SOME THEOREMS ON BERNOULLI NUMBERS OF HIGHER ORDER

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1. Introduction. We define the Bernoulli numbers of order k by means of [3, Chapter 6]

$$\left(\frac{t}{e^t - 1}\right)^k = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m^{(k)} \qquad (|t| < 2\pi);$$

in particular, $B_m = B_m^{(1)}$ denotes the ordinary Bernoulli number. Not much seems to be known about divisibility properties of $B_m^{(k)}$. Using different notation, S. Wachs [4] proved a result which may be stated in the form

$$B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}, \tag{1.1}$$

where p is a prime ≥ 3. In attempting to simplify Wachs' proof, the writer found the stronger result

$$B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3} \qquad (p > 3). \tag{1.2}$$

We remark that $B_5^{(4)} = -9$.

The proof of (1.2) depends on some well-known properties of the Bernoulli numbers and factorial coefficients; in particular, we make use of some theorems of Glaisher and Nielsen. The necessary formulas are collected in §2; the proof of (1.2) is given in §3. In §4 we prove

$$B_p^{(p)} \equiv \frac{1}{2} p^2 \pmod{p^3} \qquad (p \geq 3); \tag{1.3}$$

the proof of this result is somewhat simpler than that of (1.2). For the residue of $B_p^{(p)} \pmod{p^4}$, see (4.5) below.

In §5 we prove several formulas of a similar nature $(p > 3)$:

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$$(1.4) \qquad B_{p+1}^{(p)} \equiv -p\,\frac{B_{p+1}}{p+1} + \frac{1}{24}\,p^2 \pmod{p^3},$$

$$(1.5) \qquad B_{p+2}^{(p)} \equiv p^2\,\frac{B_{p+1}}{p+1} \pmod{p^4},$$

$$(1.6) \qquad B_{p+1}^{(p+1)} \equiv \frac{B_{p+1}}{p+1} - \frac{1}{24}\,p \pmod{p^2}.$$

In §6 we discuss the number $B_m^{(p)}$ for arbitrary m; this requires the consideration of a number of cases. In particular, we mention the following special results $(p > 3)$:

$$(1.7) \qquad B_{p^r}^{(p)} \equiv -\frac{1}{2}\,p^{r+1}(p-1)\,B_{p^r-1} \pmod{p^{r+2}}$$

for $r > 1$;

$$(1.8) \qquad B_m^{(p)} \equiv \frac{1}{2}\,p(p-1)\,B_{m-1} \pmod{p^{r+2}}$$

for $m \equiv 1 \pmod{p^r(p-1)}$.

It also follows from the results of §6 that $B_m^{(p)}$ is integral $\pmod p$, $p \geq 3$, unless $m \equiv 0 \pmod{p-1}$ and $m \equiv 0$ or $p-1 \pmod p$, in which case $pB_m^{(p)}$ is integral.

The number $B_m^{(p+1)}$ requires a more detailed discussion than $B_m^{(p)}$; this will be omitted from the present paper. However, we note the special formula

$$(1.9) \qquad B_{p^r}^{(p+1)} \equiv p^r\left\{\frac{1}{2}\,p(p+1)\,\frac{B_{p^r-1}}{p^r-1} + p!\,\frac{B_{p^r-p}}{p^r-p}\right\} \pmod{p^{r+2}}$$

for $p > 3$, $r > 1$. The residue $\pmod{p^{r+3}}$ can be specified.

2. Some preliminary results. We first state a number of formulas involving $B_m^{(k)}$ which may be found in [3, Chapter 6].

$$(2.1) \qquad B_m^{(k+1)} = (k+1)\binom{m}{k+1}\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}B_s^{(k+1)}\,\frac{B_{m-s}}{m-s},$$

$$(2.2) \qquad (x-1)(x-2)\cdots(x-m) = \sum_{s=0}^{m}\binom{m}{s}B_s^{(m+1)}\,x^{m-s}.$$

We shall require the special values

$$(2.3)\qquad B_1^{(k)} = -\frac{1}{2}k,\ B_2^{(k)} = \frac{1}{12}k(3k-1),\ B_3^{(k)} = -\frac{1}{8}k^2(k-1).$$

If we define the factorial coefficients by means of

$$(x+1)\cdots(x+m-1) = \sum_{s=0}^{m-1} C_s^{(m)} x^{m-1-s},$$

we see at once that

$$(2.4)\qquad (-1)^s \binom{m}{s} B_s^{(m+1)} = C_s^{(m+1)}.$$

We have also the recurrence formula

$$(2.5)\qquad C_s^{(m+1)} = C_s^{(m)} + mC_{s-1}^{(m)}.$$

In the next place [1, p. 325; 2, p. 328] for p a prime > 3,

$$(2.6)\qquad C_{2r}^{(p)} \equiv -p\frac{B_{2r}}{2r} \pmod{p^2} \qquad (2 \le 2r \le p-3),$$

$$(2.7)\qquad C_{2r+1}^{(p)} \equiv p^2\frac{(2r+1)B_{2r}}{4r} \pmod{p^3} \qquad (r \ge 1),$$

$$(2.8)\qquad C_{p-1}^{(p)} = (p-1)! \equiv p(-1+B_{p-1}) \pmod{p^2}.$$

It follows immediately from (2.8) and Wilson's theorem that

$$(2.9)\qquad p(p+1)B_{p-1} \equiv (p-1)! \pmod{p^2}.$$

We shall require the following special case of Kummer's congruence [2, Chapter 14]:

$$(2.10)\qquad \frac{B_{m+p-1}}{m+p-1} \equiv \frac{B_m}{m} \pmod{p} \qquad (p-1 \nmid m);$$

also, the Staudt-Clausen theorem [3, 32] which we quote in the following form:

$$pB_m \equiv -1 \pmod p \qquad (p-1 \mid m). \tag{2.11}$$

A formula of a different sort that will be used is [3, p. 146, formula (83)]

$$B_m = -\frac{1}{m} \sum_{s=1}^{m} (-1)^s \binom{m}{s} B_s B_{m-s}.$$

In particular, replacing m by $2m$, this becomes

$$(2m+1) B_{2m} + \sum_{t=1}^{m-1} \binom{2m}{2t} B_{2t} B_{2m-2t} = 0 \tag{2.12}$$

provided $m > 1$, a formula due to Euler. The formula [3, p. 145]

$$B_m^{(k+1)} = \left(1 - \frac{m}{k}\right) B_m^{(k)} - m B_{m-1}^{(k)} \tag{2.13}$$

will also be employed. In particular, we note that

$$B_m^{(m+1)} = (-1)^m \, m! \tag{2.14}$$

3. **Proof of (1.2).** Let p be a prime > 3. In (2.1), taking $k = p$, $m = p + 2$, we get

$$B_{p+2}^{(p+1)} = (p+1)(p+2) \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} \frac{B_{p+2-s}}{p+2-s} B_s^{(p+1)}$$

$$= (p+1)(p+2) \sum_{t=0}^{(p-1)/2} \binom{p}{2t+1} \frac{B_{p+1-2t}}{p+1-2t} B_{2t+1}^{(p+1)} \tag{3.1}$$

$$= (p+1)(p+2) A,$$

say. We break the sum A into several parts:

$$A = u_0 + u_1 + \sum_{t=2}^{(p-3)/2} u_t + u_{(p-1)/2}, \tag{3.2}$$

where

$$u_t = \binom{p}{2t+1} B_{2t+1}^{(p+1)} \frac{B_{p+1-2t}}{p+1-2t} \qquad (0 \le t \le p-1).$$

Then by (2.2) and (2.3) we have

$$u_0 = pB_1^{(p+1)} \frac{B_{p+1}}{p+1} = -\frac{1}{2} pB_{p+1} \; ; \tag{3.3}$$

and

$$u_{(p-1)/2} = \frac{1}{2} B_2 B_p^{(p+1)} = -\frac{1}{12} p! \tag{3.4}$$

by (2.14). As for u_1 we have, by (2.3),

$$\binom{p}{3} B_3^{(p+1)} = -\frac{1}{48} p^2 (p+1)^2 (p-1)(p-2)$$

$$\equiv -\frac{1}{48} (p^3 + 2p^2) \qquad (\mathrm{mod}\ p^4);$$

thus, by (2.8),

$$u_1 \equiv -\frac{1}{48} (p^2 + 2p) \frac{pB_{p-1}}{p-1} \equiv -\frac{1}{48} (p^2 + 2p) \frac{(p-1)!}{p^2 - 1}$$

$$\equiv \frac{1}{48} (p^2 + 2p)(p-1)! \qquad (\mathrm{mod}\ p^3). \tag{3.5}$$

In the next place, by (2.4) and (2.5),

$$\binom{p}{2t+1} B_{2t+1}^{(p+1)} = -C_{2t+1}^{(p+1)} = -C_{2t+1}^{(p)} - pC_{2t}^{(p)}$$

$$\equiv -p^2 \left(\frac{2t+1}{4t} - \frac{1}{2t} \right) B_{2t}$$

$$\equiv -p^2 \frac{2t-1}{4t} B_{2t} \qquad (\mathrm{mod}\ p^3)$$

for $2 \le t \le (p-3)/2$. Hence

$$u_t \equiv p^2 \frac{B_{2t}}{4t} B_{p+1-2t} \qquad (\mathrm{mod}\ p^3),$$

so that

$$\sum_{t=2}^{(p-3)/2} u_t \equiv p^2 \sum_{t=2}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p+1-2t} \qquad (\mathrm{mod}\ p^3). \tag{3.6}$$

On the other hand, by (2.12),

$$(p+2) B_{p+1} + \sum_{t=1}^{(p-1)/2} \binom{p+1}{2t} B_{2t} B_{p+1-2t} = 0,$$

which implies

$$(p+2) B_{p+1} + \frac{1}{6} p(p+1) B_{p-1} \equiv p(p+1) \sum_{2}^{(p-3)/2} \frac{B_{2t}}{2t} \frac{B_{p+1-2t}}{p+1-2t}$$

$$\equiv 2p \sum_{2}^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p+1-2t} \quad (\mathrm{mod}\ p^2); \tag{3.61}$$

the last cengruence is a consequence of

$$\frac{1}{2t} + \frac{1}{p+1-2t} \equiv \frac{1}{2t(p+1-2t)} \quad (\mathrm{mod}\ p).$$

Now using (3.6) we see that

$$\begin{aligned} \sum_{2}^{(p-3)/2} u_t &\equiv \frac{1}{4} p(p+2) B_{p+1} + \frac{1}{24} p^2 (p+1) B_{p-1} \\ &\equiv \frac{1}{4} p(p+2) B_{p+1} + \frac{1}{24} p(p-1)! \qquad (\mathrm{mod}\ p^3) \end{aligned} \tag{3.7}$$

by (2.9). Collecting from (3.3), (3.4), (3.5), and (3.7) we get, after some simplification,

$$A \equiv \frac{1}{4} p^2 B_{p+1} + \frac{1}{48} p^2 (p-1)!$$

$$\equiv \frac{1}{4} p^2 \left(\frac{B_{p+1}}{p+1} - \frac{B_2}{2} \right) + \frac{1}{48} p^2 + \frac{1}{48} p^2 (p-1)!$$

$$\equiv 0 \pmod{p^3}$$

by (2.10). Therefore, by (3.1), $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}$.

It would be of interest to determine the residue of $B_{p+2}^{(p+1)} \pmod{p^4}$. We have already noted that $B_5^{(4)} \not\equiv 0 \pmod{3^3}$; for small p at least, it can be verified that $B_{p+2}^{(p+1)} \not\equiv 0 \pmod{p^4}$.

4. **Proof of** (1.3). We now take $m = p > 3$, $k = p - 1$ in (2.1), so that

$$B_p^{(p)} = p \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} \frac{B_{p-s}}{p-s} B_s^{(p)}$$

(4.1)
$$= p \left\{ \frac{1}{2} pB_{p-1} - \frac{1}{2}(p-1)! - \sum_{t=1}^{(p-3)/2} \binom{p-1}{2t+1} \frac{B_{p-1-2t}}{p-1-2t} B_{2t+1}^{(p)} \right\}$$

by (2.3) and (2.4). Now, again using (2.4), we have

$$\binom{p-1}{2t+1} B_{2t+1}^{(p)} = - C_{2t+1}^{(p)}$$

$$\equiv - p^2 \frac{(2t+1) B_{2t+1}}{4t} \pmod{p^3}.$$

Hence, the sum Q in the right member of (4.1) satisfies $Q \equiv 0 \pmod{p^2}$; more precisely, we see that

(4.2)
$$Q \equiv p^2 \sum_{1}^{(p-3)/2} \frac{1}{4t} B_{2t} B_{p-1-2t} \pmod{p^3},$$

to which we return presently. Thus, it is clear that (4.1) reduces to

$$B_p^{(p)} \equiv \frac{p}{2} (pB_{p-1} - (p-1)!) \pmod{p^3}.$$

But by (2.8) this implies

(4.3)
$$B_p^{(p)} \equiv \frac{1}{2} p^2 \pmod{p^3}.$$

Since $B_3^{(3)} = -9/4 \equiv 9/2 \pmod{27}$, (4.3) holds for $p \geq 3$.

To determine the residue of $B_p^{(p)} \pmod{p^4}$ we make use of [2, p. 366, formula (10)],

$$(4.4) \qquad \sum_{t=1}^{(p-3)/2} \frac{1}{2t} B_{2t} B_{p-1-2t} \equiv \frac{1}{p}(W_p - K_p) - W_p \qquad (\text{mod } p),$$

where W_p, K_p are defined by

$$(p-1)! + 1 = pW_p, \qquad a^{p-1} - 1 = pk(a) \qquad (p \nmid a),$$

$$K_p = k(1) + k(2) + \cdots + k(p-1).$$

Then, by (4.1) and (4.2),

$$B_p^{(p)} \equiv \frac{1}{2} p \left\{ pB_{p-1} + 1 - pK_p - p^2 W_p \right\} \qquad (\text{mod } p^4);$$

since $W_p \equiv K_p \pmod{p}$, this may also be put in the form

$$(4.5) \qquad B_p^{(p)} \equiv \frac{1}{2} p^2 \left\{ B_{p-1} + \frac{1}{p} - (p+1) K_p \right\} \qquad (\text{mod } p^4).$$

That (4.5) includes (4.3) is easily verified.

5. **Proof of** (1.4), (1.5), (1.6). In the remainder of the paper let $p > 3$. In (2.13) take $k = p$, $m = p + 2$; then

$$B_{p+2}^{(p+1)} = \left(1 - \frac{p+2}{p}\right) B_{p+2}^{(p)} - (p+2) B_{p+1}^{(p)}.$$

Therefore, by (1.2),

$$(5.1) \qquad \frac{2}{p} B_{p+2}^{(p)} + (p+2) B_{p+1}^{(p)} \equiv 0 \qquad (\text{mod } p^3).$$

Now take $k = p - 1$, $m = p + 2$ in (2.1), so that

$$B_{p+2}^{(p)} = p \binom{p+2}{p} \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} B_s^{(p)} \frac{B_{p+2-s}}{p+2-s}.$$

Clearly only odd values of s need be considered; we get, using (2.4),

$$B_{p+2}^{(p)} = (p+2)\binom{p+1}{p-1}\sum_{t=0}^{(p-3)/2} C_{2t+1}^{(p)} \frac{B_{p+1-2t}}{p+1-2t}$$

$$\equiv (p+2)\binom{p+1}{p-1}\left(-\frac{1}{2}\,p\,\frac{B_{p+1}}{p+1}+\frac{1}{8}\,p^2\,\frac{B_{p-1}}{p-1}\right) \qquad (\mathrm{mod}\ p^3)$$

by (2.3) and (2.7); next, by (2.9) and (2.10), we get

$$B_{p+2}^{(p)} \equiv \frac{1}{12}\,p^2 \qquad (\mathrm{mod}\ p^3). \tag{5.2}$$

In view of (5.1) we have also

$$B_{p+1}^{(p)} \equiv -\frac{1}{12}\,p \qquad (\mathrm{mod}\ p^2). \tag{5.3}$$

However, (5.2) and (5.3) do not imply (5.1) but only the weaker result with modulus p^2.

To improve these results we follow the method of §3. Thus

$$B_{p+1}^{(p)} = p(p+1)\sum_{s=0}^{(p-1)/2} C_{2s}^{(p)} \frac{B_{p+1-2s}}{p+1-2s} = p(p+1)A, \tag{5.4}$$

and

$$A = \frac{B_{p+1}}{p+1} + C_2^{(p)}\frac{B_{p-1}}{p-1} + \sum_{t=2}^{(p-3)/2} C_{2t}^{(p)}\frac{B_{p+1-2t}}{p+1-2t} + \frac{1}{12}\,(p-1)!\,.$$

But, by (3.61),

$$\sum_{t=2}^{(p-3)/2} C_{2t}^{(p)}\frac{B_{p+1-2t}}{p+1-2t} \equiv -p\sum_{2}^{(p-3)/2}\frac{B_{2t}}{2t}\,\frac{B_{p+1-2t}}{p+1-2t}$$

$$\equiv -\frac{p+2}{p+1}\,B_{p+1} - \frac{1}{6}\,pB_{p-1} \qquad (\mathrm{mod}\ p^2),$$

so that after some simplification we get

$$A \equiv -\frac{B_{p+1}}{p+1} + \frac{1}{8}\,p \qquad (\mathrm{mod}\ p^2),$$

and therefore, by (5.4) and (2.10),

$$B_{p+1}^{(p)} \equiv -p\,\frac{B_{p+1}}{p+1} + \frac{1}{24}\,p^2 \qquad (\mathrm{mod}\ p^3). \tag{5.5}$$

In view of (5.1) this implies

$$B_{p+2}^{(p)} \equiv p^2\,\frac{B_{p+1}}{p+1} \qquad (\mathrm{mod}\ p^4). \tag{5.6}$$

That (5.5) and (5.6) include (5.3) and (5.2) is evident; also (5.5) and (5.6) imply (1.2).

We remark also that using (2.13), (5.5), and (1.3) we get

$$B_{p+1}^{(p+1)} \equiv \frac{B_{p+1}}{p+1} - \frac{1}{24}\,p \qquad (\mathrm{mod}\ p^2). \tag{5.7}$$

6. **Discussion of** $B_m^{(p)}$. Let first $m > p$ be odd, so that (2.1) implies

$$B_m^{(p)} = p\binom{m}{p}\sum_{t=0}^{(p-3)/2} C_{2t+1}^{(p)}\,\frac{B_{m-1-2t}}{m-1-2t}. \tag{6.1}$$

Now let $m \equiv a\ (\mathrm{mod}\ p)$, $0 \leq a < p$; $m \equiv b\ (\mathrm{mod}\ p-1)$, $0 \leq b < p-1$. Also, let $p^r \mid m-a$, $p^{r+1} \nmid m-a$, so that the binomial coefficient $\binom{m}{p}$ is divisible by exactly p^{r-1}. Clearly b is odd. Now by a well-known theorem [2, p. 252], if $a \neq b$, the quotient $B_{m-a}/(m-a)$ is integral (mod p). Thus, by (2.7), the right member of (6.1), except for the terms corresponding to $t = 0$, $(b-1)/2$, is a multiple of p^{r+2}. As for the exceptional terms

$$u_1 = p\binom{m}{p} C_1^{(p)}\,\frac{B_{m-1}}{m-1}, \qquad u_b = p\binom{m}{p} C_b^{(p)}\,\frac{B_{m-b}}{m-b}, \tag{6.2}$$

there are several possibilities.

(i) Suppose $b = 1$, so that the two terms in (6.2) coincide. Then if $a \neq 1$, we see that the term in question is exactly divisible by p^r. On the other hand if $a = 1$, the term is integral (mod p) but not divisible by p.

(ii) If $b \neq 1$, u_1 and u_b in (6.2) are distinct. There are several cases to consider. If $a = b$, then u_1 is divisible by p^{r+1}, while u_b is divisible by exactly p^{r+1}. Thus, in this sub-case $B_m^{(p)} = 0\ (\mathrm{mod}\ p^{r+1})$; for $m = p+2$ this is less precise than (5.2).

In the next place, let m be even and define a, b, r as above so that b is now even.. Then we have

$$(6.3)\qquad B_m^{(p)} = p\binom{m}{p} \sum_{t=0}^{(p-1)/2} C_{2t}^{(p)} \frac{B_{m-2t}}{m-2t} = \sum_{t=0}^{(p-1)/2} u_{2t}\ .$$

Then by (2.6) the right member, except for the terms u_0, u_b, u_{p-1}, is a multiple of p^{r+1}. We consider a number of cases.

(iii) If $b = 0$, there are only two distinct terms u_0, u_{p-1}. If $a = 0$, we find that pu_0 is integral (mod p); indeed $pu_0 \equiv -1 \pmod p$ by the Staudt-Clausen theorem (2.11). On the other hand, u_{p-1} is divisible by p^{r-1}; indeed $u_{p-1} \equiv m/(m-p+1) \pmod{p^r}$. If $a = p-1$, then $u_0 \equiv (m-p+1)/m \pmod{p^r}$ while $pu_{p-1} \equiv 1 \pmod p$. If $a \neq 0$ or $p-1$ then it can be verified that $u_0 + u_{p-1}$ is divisible by p^r.

(iv) If $b \neq 0$, then all three terms u_0, u_b, u_{p-1} are distinct. By means of Kummer's congruence (2.10) we find that $u_0 + u_{p-1} \equiv 0\ (p^{r+1})$; in other words;

$$(6.4)\qquad B_m^{(p)} \equiv u_b \pmod{p^{r+1}} \qquad (b \neq 0).$$

As for u_b, there are several possibilities. If $a = b$, it is easily seen that u_b is integral (mod p); moreover, by (2.6), $u_b \equiv 0 \pmod p$ if and only if $B_b \equiv 0 \pmod p$. If $a \neq b$, then u_b is divisible by p^r at least; indeed using (6.4) we get

$$(6.5)\qquad B_m^{(p)} \equiv B_b \frac{m-a}{b(m-b)} \pmod{p^{r+1}} \qquad (a \neq b,\ b \neq 0).$$

This result evidently includes (5.3) but not (5.5).

We remark that $B_m^{(p)}$ is integral (mod p) in cases (i), (ii), (iv). In case (iii), however, if $a = 0$ or $p-1$, then $B_m^{(p)}$ is no longer integral, but $pB_m^{(p)}$ is integral; indeed it is easily verified that

$$pB_m^{(p)} \equiv \begin{cases} -1 \pmod p & (a = 0), \\ +1 \pmod p & (a = p-1). \end{cases}$$

7. **Some special cases.** Clearly $m = p^r$, $r > 1$, falls under (i) above with $a = 0$, $b = 1$. Thus,

$$(7.1)\qquad B_{p^r}^{(p)} \equiv -\frac{1}{2} p^{r-1}(p-1) B_{p^r-1} \pmod{p^{r+2}},$$

and in particular,

$$B_{p^r}^{(p)} \equiv -\frac{1}{2} p^r \qquad (\mathrm{mod}\ p^{r+1}).$$

For $m \equiv 1 \pmod{p^r(p-1)}$, we have $a = b = 1$ which also falls under (i); we now have

$$B_m^{(p)} = \frac{1}{2}(p-1)pB_{m-1} \qquad (\mathrm{mod}\ p^{r+2}). \tag{7.3}$$

For $m = cp^r$, where c is odd, $p \nmid c$, we have $a = 0$, $c \equiv b \pmod{p-1}$, which evidently falls under (i) or (ii). Thus, we get ($r \geq 1$)

$$B_{cp^r}^{(p)} = \frac{1}{2} cp^{r+1} \frac{p-1}{cp^r - 1} B_{cp^r-1} \qquad (\mathrm{mod}\ p^{r+2}) \tag{7.4}$$

for $c \equiv 1 \pmod{p-1}$;

$$B_{cp^r}^{(p)} \equiv -\frac{1}{2} cp^{r+1} \left(B_{cp^r-b} + \frac{1}{b-1} \right) \qquad (\mathrm{mod}\ p^{r+2}) \tag{7.5}$$

for $c \equiv b \pmod{p-1}$, $b \neq 1$.

Similarly, for $m = cp^r$, c even, $p \nmid c$, we have $a = 0$, $c \equiv b \pmod{p-1}$, which falls under (iii) or (iv). We consider only the case $p - 1 \nmid c$; that is, $b \neq 0$. Then, by (6.5), we have

$$B_{cp^r}^{(p)} \equiv -\frac{c}{b^2} p^r B_b \qquad (\mathrm{mod}\ p^{r+1}). \tag{7.6}$$

Again for $m = cp^r + a$, c odd, a even, we find

$$B_m^{(p)} \equiv \frac{1}{2} cp^{r+1} (p-1) \frac{B_{m-1}}{m-1} \qquad (\mathrm{mod}\ p^{r+2}) \tag{7.7}$$

for $b = 1$, while

$$B_m^{(p)} \equiv -\frac{1}{2} cp^{r+1} \left(\frac{B_{m-1}}{a-1} - \frac{b}{(b-1)(b-a)} \right) \qquad (\mathrm{mod}\ p^{r+2}) \tag{7.8}$$

for $b \neq 1$; in these two formulas we have $0 < a < p-1$, $b \equiv c + a \pmod{p-1}$. For c even, a odd, (7.7) holds; but (7.8) requires modification. For c and a both odd or both even, there are several cases; in particular, by (6.5) we have

$$B_m^{(p)} \equiv \frac{cp}{b(a-b)} \pmod{p^{r+1}} \tag{7.9}$$

for $a \neq b$, $b \neq 0$.

For $p = 2, 3$ it follows at once from (2.1) that

$$B_m^{(2)} = -m(m-1)\left(\frac{B_m}{m} + \frac{B_{m-1}}{m-1}\right),$$

$$B_m^{(3)} = \frac{1}{2}m(m-1)(m-2)\left(\frac{B_m}{m} + 3\frac{B_{m-1}}{m-1} + 2\frac{B_{m-2}}{m-2}\right),$$

by means of which numerous special formulas can easily be obtained, for example,

$$B_m^{(2)} = \begin{cases} -(m-1)B_m & (m \text{ even} > 2), \\ -mB_{m-1} & (m \text{ odd}), \end{cases}$$

$$B_m^{(3)} = \frac{3}{2}m(m-2)B_{m-1} \qquad (m \text{ odd} > 1).$$

REFERENCES

1. J. W. L. Glaisher, *On the residues of the sums of products of the first $p-1$ numbers, and their powers, to modulus p^2 or p^3*, Quarterly Journal of Pure and Applied Mathematics, **31** (1900), 321-353.

2. Niels Nielsen, *Traité élémentaire des nombres de Bernoulli*, Paris, (1923).

3. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, (1924).

4. S. Wachs, *Sur une propriété arithmétique des nombres de Cauchy*, Bull. Sci. Math. (2) **71** (1947), 219-232.

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