A SHORT PROOF OF PILLAI'S THEOREM ON NORMAL NUMBERS

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1. Introduction. The object of this paper is to give a short proof of the Pillai theorem [2] on normal numbers using the Niven-Zuckerman result [1] as a tool.

DEFINITION 1. A number σ is *simply normal* to the base r if, in the expansion to the base r of the fractional part of σ, we have $\lim_{n\to\infty} n_c/n = 1/r$ for all c, where n_c is the number of occurrences of the digit c in the first n digits of σ.

DEFINITION 2. A number σ is *normal* to the base r if σ, $r\sigma$, $r^2\sigma$, $\cdots$ are each simply normal to all the bases r, r^2, r^3, $\cdots$.

THEOREM (Pillai). *A necessary and sufficient condition that a number σ be normal to the base r is that it be simply normal to the bases r, r^2, r^3, $\cdots$.*

2. Proof. The necessity of the condition follows from the definition of normality.

To prove sufficiency, assume that σ is simply normal to the bases $r, r^2, \cdots$. Let $A = (a_1 a_2 \cdots a_v)$ be any fixed sequence of digits (to base r), where $v = hr - s$, $h > 0$, $0 \le s < r$; and consider the occurrence of A in σ. Count the number of occurrences of A in the collection of sequences of length hr. There are s digits free after v of the hr digits are fixed. Thus there are $(s+1)r^s$ different occurrences of A in these sequences.

For any positive integer n, define $f_n(A)$ to be the frequency of the occurrences of A in σ except in places where A will straddle the middle of sequences of length $2h2^{n-1}r$ starting in places congruent to 1 (mod $2h2^{n-1}r$), or where A will straddle the middle of sequences of length $4h2^{n-1}r$ starting in places congruent to 1 (mod $4h2^{n-1}r$), or $\cdots$, or where A will straddle the middle of sequences of length $2^s h2^{n-1}r$ starting in places congruent to 1 (mod $2^s h 2^{n-1}r$), and so on.

Certainly $\lim_{n\to\infty} f_n(A)$, if it exists, will be equal to $f(A)$, the frequency of A in σ.

We have

$$f_1(A) = \frac{(s+1)\,r^s}{hr\,r^{hr}} = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}},$$

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since there are hr digits of σ to base r in each digit of σ to base r^{hr}, and σ is simply normal to the base r^{hr}. The number of occurrences of A straddling the middle of blocks of length $2hr$ is $(v-1)r^{2hr+s}$. The frequency of these in σ, where the sequence of length $2hr$ starts in a place congruent to 1 (mod $2hr$), is

$$\frac{(v-1)r^{2hr+s}}{2hr\, r^{2hr}} = \frac{v-1}{2hr^{v+1}},$$

since there are $2hr$ digits of σ to base r to each digit of σ to base r^{2hr}.

Thus

$$f_2(A) = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{2hr^{v+1}}.$$

Similarly,

$$f_3(A) = f_2(A) + \frac{v-1}{4hr^{v+1}} = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{hr^{v+1}}\left[\frac{1}{2} + \frac{1}{4}\right]$$

and

$$f_n(A) = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{hr^{v+1}} \sum_{i=1}^{n-1} 1/2^i.$$

It follows that

$$\lim_{n\to\infty} f_n(A) = 1/r^v.$$

Accordingly, by the Niven-Zuckerman result [1], stating that a necessary and sufficient condition in order that a number σ be normal is that every fixed sequence of v digits occur in the expansion of σ with the frequency $1/r^v$, we see that σ is normal to the scale r.

References

1. Ivan Niven and H. S. Zuckerman, *On the definition of normal numbers*. Pacific J. Math. 1 (1951), 103-109.

2. S. S. Pillai, *On normal numbers*, Proceedings of the Indian Acad. Sci., Section A, 12 (1940), 179-184.

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