ON A PAPER OF NIVEN AND ZUCKERMAN

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1. **Introduction.** Let 'digit' mean an integer in the range $0 \leq a < 10$. For digits $a_1, a_2, \cdots, a_r$; $b_1, b_2, \cdots, b_s$ $(s \geq r)$ and integer m, denote by

$$R_m(a_1, \cdots, a_r, b_1, \cdots, b_s)$$

the number of solutions of

$$b_n = a_1, b_{n+1} = a_2, \cdots, b_{n+r-1} = a_r \qquad (0 < n < n+r \leq s;\ n \equiv m \bmod r),$$

so that

(1) $$0 \leq R_m(a_1, \cdots, a_r;\ b_1, \cdots, b_s) \leq s - r + 1.$$

Suppose that

$$x_1, x_2, \cdots$$

is an infinite sequence of digits. It has been shown [2] that if

(2) $$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{r} R_m(a_1, \cdots, a_r; x_1, \cdots, x_N) = 10^{-r}$$

for all integers r and digits $a_1, \cdots, a_r$, then

(3) $$\lim_{N \to \infty} \frac{1}{N} R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) = r^{-1}\ 10^{-r}$$

for all integers r, m, and digits $a_1, \cdots, a_r$. A possibly simpler proof is as follows.

2. **Proof.** Let $\epsilon > 0$ and digits $a_1, \cdots, a_r$ be given. The simple argument of Hardy-Wright [1] shows that if the integer s is fixed large enough, then

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$$\text{(4)} \qquad \operatorname*{Max}_{\mu} \left| R_\mu(a_1, \cdots, a_r;\ b_1, \cdots, b_s) - \frac{s-r+1}{r\,10^r} \right| < \epsilon(s-r+1)$$

except for at most $\epsilon 10^s$ sets of digits $b_1, \cdots, b_s$. ('Exceptional' sets.) Thus, by (2) with $b_1, \cdots, b_s$ for $a_1, \cdots, a_r$, the number of exceptional sets

$$\text{(5)} \qquad x_t, x_{t+1}, \cdots, x_{t+s-1} \qquad (1 \le t \le N-s+1)$$

is at most $2\epsilon N$ for all large enough N.

On the other hand,

$$\text{(6)} \qquad (s-r+1)\, R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N)$$

differs from

$$\text{(7)} \qquad \sum_{t=1}^{N-s+1} R_{m-t+1}(a_1, \cdots, a_r;\ x_t, \cdots, x_{t+s-1})$$

by at most $2s^2$, since each solution of

$$a_1 = x_n,\ a_2 = x_{n+1}, \cdots,\ a_r = x_{n+r-1} \quad (s \le n \le N-s;\ n \equiv m \bmod r)$$

contributes exactly $s-r+1$ both to (6) and to (7). Hence, using the estimate (3) for the at most $2\epsilon N$ exceptional sets (5), and the estimate (4) for the others, we have

$$\left| R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) - \frac{N-s+1}{r\,10^r} \right|$$
$$\le \frac{2s^2}{s-r+1} + \epsilon(N-s+1) + 2\epsilon N,$$

and so

$$\limsup_N \left| \frac{1}{N} R_m(a_1, \cdots, a_r;\ x_1, \cdots, x_N) - r^{-1}\, 10^{-r} \right| \le 3\epsilon.$$

Since ϵ is arbitrarily small, this proves (3) as required.

References

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, First Edition, Oxford, 1938, §9.13.

2. I. Niven and H. S. Zuckerman, *On the definition of normal numbers*, Pacific J. Math. 1 (1951), 103-110.

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