|$$

$$\leqq K(|p_n(x)|^s + 1)\ (|x - x_k^*(x)| + |z_0(x) - z_k^*(x)|) \tag{3.8}$$

$$+ K(h_s\ |p_n(x)|^{s-1} + h_s\ |p_k^*(x) + \pi_n(x)|^{s-1} + 1) \cdot |p_0(x) - p_k^*(x)|;$$

$$|f[x, z_n(x),\ p_n(x)] - f[x, z_0(x),\ p_n(x)]| \tag{3.9}$$

$$\leqq K(|p_n(x)|^s + 1) \cdot |z_n(x) - z_0(x)|\ .$$

Now, by the Hölder inequality on each R_{ki}, we see that

$$\int_{D_{k_1}} |p_k^*(x)|^s\ dx \leqq \int_{D_{k_1}} |p_0(x)|^s\ dx\ . \tag{3.10}$$

By applying the Minkowski inequality, we see that the integrals

$$\int_{D_{k_1}} |\pi_n(x)|^s\ dx, \quad \int_{D_{k_1}} |p_k^*(x) + \pi_n(x)|^s\ dx \tag{3.11}$$

are uniformly bounded. Finally,

$$\lim_{k\to\infty} \int_{D_{k_1}} |p_0(x) - p_k^*(x)|^s\ dx = 0. \tag{3.12}$$

Hence, using (3.7)-(3.12), we may choose a k so large that

$$\int_{D_{k_1}} |f[x, z_0(x),\ p_0(x)] - f[x_k^*(x),\ z_k^*(x),\ p_k^*(x)]|\ dx < \epsilon/5, \tag{3.13}$$

$$\int_{D_{k_1}} |f[x, z_0(x),\ p_n(x)] - f[x_k^*(x),\ z_k^*(x)\ p_k^*(x) + \pi_n(x)]|\ dx < \epsilon/5 \tag{3.14}$$

$$(n = 1, 2, \cdots),$$

and then choose n_1 so large that

$$\int_{D_{k_1}} |f[x, z_n(x),\ p_n(x)] - f[x, z_0(x),\ p_n(x)]|\ dx < \epsilon/5, \qquad n > n_1\ . \tag{3.15}$$

Since $x_k^*(x)$, $z_k^*(x)$, $p_k^*(x)$ are constant on each R_{ki}, it follows from Lemma 3.4 that

$$\liminf_{n\to\infty} \int_{D_{k_1}} \overset{*}{f}[x_k^*(x),\ z_k^*(x),\ p_k^*(x) + \pi_n(x)]\ dx \tag{3.16}$$

$$\geqq \int_{D_{k_1}} f[x_k^*(x),\ z_k^*(x),\ p_k^*(x)]\, dx.$$

Using (3.6) and (3.13)-(3.16), we see that

$$\liminf_{n\to\infty} I(z_n, D) \geqq I(z_0, D) - \epsilon.$$

The result follows in this case.

We now consider the alternative (b). For each natural number q, we define

$$f_q(x,z,p) = [1 - a_q(x,z)]\, f(x,z,p) + k \cdot a_q(x,z),$$

$$a_q(x,z) = \begin{cases} 0 & (0 \leqq R \leqq q), \\ 3(R-q)^2 - 2(R-q)^3 & (q \leqq R \leqq q+1), \\ 1 & (R \geqq q+1), \quad R = (|x|^2 + |z|^2)^{1/2}. \end{cases}$$

Remembering (3.3)-(3.5), we see that each f_q satisfies hypotheses (i)-(iv) with the same k and some K_q. Moreover f_q is independent of (x,z) for $R \geqq q+1$, and also

$$f_q(x,z,p) \leqq f_{q+1}(x,z,p), \quad \lim_{q\to\infty} f_q(x,z,p) = f(x,z,p).$$

Thus it is sufficient to prove the lower semicontinuity for each q.

For a fixed q, we note that we may replace $|z_0(x) - z_k^*(x)|$ by $\phi_k(x)$ in (3.7) and (3.8) and $|z_n(x) - z_0(x)|$ by $\psi_n(x)$ in (3.9), where

$$\phi_k(x) = \min(|z_0(x) - z_k^*(x)|,\ 2q+2),$$

$$\psi_n(x) = \min(|z_n(x) - z_0(x)|,\ 2q+2).$$

From the uniform boundedness of the ϕ_k and ψ_n (q fixed), the uniform absolute continuity of the set function $D_s(z_n, e)$, and the facts that

$$\lim_{k\to\infty} \phi_k(x) = 0, \quad \lim_{n\to\infty} \psi_n(x) = 0$$

almost everywhere, it follows that the argument can be carried through as before for each fixed q.

THEOREM 3.2. *Suppose* $s > \nu$ *and suppose* f *satisfies the hypotheses of Theorem* 3.1 *with* (i) *replaced by*

$$(\mathrm{i}') \qquad f(x,z,p) \geqq m|p|^s + k \qquad (m > 0).$$

If z^* is any function of class $\mathfrak{P}_s$ on the bounded domain D, then there is a function z_0 of class $\mathfrak{P}_s$ which coincides with z^* on D^* and minimizes $I(z,D)$ among all such functions.

Proof. Let z_n be a minimizing sequence. It follows from (i′) that $D_s(z_n, D)$ is uniformly bounded. From [2], Theorem 9.4, it follows that $\overline{D}_s(z_n,D)$ is uniformly bounded. But then a subsequence, still called $\{z_n\}$, converges weakly in $\mathfrak{P}_s$ to some function z_0 of class $\mathfrak{P}_s$ which coincides with z^* on D^* by [2], Theorem 9.2. But, from [3], Chapter II, Theorem 2.1, it follows that the equivalent functions $\overline{z}_n$ and $\overline{z}_0$ are equicontinuous on closed sets interior to D. Hence z_n converges uniformly to $\overline{z}_0$ on each closed set interior to D. Hence, from the preceding theorem, z_0 is a desired solution.

More general theorems involving variable boundary values, similar to those in [3], Chapter III, §5, with $s > \nu$, can be proved.

4. Necessary conditions for quasi-convexity. In the two preceding sections, we have established the connection between quasi-convexity and lower semicontinuity. In this section, we shall establish some necessary conditions for quasi-convexity. In the next section, we establish some sufficient conditions which are also necessary when f has certain interesting special forms. Unfortunately, the writer is unable to establish conditions which are both necessary and sufficient in the general case.

LEMMA 4.1. *Suppose f is continuous, Q is the cell*

$$|x^\alpha| \leqq 1 \quad (\alpha = 1, \cdots, \nu), \quad \delta > 0,$$

and suppose

$$(4.1) \qquad \int_Q f[p + \pi(x)]\, dx \geqq f(p) \cdot m(Q)$$

for every function ζ which satisfies a Lipschitz condition with coefficient $< \delta$ on $\overline{Q}$ and vanishes on Q^*. Then (4.1) *also holds with Q replaced by any bounded domain D.*

Proof. Suppose ζ satisfies the conditions on the bounded domain $\overline{D}$. Let $\overline{R}$ be a hypercube of side h which contains $\overline{D}$, and extend ζ to $\overline{R}$:

$$x_0^\alpha \leqq x^\alpha \leqq x_0^\alpha + h$$

by defining $\zeta = 0$ on $\overline{R} - \overline{D}$. Then ζ satisfies the conditions on $\overline{R}$, and

$$\zeta^*(x) = h^{-1}\,\zeta(x_0 + hx)$$

satisfies the conditions on $\overline{Q}$, and

$$\zeta^{*i}_{x^\alpha}(x) = \zeta^i_{x^\alpha}(x_0 + hx)\,.$$

DEFINITION 4.1. The function f is said to be *weakly quasi-convex* if with each p is associated a $\delta_p > 0$ such that (4.1) holds for all D and all ζ satisfying a Lipschitz condition with coefficient $< \delta_p$ and vanishing on D^*.

In other words, f is *weakly quasi-convex* if and only if each linear function furnishes a *weak relative minimum* among all Lipschitzian functions coinciding with it on the boundary, whereas f is *quasi-convex* if and only if any linear function furnishes the *absolute minimum* among all such functions. Thus we have the following result.

THEOREM 4.1. *If f is continuous and quasi-convex, it is weakly quasi-convex.*

We shall see that if f is weakly quasi-convex and continuous, then f satisfies a uniform Lipschitz condition on any bounded set in p-space and satisfies a *generalized Weierstrass condition* (see Theorem 4.3) which reduces to the *ordinary Weierstrass condition* if f is of class C' (see (4.7)) and is equivalent to the *Legendre-Hadamard condition* (see (4.8)) if f is of class C''.

LEMMA 4.2. *Suppose ϕ is continuous, and suppose corresponding to any point λ_0 in E_ν there is a $\delta > 0$ such that for any unit vector μ we have*

$$k\phi(\lambda_0 - h\mu) + h\phi(\lambda_0 + k\mu) \geqq (h + k)\,\phi(\lambda_0) \qquad (0 < h < \delta,\ 0 < k < \delta).$$

Then ϕ is convex in λ.

Proof. Let λ_0 be any point, and μ any point with $|\mu| = 1$. We shall show that

$$\psi(t) = \phi(\lambda_0 + \mu t)$$

is convex in t. From the hypothesis, it follows that for each t_0, there is a $\delta(t_0) > 0$ such that

$$(4.2) \quad k\psi(t_0 - h) + h\psi(t_0 + k) \geqq (h + k)\,\psi(t_0) \quad (0 \leqq h \leqq \delta,\ \ 0 \leqq k \leqq \delta).$$

Now, suppose $t_1 < t_2$. Let

$$\chi(t) = \psi(t) - \psi(t_1) - \frac{t - t_1}{t_2 - t_1}[\psi(t_2) - \psi(t_1)].$$

Then $\chi(t)$ satisfies (4.2) and $\chi(t_1) = \chi(t_2) = 0$. Suppose $M = \max \chi(t)$ $(t_1 \leqq t \leqq t_2)$, and suppose $M > 0$. Let t_0 be the smallest value of t such that $\chi(t) = M$, and let the number $\delta(t_0)$ be chosen as above. Clearly $t_1 < t_0 < t_2$. Choose t_3 and t_4 with

$$|t_3 - t_0| < \delta, \; |t_4 - t_0| < \delta \quad (t_1 \leqq t_3 < t_0 < t_4 \leqq t_2).$$

Then $\chi(t_3) < M$, $\chi(t_4) \leqq M$, so that

$$(t_4 - t_0)\chi[t_0 - (t_0 - t_3)] + (t_0 - t_3)\chi[t_0 + (t_4 - t_0)] < (t_4 - t_3)\chi(t_0),$$

which contradicts the hypothesis. Thus $\chi(t) \leqq 0$, so that

$$\psi(t) \leqq \psi(t_1) + \frac{t - t_1}{t_2 - t_1}[\psi(t_2) - \psi(t_1)].$$

Since t_1 and t_2 were arbitrary with $t_1 < t_2$, the function ψ is convex in t. Thus ϕ is convex in λ.

THEOREM 4.2. *If f is weakly quasi-convex, then $f(p^i_\alpha + \lambda_\alpha \xi^i)$ is convex in λ for each fixed p and ξ.*

Proof. Let p^i_α, ξ^i and $\lambda_{0\alpha}$ be fixed and let μ_1 be any unit vector, and suppose $h > 0$, $k > 0$. Choose $\delta(p^i_\alpha, \xi^i, \lambda_{0\alpha}) > 0$ but so small that, for any bounded domain G,

$$\int_G f[p^i_\alpha + \lambda_{0\alpha}\xi^i + \zeta^i_{x^\alpha}(x)]\,dx \geqq m(G)\, f(p^i_\alpha + \lambda_{0\alpha}\xi^i) \tag{4.3}$$

for all ζ satisfying a Lipschitz condition of constant $< \delta$ on G and vanishing on G^*. Let $(\mu_1, \cdots, \mu_\nu)$ be a normal orthogonal set of unit vectors. If $\xi = 0$, the result is obvious. If $\xi \neq 0$, choose h and k with $0 < h\,|\xi| < \delta$, $0 < k\,|\xi| < \delta$, and let ρ be any number $> |\xi|/\delta$. Let $H = (1/\rho)\,k$, $K = (1/\rho)\,h$, and let R be the rectangular parallelepiped

$$-\rho H \leqq y^1 \leqq \rho K, \quad |y^\beta| \leqq \rho \quad (\beta = 2, \cdots, \nu)$$

where

$$y^\beta = x, \mu_\beta \ .$$

Let F_1^- be the face $y^1 = -\rho H$, F_1^+ be the face $y^1 = \rho K$, F_β^- be the face $y^\beta = -\rho$, F_β^+ be the face $y^\beta = \rho$, and let Π_β^- and Π_β^+ be the pyramids with vertex at the origin and base F_β^- and F_β^+, respectively. Let ζ be defined on $\overline{R}$ to be continuous on $\overline{R}$, zero on R^*, linear on each Π_β and Π_β^+, with $\zeta(0) = \xi$. Then

$$\text{(4.4)} \qquad \zeta^i_{x^\alpha} = \begin{cases} (\rho H)^{-1} \mu_{1\alpha} \xi^i = k\mu_{1\alpha} \xi^i & , \text{ on } \Pi_1^- \\ -(\rho K)^{-1} \mu_{1\alpha} \xi^i = -h\mu_{1\alpha} \xi^i, & \text{ on } \Pi_1^+ \\ \rho^{-1} \mu_{\beta\alpha} \xi^i & , \text{ on } \Pi_\beta^- \\ -\rho^{-1} \mu_{\beta\alpha} \xi^i & , \text{ on } \Pi_\beta^+ \end{cases}$$

Also

$$m(\Pi_1^-) = \nu^{-1} 2^{\nu-1} \rho^\nu H, \ m(\Pi_1^+) = \nu^{-1} 2^{\nu-1} \rho^\nu K, \ m(R) = 2^{\nu-1} \rho^\nu (H+K)$$

(4.5)

$$m(\Pi_\beta^-) = m(\Pi_\beta^+) = \nu^{-1} 2^{\nu-2} \rho^\nu (H+K) \qquad (\beta = 2, \cdots, \nu).$$

Then, by applying (4.3), (4.4), and (4.5), we obtain

$$\frac{1}{2\nu} \left\{ \frac{2k}{h+k} f[p_\alpha^i + (\lambda_{0\alpha} - h\mu_{1\alpha}) \xi^i] + \frac{2h}{h+k} f[p_\alpha^i + (\lambda_{0\alpha} + k\mu_{1\alpha}) \xi^i] \right.$$

$$\left. + \sum_{\beta=2}^{\nu} \left(f[p_\alpha^i + (\lambda_{0\alpha} - \rho^{-1} \mu_{\beta\alpha}) \xi^i] + f\left[p_\alpha^i + (\lambda_{0\alpha} + \rho^{-1} \mu_{\beta\alpha}) \right] \right) \right\}$$

$$\geqq f(p_\alpha^i + \lambda_{0\alpha} \xi^i).$$

Letting $\rho \longrightarrow \infty$, we obtain

$$kf[p_\alpha^i + (\lambda_{0\alpha} - h\mu_{1\alpha}) \xi^i] + hf[p_\alpha^i + (\lambda_{0\alpha} + k\mu_{1\alpha}) \xi^i] \geqq (h+k) f(p_\alpha^i + \lambda_{0\alpha} \xi^i)$$

From the preceding lemma, it follows that $f(p_\alpha^i + \lambda_\alpha \xi^i)$ is convex in λ for each ξ and p.

THEOREM 4.3. *Suppose f is continuous and convex in λ for p and ξ.*

Then f satisfies a uniform Lipschitz condition on each bounded closed set, and for each fixed p there exists a set of constants A_i^α such that

$$f(p_\alpha^i + \lambda_\alpha \xi^i) \geqq f(p) + A_i^\alpha \lambda_\alpha \xi^i \tag{4.6}$$

for all λ and ξ. If f is of class C', (4.6) holds if and only if $A_i^\alpha = f_{p_\alpha^i}$, that is,

$$f(p_\alpha^i + \lambda_\alpha \xi^i) \geqq f(p) + f_{p_\alpha^i}(p) \lambda_\alpha \xi^i. \tag{4.7}$$

If f is of class C'', (4.7) holds for all p, λ, ξ if and only if

$$f_{p_\alpha^i p_\beta^j}(p) \lambda_\alpha \lambda_\beta \xi^i \xi^j \geqq 0 \tag{4.8}$$

for all λ, ξ, p.

Proof. Suppose, first, that f is of class C'. Let p and ξ be fixed. Then (4.7) follows from the convexity in λ. Moreover, since each unit vector e_α^i in the p-space is of the form $\lambda_\alpha \xi^i$, we see from the convexity in λ that

$$f(p) - f\left(p - e_\alpha^i\right) \leqq f_{p_\alpha^i}(p) \leqq f\left(p + e_\alpha^i\right) - f(p) \tag{4.9}$$

for all p. Thus the derivatives of f are uniformly bounded by these *differences in the values of f* on any bounded part of space. Moreover, in this case, if constants A_i^α satisfy (4.6), we must have

$$A_i^\alpha = f_{p_\alpha^i}(p).$$

Now, if f is of class C'', equation (4.8) with p replaced by $p_\alpha^i + \lambda_\alpha \xi^i$ is equivalent to the condition that f is convex in λ for each fixed p and ξ.

Finally, if f is continuous and has this stated convexity property, it is clear that the h-average function also does, and f_h is of class C'. By letting $h \longrightarrow 0$, we see that f satisfies a uniform Lipschitz condition on any bounded closed set. Now, choose $h_n = n^{-1}$ and choose p fixed. From (4.9) and the uniform convergence of f_h to f on any bounded part of space, we conclude that the derivatives $f_{h_n p_\alpha^i}(p)$ are uniformly bounded. We may therefore choose a subsequence, still called h_n, such that

$$\lim_{n \to \infty} f_{h_n p_\alpha^i}(p) = A_i^\alpha.$$

Since (4.7) holds for all λ and ξ for each n, (4.6) holds in the limit.

5. Sufficient conditions for quasi-convexity. In this section we prove one general sufficient condition and then give conditions which are necessary and sufficient when f has certain interesting special forms.

LEMMA 5.1. *Suppose ζ satisfies a uniform Lipschitz condition on the closure $\overline{D}$ of the bounded domain D and suppose $\zeta = 0$ on D^*. If*

$$1 \leqq \mu \leqq \nu, \ 1 \leqq i_1, \cdots, i_\mu \leqq N, \ 1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_\mu \leqq \nu,$$

then

$$\int_D \frac{\partial(\zeta^{i_1}, \cdots, \zeta^{i_\mu})}{\partial(x^{\alpha_1}, \cdots, x^{\alpha_\mu})} dx = 0.$$

Proof. Choose a large cell R containing $\overline{D}$ in its interior, and extend ζ by defining it to be zero outside $\overline{D}$. Then the second h-average function ζ_{hh} is of class C'' on $\overline{R}$ and vanishes on and near R^*. Since any integral of the above type formed for ζ_{hh} tends to that for ζ as $h \longrightarrow 0$, we need prove the theorem only for functions ζ of class C'' on cells $\overline{R}$.

As an example, take $i_\beta = \alpha_\beta = \beta$, $\beta = 1, \cdots, \mu$, $D = R$. Then

$$\int_R \frac{\partial(\zeta^1, \cdots, \zeta^\mu)}{\partial(x^1, \cdots, x^\mu)} dx = \int_R \sum_{\alpha=1}^{\mu} (-1)^{\mu+\alpha} \zeta^\mu_{x^\alpha} Q \, dx$$

$$= \int_{R^*} \zeta^\mu \sum_{\alpha=1}^{\mu} (-1)^{\mu+\alpha} Q \, dx'_\alpha$$

$$- \int_R (-1)^\mu \zeta^\mu \sum_{\alpha=1}^{\mu} (-1)^\alpha \frac{\partial}{\partial x^\alpha} Q \, dx,$$

where

$$Q = \frac{\partial(\zeta^1, \cdots, \zeta^{\alpha-1}, \zeta^\alpha, \cdots, \zeta^{\mu-1})}{\partial(x^1, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^\mu)},$$

the last equality holding by Green's theorem. But the boundary integral vanishes since $\zeta = 0$ on R^*, and the integrand in the second integral vanishes on R (see [3], Chapter II, Lemma 1.1, for instance).

THEOREM 5.1. *A sufficient condition that f be quasi-convex is that for each p there exist alternating forms*

$$A_i^\alpha \pi_\alpha^i, \; A_{ij}^{\alpha\beta} \pi_\alpha^i \pi_\beta^j, \; \cdots, \; A_{i_1,\cdots,i_\nu}^{\alpha_1,\cdots,\alpha_\nu} \pi_{\alpha_1}^{i_\nu} \cdots \pi_{\alpha_\nu}^{i_\nu}$$

such that for all π we have

$$f(p+\pi) \geqq f(p) + A_i^\alpha \pi_\alpha^i + \cdots + A_{i_1,\cdots,i_\nu}^{\alpha_1,\cdots,\alpha_\nu} \pi_{\alpha_1}^{i_1} \cdots \pi_{\alpha_\nu}^{i_\nu}.$$

Proof. This is an immediate consequence of the preceding lemma.

THEOREM 5.2. *If the $a_{ij}^{\alpha\beta}$ are constants and*

$$f(p) = a_{ij}^{\alpha\beta} p_\alpha^i p_\beta^j, \tag{5.1}$$

a necessary and sufficient condition that f be quasi-convex is that

$$a_{ij}^{\alpha\beta} \lambda_\alpha \lambda_\beta \xi^i \xi^j \geqq 0 \tag{5.2}$$

for all λ and ξ.

Proof. If $\zeta = 0$ on D^*, we see from Lemma 5.1 that

$$\int_D f[p+\pi(x)]\,dx = f(p)\, m\,(D) + \int_D a_{ij}^{\alpha\beta} \pi_\alpha^i(x)\, \pi_\beta^j(x)\,dx.$$

But Van Hove [6] has shown that the condition (5.2) is necessary (this also follows from Theorem 4.3) and sufficient for the second integral to be $\geqq 0$ for all ζ of class D' on D which vanish on D^* (hence this is true also for all ζ of class $\mathfrak{P}_2$ on D and vanishing on D^*).

LEMMA 5.2. *Suppose*

$$\sum_{i,j=1}^{n} a_{ij} x^i y^j = 0$$

for all x and y for which

$$\sum_{i,j=1}^{n} b_{ij} x^i y^j = 0.$$

Then there is a constant K such that

$$a_{ij} = Kb_{ij} \quad (i,\ j = 1, \cdots, n).$$

Proof. We may introduce new variables ξ and η by

$$x = c\xi, \quad y = d\eta,$$

c and d being $n \times n$ nonsingular matrices. Let a and b be the matrices of the original forms and A and B those of the transformed forms. Then

$$A = c'ad, \quad B = c'bd \quad (c'_{ij} = c_{ji}).$$

We shall show that there is a scalar K such that $A = KB$. We may assume that

$$B_{ii} = 1 \quad (i = 1, \cdots, r); \quad B_{ij} = 0 \text{ otherwise}, \qquad r \leqq n,$$

unless $B = 0$ in which case $A = 0$ also and the theorem holds. By taking $\eta^s = 1$, $\eta^j = 0 \quad (j \neq s,\ s = 1, \cdots, n)$ in turn we see that

$$A_{is} = 0 \ (i = 1, \cdots, n,\ s > r); \ A_{is} = 0 \quad (i \neq s,\ s = 1, \cdots, r,\ i = 1, \cdots, n).$$

Then, by choosing $1 \leqq s < t \leqq r$ and setting $\eta^s = \eta^t = 1$, $\eta^j = 0$, $j \neq s$, $j \neq t$, we have

$$(A_{is} + A_{it})\,\xi^i = 0 \qquad \text{for all } \xi \text{ with } \xi^s + \xi^t = 0.$$

Thus there exists a constant $K(s,t)$ such that

$$A_{ss} + A_{st} = K(s,t), \qquad A_{ts} + A_{tt} = K(s,t).$$

Hence

$$A_{11} = A_{22} = \cdots = A_{rr} = K,$$

so that $A = KB$.

THEOREM 5.3. *Suppose that* $N = \nu + 1$ *and*

$$f(p) = F(X_1, \cdots, X_{\nu+1}), \tag{5.3}$$

where F *is positively homogeneous of the first degree in the* X_i *and*

$$X_i = -\det M_i \quad (i = 1, \cdots, \nu), \quad X_{\nu+1} = \det M_{\nu+1},$$

$$M_{\nu+1} = \| p_\alpha^1, \cdots, p_\alpha^\nu \|, \ M_i = \| p_\alpha^1, \cdots, p_\alpha^{i-1}, p_\alpha^{\nu+1}, p_\alpha^{i+1}, \cdots, p_\alpha^\nu \|$$

$$(i = 1, \cdots, \nu).$$

Then f is quasi-convex in p if and only if F is convex in the X_i.

Proof. If F is convex in the X_i, it follows from Theorem 5.1 that f is quasi-convex in p.

Hence suppose f is given by (5.3) and is quasi-convex in p. If

$$\Delta X_k = X_k\left(p^i_\alpha + \lambda_\alpha \xi^i\right) - X_k\left(p^i_\alpha\right),$$

then

$$\Delta X_k = X_{k p^i_\alpha} \lambda_\alpha \xi^i. \tag{5.4}$$

Also, since

$$p^k_\beta X_k = 0 \quad (\beta = 1, \cdots, \nu),$$

we have

$$p^k_\beta X_{k p^i_\alpha} = - \delta^\alpha_\beta X_i . \tag{5.5}$$

Now, choose a set of X_i not all zero and choose any p such that

$$X_i(p) = X_i .$$

Since f is quasi-convex and hence weakly so, there are constants A^i_α such that

$$f\left(p^i_\alpha + \lambda_\alpha \xi^i\right) \geqq f(p) + A^\alpha_i \lambda_\alpha \xi^i.$$

Since f depends only on the X_i, we must have

$$A^\alpha_i \lambda_\alpha \xi^i \leqq 0 \text{ for all } \lambda,\ \xi \text{ with } X_{k p^i_\alpha} \lambda_\alpha \xi^i = 0 \quad (k = 1, \cdots, \nu + 1). \tag{5.6}$$

Obviously, then, the equality must hold in (5.6). Using (5.4) and (5.5), we see that

$$p^k_\beta \Delta X_k = - \lambda_\beta\left(X_i \xi^i\right) \qquad (\beta = 1, \cdots, \nu). \tag{5.7}$$

Hence, we must have

$$A^\alpha_i \lambda_\alpha \xi^i = 0 \tag{5.8}$$

for all λ, ξ for which

$$X_i \xi^i = 0 \text{ and } D^\alpha_i \lambda_\alpha \xi^i = 0, \quad D^\alpha_i = X_k X_{k p^i_\alpha}. \tag{5.9}$$

Now, since not all the X_i are zero, assume $X_k \neq 0$. Then

$$\sum_{i \neq k} \left(A_i^\alpha X_k - A_k^\alpha X_i\right) \lambda_\alpha \xi^i = 0 \tag{5.10}$$

for all λ, ξ for which

$$\sum_{i \neq k} \left(D_i^\alpha X_k - D_k^\alpha X_i\right) \lambda_\alpha \xi^i = 0. \tag{5.11}$$

From the preceding lemma, it follows that there is a constant K such that

$$A_i^\alpha X_k - A_k^\alpha X_i = K\left(D_i^\alpha X_k - D_k^\alpha X_i\right). \tag{5.12}$$

Hence

$$A_i^\alpha = KD_i^\alpha + L^\alpha X_i, \quad L^\alpha = X_k^{-1}\left(A_k^\alpha - KD_k^\alpha\right). \tag{5.13}$$

From (5.7) and (5.13) it follows that

$$A_i^\alpha \lambda_\alpha \xi^i = KD_i^\alpha \lambda_\alpha \xi^i + L^\alpha \lambda_\alpha X_i \xi^i = C^k \Delta X_k, \; C^k = \left(KX_k - L^\alpha p_\alpha^k\right) \tag{5.14}$$

Finally, if we are given any values of the ΔX_k, the quantities

$$h_i = p_i^k \Delta X_k \; (i = 1, \cdots, \nu) \text{ and } h_{\nu+1} = X_k \Delta X_k$$

are determined and the ΔX_k are also uniquely determined by the h_i. Using (5.7), we may determine the λ_α in terms of the h_i $(i = 1, \cdots, \nu)$, and substitute them into

$$h_{\nu+1} = X_k \Delta X_k = D_i^\alpha \lambda_\alpha \xi^i,$$

and we merely have to choose the ξ^i to satisfy the equation

$$\left(D_i^\alpha h_\alpha + h_{\nu+1} X_i\right) \xi^i = 0 \text{ with } X_i \xi^i \neq 0;$$

this is always possible unless all the $D_i^\alpha h_\alpha = 0$. Thus, unless these linear relations in the ΔX_i hold, we have

$$F(X + \Delta X) = f\left(p_\alpha^i + \lambda_\alpha \xi^i\right) \geqq f(p) + A_i^\alpha \lambda_\alpha \xi^i = F(X) + C^k \Delta X_k. \tag{5.15}$$

The result follows in this case by continuity.

Finally, since F is homogeneous of the first degree, we see by taking

$$\Delta X = hX, \quad h > -1,$$

that

$$F[(1+h)X] = (1+h)F(X) \geqq F(X) + hC^k X_k,$$

or

$$h\left[F(X) - C^k X_k\right] \geqq 0, \quad h > -1.$$

Hence $F(X) = C^k X_k$. Then by setting $X = hX_0$, $X_0 \neq 0$, choosing the C^k for this X_0, and then letting $h \longrightarrow 0$, we see that (5.15) holds for some C^k even if $X = 0$.

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