STRUCTURED THEOREMS FOR RELATIVELY COMPLEMENTED LATTICES

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Introduction. In a previous paper [3] a study was made of the projectivities between the points of a simple relatively complemented lattice of finite dimension. It was shown that for a given dimension there is an upper bound for the number of transposes required to establish the projectivities between the points. The examples given in which this upper bound is attained have a particularly simple structure — they are closely related to a direct union. We shall prove here some general structure theorems for relatively complemented lattices and then apply these to the case of maximal projectivities.

The notation will be that of [3]. The lattice L to which we refer is always relatively complemented.

1. Structure Theorems. Our arguments depend heavily upon the simplicity or indecomposability [2] of L, and it is convenient to have the following characterization of a direct union:

Theorem 1.1. *If L has dimension n, and a, b are two elements of L, then $L \cong a/z \vee b/z$ if and only if*

(1) $\rho(a) + \rho(b) \leq n$, *and*

(2) $p \subseteq a$ *if and only if* $p \not\subseteq b$ *for all points* $p \in L$.

Proof. Certainly if $L \cong a/z \vee b/z$, conditions (1) and (2) will hold. Suppose (1) and (2) hold in L. We shall proceed by induction on n. The theorem is true when $n = 1, 2$. Suppose it is true for all lattices of dimension less than n, but $L \not\cong a/z \vee b/z$.

It is clear that

$$x = (a \cap x) \cup (b \cap x)$$

for all $x \in L$. Consider the mapping

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$$x \longrightarrow \sigma x = (a \cap x,\ b \cap x) \in a/z \vee b/z.$$

Now $x \supseteq y$ if and only if

$$a \cap x \supseteq a \cap y \text{ and } b \cap x \supseteq b \cap y;$$

and the latter occurs if and only if $\sigma x \supseteq \sigma y$. Hence L is isomorphic, as a partially ordered set, to a subset of $a/z \vee b/z$, where

$$\sigma u = (a, b),\ \sigma z = (z, z).$$

These remarks show that if any two elements a, b of L satisfy (2), we must have

$$\rho(a) + \rho(b) \geq n.$$

If $L \not\cong a/z \vee b/z$, there are points $p \subseteq a$ and $q \subseteq b$ such that $p/z\ P_2\ q/z$. Hence there is a maximal element m such that $m \not\supseteq p$, $m \not\supseteq q$. Then s_1 and s_2 exist with

$$a > s_1 \supseteq m \cap a,\ b > s_2 \supseteq m \cap b.$$

Furthermore,

$$a \cup s_2 = b \cup s_1 = u.$$

Let $u = x_0 > x_1 > \cdots > x_{n-1} > x_n = z$ be a complete chain in L. This chain maps onto

$$\sigma u = (a, b) > \sigma x_1 > \cdots > \sigma x_{n-1} > \sigma x_n = (z, z).$$

Either (i) $\sigma x_1 = (a, t_2)$, where $b > t_2$, or (ii) $\sigma x_1 = (t_1, b)$, where $a > t_1$. Suppose the former is true. The points of x_1 are in either a or t_2, but not both. Then a and t_2 satisfy (2) in x_1/z, and since

$$\rho(x_1) = n - 1,$$

we have

$$\rho(a) + \rho(t_2) \geq n - 1.$$

But

$$\rho(a) + \rho(b) \leq n, \text{ so } \rho(t_2) = \rho(b) - 1.$$

Then by the induction hypothesis, $x_1/z \cong a/z \vee t_2/z$. This gives the existence of a chain from s_1 through a to u of length $1 + \rho(b) - 1 + 1$, or $\rho(b) + 1$. By Lemma 3.6 of [3], there is a chain from b to z of length at least $\rho(b) + 1$, which is a contradiction. A similar contradiction arises if $\sigma x_1 = (t_1, b)$. Therefore $L \cong a/z \vee b/z$, and thus the theorem is proved.

The following theorem gives more information about the quotient lattices a_p^k/z introduced in Lemma 3.5 of [3].

THEOREM 1.2. *Let L be simple of dimension $n > 1$. If p is any point and k is a nonnegative integer such that $k < [(n+1)/2]$, then a_p^k/z has dimension at least $2k+1$.*

Proof. The theorem is true when $k = 0$. Suppose it is true for all k less than the one in which we are interested. Then a_p^{k-1} has dimension at least $2k - 1$, and $a_p^k \supseteq a_p^{k-1}$. If $a_p^k = u$, we are through, so assume $u \supset a_p^k$. Then there is a point $s \in L$ with $s \nsubseteq a_p^k$, but $s/z\ P_2\ t/z$ for some $t \in C_p^k$. Hence there is a maximal element m *such* that $m \nsupseteq s$, $m \nsupseteq a_p^k$. Since $s \in C_p^k$, we have $m \supseteq a_p^{k-1}$. Therefore $a_p^k \supset a_p^{k-1}$, and the dimension of a_p^k/z is at least $2k$. Suppose $\dim(a_p^k/z) = 2k$. Let b be the join of all points of L which are not in a_p^k. All of these points are in $x_p^k = \cap M_p^k$, where

$$M_p^k \equiv \{m \in L \mid u > m \nsupseteq a_p^{k-1}\}.$$

(See proof of Lemma 3.5 of [3].) Hence $x_p^k \supseteq b$ and $b \cap a_p^k = z$. The latter follows from the assumption $\dim(a_p^k/z) = 2k$, since, by Theorem 3.1 of [3] for any point q we would have $q \subseteq a_p^k$ if and only if $q \in C_p^k$. On the other hand, it is shown, in the proof of Lemma 3.5 of [3], that $q \in C_p^k$ if and only if $q \nsubseteq x_p^k$.

Since L is simple, there exists an x such that

$$u > x,\ x \nsupseteq a_p^k,\ x \nsupseteq b.$$

But $x \nsupseteq b$ implies $x \supseteq a_p^{k-1}$. Then

$$x = a_p^{k-1} \cup (b \cap x), \text{ and } u = b \cup x = a_p^{k-1} \cup b.$$

Hence if $u > m$ we have $m \supseteq a_p^{k-1}$, if and only if $m \nsupseteq b$. Therefore a_p^k, a_p^{k-1}, and b satisfy the conditions of Lemma 3.6 of [3], and there exists a chain of length at least $2k$ from u to b. Then

$$\rho(b) \leq n - 2k, \text{ so } \rho(a_p^k) + \rho(b) \leq n.$$

But by Theorem 1.1 we would have $L = a_p^k/z \vee b/z$, contrary to the simplicity of L. Therefore $\rho(a_p^k) \geq 2k + 1$ for all $k < [(n+1)/2]$.

Let $\mathfrak{P}$ denote the partially ordered subset of L consisting of u, the maximal elements, the points, and z. Let $\mathfrak{P}_\nu$ be the normal completion of $\mathfrak{P}$. Consider the mapping $A \longrightarrow \cup A$ from $\mathfrak{P}_\nu$ into L. (A is a normally closed subset of L.) If $A \supseteq B$, then $\cup A \supseteq \cup B$. Suppose $\cup A \supseteq \cup B$; then $x \in A^*$ implies $x \supseteq a$, all $a \in A$ implies $x \supseteq \cup A$, so $x \supseteq \cup B$, and hence $x \supseteq b$ all $b \in B$ and $x \in B^*$; therefore $A^* \subseteq B^*$, so $(A^*)_* \supseteq (B^*)_*$, or $A \supseteq B$. Thus the mapping is order preserving both ways.

Suppose $a \in L$, $a \neq u$, $a \neq z$. Set

$$P(a) \equiv \{p \in \mathfrak{P} \mid a \supseteq p > z\},$$

$$M(a) \equiv \{m \in \mathfrak{P} \mid u > m \supseteq a\}.$$

Now $x \supseteq p$, all $p \in P(a)$, if and only if $x \supseteq a$, so $M(a) \subseteq (P(a))^*$. Also $P(a) \subseteq (P(a)^*)_*$. Suppose $y \in (P(a)^*)_*$; then $y \subseteq x$, all $x \in P(a)^*$ implies $y \subseteq m$, all $m \in M(a)$ implies $y \subseteq a$. Suppose $a' \supseteq y$, all $y \in (P(a)^*)_*$; then $a' \supseteq p$, all $p \in P(a)$ implies $a' \supseteq a$, so $a = \cup(P(a)^*)_*$. If $a = u$, then $a = \cup(u)$; if $a = z$ then $a = \cup(z)$. (Here (x) denotes the principal ideal generated by x.) Hence each $a \in L$ has an inverse image under the above mapping, and $\mathfrak{P}_\nu \cong L$; see [2]. This proves the following:

THEOREM 1.3. *The structure of L is completely determined by the structure of $\mathfrak{P}$.*

REMARK. From the nature of the proof it is seen that the above theorem will be true for any lattice each of whose elements is a join of points and the meet of maximal elements.

2. Lattices with maximal projectivities. In this section we shall study simple lattices of odd dimension in which there occurs a maximal projectivity. We shall show that these lattices are quite close to a direct union in the sense that their structure can be completely described in terms of sublattices. Throughout this section L will be a simple lattice of dimension $2n + 1$, and p, q are two points in L such that $p/z \;P\; q/z$ requires $2n + 2$ transposes. Then we have:

THEOREM 2.1. *If $k \leq n$, the following statements are true:*

(1) $\rho(a_p^k) = 2k + 1$, $\rho(a_q^{n-k}) = 2n - 2k + 1$;

(2) $x_p^k = a_q^{n-k}$, $x_q^{n-k} = a_p^k$;

(3) *a_p^k/z has a maximal projectivity if and only if a_q^{n-k}/z has a maximal projectivity;*

(4) *if* a_p^k/z *has a maximal projectivity then* $a_p^k \cap a_q^{n-k} = r > z$, *otherwise* $a_p^k \cap a_q^{n-k} = z$.

Proof. Note that $s \in C_q^{n-k}$ implies $s \notin C_p^k$ implies $s \subseteq x_p^k$ implies $a_q^{n-k} \subseteq x_p^k$. Suppose there is a maximal element m such that $m \supseteq a_p^{k-1}$, $m \supseteq x_p^k$. If m/z is simple, we contradict the assumption of a maximal projectivity between p/z and q/z, since $\rho(m) \leq 2n$. Write

$$m/z = L_1 \vee L_2 \vee \cdots \vee L_\nu,$$

where the L_i are simple nontrivial quotient lattices, and $\nu > 1$. Now a_q^{n-k}/z and a_p^{k-1}/z are both simple; if they are in the same L_i, we again contradict our maximal projectivity assdumption. Hence they are in different components and we must have

$$\rho(a_p^{k-1}) + \rho(a_q^{n-k}) \leq 2n.$$

By Theorem 1.2,

$$\rho(a_p^{k-1}) \geq 2k - 1; \ \rho(a_q^{n-k}) \geq 2n - 2k + 1.$$

Therefore

$$\rho(a_q^{n-k}) = 2n - 2k + 1.$$

The elements a_p^{k-1} and a_q^{n-k} are in different L_i, so

$$\rho(a_p^{k-1} \cup a_q^{n-k}) = \rho(a_p^{k-1}) + \rho(a_q^{n-k}) \geq 2n,$$

and hence

$$m = a_p^{k-1} \cup a_q^{n-k} \ \text{ or } \ m/z = a_p^{k-1}/z \vee a_q^{n-k}/z.$$

Now let $s > z$, $s \subseteq x_p^k$. Then $s \notin C_p^k$, so $s \not\subseteq a_p^{k-1}$. But $m \supseteq x_p^k$, so $m \supseteq s$, and therefore $s \subseteq a_q^{n-k}$. This shows that $x_p^k \subseteq a_q^{n-k}$, and hence $x_p^k = a_q^{n-k}$. Thus we have shown that if $a_p^{k-1} \cup x_p^k \neq u$, then $x_p^k = a_q^{n-k}$ and $\rho(a_q^{n-k}) = 2n - 2k + 1$.

Suppose $a_p^{k-1} \cup x_p^k = u$. Then for each maximal element m, $m \supseteq a_p^{k-1}$ if and only if $m \not\supseteq x_p^k$. We have $\rho(a_p^{k-1}) \geq 2k - 1$, so dim $(u/a_p^{k-1}) \leq 2n + 2 - 2k$. Since L is simple, dim $(u/x_p^k) \geq 2k$, by Theorem 1.1. Hence $\rho(x_p^k) \leq 2n - 2k + 1$. But $x_p^k \supseteq a_q^{n-k}$, and $\rho(a_q^{n-k}) \geq 2n - 2k + 1$. Hence, in all cases, $x_p^k = a_q^{n-k}$ and $\rho(a_q^{n-k}) = 2n - 2k + 1$. By a similar argument, $x_q^{n-k} = a_p^k$ and $\rho(a_p^k) = 2k + 1$. This demonstrates (1) and (2).

Suppose $r > z$, $r \subseteq a_p^k$ such that $r/z\ P\ p/z$ requires $2k+2$ transposes. Now $r \nsubseteq C_p^k$ implies $r \subseteq x_p^k = a_q^{n-k}$. Furthermore, $r \subseteq a_p^k = x_q^{n-k}$ implies $r \nsubseteq C_q^{n-k}$ implies that $r/z\ P\ q/z$ requires $2n-2k+2$ transposes. The argument is symmetric in p and q, and this proves (3).

Suppose $s > z$ and $s/z\ P\ p/z$ requires $2n+2$ transposes. Then $x_p^n = a_s^0 = s = a_q^0 = q$, so there is at most one point q such that $p/z\ P\ q/z$ requires $2n+2$ transposes. This shows that the r in the preceding paragraph, if it exists, is unique, and we have (4).

We are now in a position to characterize the maximal elements of L in terms of the structure of a_p^k/z and a_q^{n-k}/z. When we know these maximal elements, we will know the structure of L, by Theorem 1.3. First we prove two useful lemmas.

LEMMA 2.1. *There is a chain of length $2n+1$ through a_p^k.*

Suppose $a_p^k \cup a_q^{n-k-1} = u$. Then the maximal elements of L are in two disjoint classes—those containing a_p^k and those containing a_q^{n-k-1}; and by Theorem 1.1,

$$\dim(u/a_p^k) + \dim(u/a_q^{n-k-1}) > 2n+1.$$

But

$$\dim(u/a_p^k) \leq 2n + 1 - (2k+1);$$

$$\dim(u/a_q^{n-k-1}) \leq 2n + 1 - (2n-2k-1).$$

Hence $\dim(u/a_p^k) = 2n - 2k$.

Suppose $u > m \supseteq a_p^k \cup a_q^{n-k-1}$. Now m/z is not simple, since $\rho(m) \leq 2n$ and $m \supseteq p$, $m \supseteq q$. Suppose

$$m/z = L_1 \vee L_2 \vee \cdots \vee L_\nu,$$

where $\nu > 1$. Then a_p^k/z and a_q^{n-k-1}/z are in different components and again there is a chain from a_p^k to u of length at least $2n-2k$ since $\rho(a_q^{n-k-1}) = 2n - 2k - 1$. This proves the lemma.

LEMMA 2.2. *If $s > z$, $a \nsupseteq s$, $b \nsupseteq s$, but $a \cup b \supseteq s$, then there are points $s_1 \subseteq a$, $s_2 \subseteq b$ such that $s_1/z\ P_2\ s/z$ and $s_2/z\ P_2\ s/z$.*

Let $s \cup b > x \supseteq b$, and let x' be a relative complement of $s \cup b$ in $a \cup b/x$ such that $a \cup b > x'$. Then $x' \nsupseteq a$, $x' \nsupseteq s$; hence $x' \nsupseteq s_1$, for some point $s_1 \subseteq a$. Therefore $s/z\ T\ a \cup b/x'\ T\ s_1/z$. Similarly we can show the existence of s_2.

proving the lemma.

LEMMA 2.3. *The following relation holds*: $\dim(a^k/a_p^{k-1}) = 2$.

For since L is simple there is a maximal m_0 such that $m_0 \not\supseteq a_p^k$, $m_0 \not\supseteq a_q^{n-k}$. Then $m_0 \supseteq a_p^{k-1}$, $m_0 \supseteq a_q^{n-k-1}$. Assume $a_p^k > a_p^{k-1}$. Then $m_0 \cap a_p^k = a_p^{k-1}$. Set $w = a_q^{n-k} \cap m_0$. Then y exists such that $a_q^{n-k} > y \supseteq w$. Since $m_0 = a_p^{k-1} \cup w$, we have $u = a_p^{k-1} \cup y = w \cup a_p^k$. Since there is a chain of length $2k$ from a_q^{n-k} to u, there exists a maximal m such that $m \not\supseteq a_q^{n-k}$ and such that there exists a chain of length at least $2k$ from m to y. Now $m \not\supseteq a_p^k$ since $a_p^k \cup y = u$. But $m \supseteq a_p^{k-1}$ and $m/z = a_p^{k-1}/z \vee y/z$ in contradiction with the length of the chain from y to m. Hence $a^k \not> a_p^{k-1}$, and we must have $\dim(a_p^k/a_p^{k-1}) = 2$.

COROLLARY. *The following relation holds*:

$$\dim(a_q^{n-k}/a_q^{n-k-1}) = 2.$$

This follows by symmetry.

3. Maximal elements when $a_p^k \cap a_q^{n-k} \neq z$. The following theorem gives the possibilities for maximal elements when a_p^k/z and a_q^{n-k}/z each have a maximal projectivity. We assume throughout that $1 \leq k \leq n-1$.

THEOREM 3.1. *Let $a_p^k \cap a_q^{n-k} = r > z$, and let $u > m$. If $m \supseteq r$, either*

(1) $m \supseteq a_p^k$ *and* $a_q^{n-k} > m \cap a_q^{n-k}$,

or

(2) $a_p^k > m \cap a_p^k$ *and* $m \supseteq a_q^{n-k}$.

If $m \not\supseteq r$, then $a_p^k > a_p^k \cap m$ and $a_q^{n-k} > a_q^{n-k} \cap m$.

Proof. Let $u > m \supseteq r$, and suppose $m \not\supseteq a_p^k$, $m \not\supseteq a_q^{n-k}$. Then $m \supseteq a_p^{k-1}$, and $m \supseteq a_q^{n-k-1}$, for otherwise we would not have a maximal projectivity in L. For the same reason, we have $r \not\subseteq a_p^{k-1}$, $r \not\subseteq a_q^{n-k-1}$. Then since

$$\rho(a_p^{k-1}) = 2k-1, \qquad \rho(a_p^k) = 2k+1,$$

$$\rho(a_q^{n-k-1}) = 2n-2k-1, \qquad \rho(a_q^{n-k}) = 2n-2k+1,$$

we must have

$$a_p^k > m \cap a_p^k = r \cup a_p^{k-1} \text{ and } a_q^{n-k} > m \cap a_q^{n-k} = r \cup a_q^{n-k-1}.$$

Hence

$$m = (r \cup a_p^{k-1}) \cup (r \cup a_q^{n-k-1}) \text{ and } u = a_p^k \cup m = a_p^k \cup a_q^{n-k-1}.$$

Similarly, $u = a_q^{n-k} \cup a_p^{k-1}$.

By Lemma 2.1, there is a chain from a_p^k to u of length $2n-2k$. Since L is relatively complemented, it is easy to see that v exists such that $u > v$, $v \cap a_p^k = r \cup a_p^{k-1}$, and there is a chain from $r \cup a_p^{k-1}$ to v of length at least $2n - 2k$. There is an $s \in C_p^k$ such that $s \nsubseteq a_p^{k-1} \cup r$. Hence $s \nsubseteq v$, and this implies $v \supseteq a_q^{n-k-1}$. Therefore $v \supseteq m$ and $v = m$. Then by Theorem 1.1,

$$m/z = a_p^{k-1} \cup r/z \vee a_q^{n-k-1}/z;$$

but this contradicts the existence of a chain from $a_p^{k-1} \cup r$ to m of length $2n - 2k$. Hence we must have either $m \supseteq a_p^k$, or $m \supseteq a_q^{n-k}$.

Suppose $m \supseteq a_q^{n-k}$, but $a_p^k > x \supset m \cap a_p^k$. Let y be a relative complement of x in $a_p^k/a_p^k \cap m$. Then $y \supset a_p^k \cap m$, since $a_p^k > x$. Hence $m \nsupseteq x$, $m \nsupseteq y$, so

$$x \cup a_q^{n-k} = y \cup a_q^{n-k} = u.$$

Since

$$\rho(a_p^k) = \rho(a_p^{k-1}) + 2 \text{ and } r \cup a_p^{k-1} \supset a_p^{k-1},$$

it follows that $m \nsupseteq a_p^{k-1}$. Hence either $x \nsupseteq a_p^{k-1}$ or $y \nsupseteq a_p^{k-1}$. Suppose the latter is the case. Then there is an $s \in C_p^{k-1}$ such that $s \nsubseteq y$. But $s \subseteq u = y \cup a_q^{n-k}$. Hence, by Lemma 2.2, $s/z \; P_{2n-2k+2} \; q/z$ and $p/z \; P_{2n} \; q/z$ contrary to our assumption of a maximal projectivity between p/z and q/z. A similar contradiction arises if $x \nsupseteq a_p^{k-1}$. Hence $a_p^k > a_p^k \cap m$. The roles of p and q are symmetric, so if $m \supseteq a_p^k$, then $a_q^{n-k} > m \cap a_q^{n-k}$.

Now let $u > m \nsupseteq r$. Since $m \nsupseteq r$, we have $m \supseteq a_p^{k-1}$ and $m \supseteq a_q^{n-k-1}$. Suppose

$$a_p^k > x > a_p^{k-1} = a_p^k \cap m \text{ and } a_q^{n-k} > y > a_q^{n-k-1} = a_q^{n-k} \cap m.$$

Let x' be a relative complement of x in a_p^k/a_p^{k-1}. Suppose $x' \nsupseteq r$, and let x'' be a relative complement of a_p^k in u/x'. Since $a_p^k > x'$, we can assume $u > x''$. Now $x'' \nsupseteq r$, so $x'' \supseteq a_q^{n-k-1}$. Hence $x'' = m$, contrary to $a_p^{k-1} = m \cap a_p^k$. A similar contradiction arises if $x \nsupseteq r$; and since $a_p^{k-1} \nsupseteq r$, we must have $a_p^k > m \cap a_p^k$. Therefore either

$$a_p^k > m \cap a_p^k \text{ or } a_q^{n-k} > m \cap a_q^{n-k}.$$

Suppose

$$a_p^k > m \cap a_p^k > a_p^{k-1} \text{ but } a_q^{n-k} > y > a_q^{n-k-1} = m \cap a_q^{n-k}.$$

As before, v exists with $u > v$, $v \cap a_p^k = m \cap a_p^k$, and there is a chain from $m \cap a_p^k$ to v of length at least $2n - 2k$. There is a point $s \in C_p^k$ such that $s \nsubseteq m \cap a_p^k$, and hence $s \nsubseteq v$. Therefore $v \supseteq a_q^{n-k-1}$, so $v = m$. But m/z, by Theorem 1.1, is equal to $m \cap a_p^k/z \vee a_q^{n-k-1}/z$ in contradiction with 'the length of the chain from $m \cap a_p^k$ to $v = m$. Hence $a_q^{n-k} > m \cap a_q^{n-k}$; and whenever $u > m \nsupseteq r$, we have

$$a_p^k > m \cap a_p^k, \ a_p^{n-k} > m \cap a_q^{n-k}.$$

The converse of this theorem is not true; however we do have the following result:

THEOREM 3.2. *If $a_p^k > x \supseteq r$, then $u > x \cup a_q^{n-k}$, while if $a_q^{n-k} > y \supseteq r$, then $u > a_p^k \cup y$. If $a_p^k > x \nsupseteq r$ and $a_q^{n-k} > y \nsupseteq r$, then $u > x \cup y$ if and only if for any points $t \subseteq x$, $s \subseteq y$, we have $t \cup s \nsupseteq r$.*

Proof. Let $a_p^k > x \supseteq r$, and let x' be a relative complement of a_p^k in u/x such that $u > x'$. Then by Theorem 3.1 we have $x' \supseteq a_q^{n-k}$ and $x' = x \cup a_q^{n-k}$. A similar argument shows that if $a_q^{n-k} > y \supseteq r$, then $u > a_p^k \cup y$.

Suppose

$$a_p^k > x \nsupseteq r, \ a_q^{n-k} \supseteq y \nsupseteq r.$$

If

$$x \supseteq t > z \text{ and } y \supseteq s > z,$$

such that $s \cup t \supseteq r$, then

$$x \cup y \supseteq r \text{ and } x \cup y = (x \cup r) \cup (y \cup r) = u.$$

Suppose $x \cup y = u$. Since

$$x \nsupseteq r, y \nsupseteq r,$$

it follows that

$$x \supseteq a_p^{k-1}, \ y \supseteq a_q^{n-k-1}.$$

If $x = a_p^{k-1}$ or $y = a_q^{n-k-1}$, Lemma 2.2 tells us that $r \in C_p^k$ or $r \in C_q^{n-k}$. Hence

$$x > a_p^{k-1} \text{ and } y > a_q^{n-k-1}.$$

So points s and t exist such that $x = t \cup a_p^{k-1}$ and $y = s \cup a_q^{n-k-1}$. Therefore

$$a_p^{k-1} \cup t \cup s \cup a_q^{n-k-1} \supseteq r;$$

and applying Lemma 2.2 twice we get $t \cup s \supseteq r$. All that is required to finish the proof of the theorem is to show that if $u > m \supseteq x \cup y$, then $m = x \cup y$. Suppose $m \supseteq r$; then

$$m \supseteq (x \cup r) \cup (y \cup r) = u.$$

So $m \not\supseteq r$. Hence, by Theorem 3.1, $m = x_1 \cup y_1$, where

$$a_p^k > x_1 \not\supseteq r \text{ and } a_q^{n-k} > y_1 \not\supseteq r.$$

But this implies $x = x_1$, $y = y_1$, and $m = x \cup y$.

4. Maximal elements when $a_p^k \cap a_q^{n-k} = z$. Here as before we assume that $1 \leq k \leq n-1$.

THEOREM 4.1. *If $u > m$ then m is one of the following three types*:

(1) $m \supseteq a_p^k$, $a_q^{n-k} > a_q^{n-k} \cap m \not\supseteq a_q^{n-k-1}$, *or dually*;

(2) $m \supseteq a_p^k$, $a_q^{n-k} \cap m = a_q^{n-k-1}$, *or dually*;

(3) $a_p^k > m \cap a_p^k \supseteq a_p^{k-1}$, *and* $a_q^{n-k} > m \cap a_q^{n-k} \supseteq a_q^{n-k-1}$.

Proof. Suppose

$$u > m \supseteq a_p^k,\ m \not\supseteq a_q^{n-k-1}, \text{ but } a_q^{n-k} > x \supset a_q^{n-k} \cap m.$$

Then not all elements of a_q^{n-k}/z covering $m \cap a_q^{n-k}$ will contain a_q^{n-k-1}. On the other hand,

$$m = (m \cap a_q^{n-k}) \cup a_p^k,$$

so for any point

$$s \subseteq a_q^{n-k},\ s \not\subseteq m \cap a_q^{n-k},$$

we have

$$s \cup (m \cap a_q^{n-k}) \cup a_p^k \supseteq a_q^{n-k-1}.$$

Then by Lemma 2.2 we must have

$$s \cup (m \cap a_q^{n-k}) \supseteq a_q^{n-k-1},$$

contrary to the above assertion. Therefore if

$$u > m \supseteq a_p^k \text{ and } m \not\supseteq a_q^{n-k-1},$$

then $a_q^{n-k} > m \cap a_q^{n-k}$.

Now suppose $u > m \supseteq a_p^k$ and $m \supseteq a_q^{n-k-1}$. If m/z is simple, we contradict our maximal projectivity assumption; but arguing as before on the direct split of m/z, we see that

$$m/z = a_p^k/z \vee a_q^{n-k-1}/z,$$

and hence $m \cap a_q^{n-k} = a_q^{n-k-1}$.

Finally suppose $u > m$, but $m \not\supseteq a_p^k$, $m \not\supseteq a_q^{n-k}$. Then $m \supseteq a_p^{k-1}$ and $m \supseteq a_q^{n-k-1}$. Assume $m \cap a_p^k = a_p^{k-1}$, and let $a_p^k > x > a_p^{k-1}$, by Lemma 2.3. Let v be a relative complement of a_p^k in u/x such that $u > v$. Since $v \not\supseteq a_p^k$, we have $v \supseteq a_q^{n-k-1}$. Now $v \neq m$, so $a_q^{n-k} > m \cap a_q^{n-k}$. Then m' exists such that $u > m'$, $m' \not\supseteq a_q^{n-k}$, and there is a chain from $m \cap a_q^{n-k}$ to m' of length at least $2k$. Since $m' \not\supseteq a_q^{n-k}$, it follows that $m' \supseteq a_p^{k-1}$, and hence $m' = m$. But m/z is not simple; a_p^{k-1} and $a_q^{n-k} \cap m$ are in different components. This is contrary to the length of the above chain, since $\rho(a_p^{k-1}) = 2k - 1$. Hence we must have $a_p^k > m \cap a_p^k$, and dually $a_q^{n-k} > m \cap a_q^{n-k}$.

Examples show that it is impossible from the structures of a_p^k/z and a_q^{n-k}/z to tell whether $u > a_p^k \cup a_q^{n-k-1}$ or $u = a_p^k \cup a_q^{n-k-1}$, and dually. However, for the other maximal elements we have:

THEOREM 4.2. *If*

$$a_q^{n-k} > y \not\supseteq a_q^{n-k-1},$$

then $u > y \cup a_p^k$, *and dually. If*

$$a_p^k > x \supseteq a_p^{k-1} \text{ and } a_q^{n-k} > y \supseteq a_q^{n-k-1},$$

then $u > x \cup y$ *if and only if for every pair of points* $s \subseteq x$, $t \subseteq y$ *the lattice* $s \cup t/z$ *is a Boolean algebra.*

Proof. Suppose

$$a_q^{n-k} > y \supseteq a_q^{n-k-1} \text{ and } u = a_p^k \cup y.$$

Then there is a point $t \subseteq a_q^{n-k-1}$ such that $t \not\subseteq y$, $t \not\subseteq a_p^k$, but $t \subseteq a_p^k \cup y$; and using Lemma 2.2 we obtain a contradiction of our maximal projectivity hypothesis. On the other hand, if $u > m \supseteq a_p^k \cup y$, then by Theorem 4.1 we get $m = a_p^k \cup y$.

Let $a_p^k > x \supseteq a_p^{k-1}$ and $a_q^{n-k} > y \supseteq a_q^{n-k-1}$. By Theorem 4.1, either $u = x \cup y$

or $u > x \cup y$. If $u = x \cup y$, there are points $s \subseteq x$, $t \subseteq y$ such that

$$t \cup a_p^{k-1} \cup s \cup a_q^{n-k-1} \supseteq a_q^{n-k}.$$

Then by Lemma 2.2, we have

$$t \cup s \cup a_q^{n-k-1} \supseteq a_q^{n-k};$$

thus $t \cup a_q^{n-k}/z$ is not a direct union, so there is another point $r \subseteq a_p^k$ such that $t \cup s \cup a_q^{n-k-1} \supseteq r$, and hence $t \cup s \supseteq r$. But this tells us that $t \cup s/z$ is not a Boolean algebra.

If $u > x \cup y$, we must have $x \cup y/z = x/z \vee y/z$, and the condition is satisfied.

Here again, then, save for the one exception, the structure of L is determined by the structure of sublattices and the relations between points.

References

1. R. P. Dilworth, *The structure of relatively complemented lattices*, Ann. of Math. (2) 5 (1950), 348-359.

2. R. P. Dilworth and Morgan Ward, *Note on paper by C. E. Rickart*, Bull. Amer. Math. Soc. 55 (1949), 1141.

3. J. E. McLaughlin, *Projectivities in relatively complemented lattices*, Duke Math. J. 18 (1951), 73-84.

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