VOLUME IN TERMS OF CONCURRENT CROSS-SECTIONS

HERBERT BUSEMANN

1. Of the two expressions

$$|M| = \frac{1}{2}\int_0^{2\pi} r^2(\omega)\,d\omega = \frac{1}{2}\int_0^{2\pi}\left(\int_{-r(\omega-\pi/2)}^{r(\omega+\pi/2)} |\rho|\,d\rho\right)d\omega$$

for the area $|M|$ of a plane domain M, given in polar coordinates ρ, ω by the inequalities $0 \leq \rho \leq r(\omega)$, $0 \leq \omega \leq 2\pi$, the first has the well-known extension

$$(1) \qquad |M| = \frac{1}{n}\int_{\Omega_n} r^n(u)\,d\omega_u^n$$

to n dimensions. Here Ω_n is the surface of the unit sphere in the n-dimensional Euclidean space, $d\omega_u^n$ is its area element at the point u, and M is given by $0 \leq \rho \leq r(u)$, $u \in \Omega_n$.

In the second expression, $|\rho|$ may be interpreted as (1-dimensional) volume of the simplex with one vertex at the origin z and the other at a variable point $p = (\rho, \omega \pm \pi/2)$ in the cross-section of M with the line normal to ω. The purpose of the present note is *the proof and the application of the following extension of this second expression to* $n-1$ *sets* $M_1, \cdots, M_{n-1}$ *in* E_n:

$$(2) \quad |M_1|\cdots|M_{n-1}|$$

$$= \frac{(n-1)!}{2}\int_{\Omega_n}\left(\int_{M_1(u)}\cdots\int_{M_{n-1}(u)} T(p_1, \cdots, p_{n-1}, z)\,dV_{p_1}^{n-1}\cdots dV_{p_{n-1}}^{n-1}\right)d\omega_u^n.$$

Here $M_j(u)$ is the cross-section of M_j with the hyperplane $H(u)$ through z normal to the unit vector u, the point p_j varies in $M_j(u)$, the differential $dV_{p_j}^{n-1}$ is the ($(n-1)$-dimensional) volume element of $M_j(u)$ at p_j, and $T(p_1, \cdots, p_{n-1}, z)$ is the volume of the simplex with vertices $p_1, \cdots, p_{n-1}, z$.

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Replacing the sets $M_{n-r+1}, \cdots, M_{n-1}$ by the unit sphere U_n with center z yields expressions for $|M_1| \cdots |M_{n-r}|$ in terms of the volume $T(p_1, \cdots, p_{n-r}, z)$, in particular (1) for $r = n - 1$.

With the notation

$$\kappa_n = |U_n| = \pi^{n/2}/\Gamma\left(\frac{n}{2} + 1\right),$$

Steiner's symmetrization leads from (2) to the following result:

If $M_1, \cdots, M_{n-1}$ are convex bodies in E_n $(n \geq 3)$ with interior points, z is a given point in E_n, and $M_j(u)$ the cross-section of M_j with the plane normal to u and through z, then

$$(3) \quad |M_1| \cdots |M_{n-1}| \geq \frac{1}{n} \frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{\Omega_n} |M_1(u)|^{n/(n-1)} \cdots |M_{n-1}(u)|^{n/(n-1)} d\omega_u^n,$$

and the equality sign holds only when the M_j are homothetic ellipsoids with center z.

It follows in particular for a convex body M that, for $n \geq 3$,

$$(4) \qquad |M|^{n-1} \geq \frac{1}{n} \frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{\Omega_n} |M(u)|^n \, d\omega_u^n,$$

with the equality (if $|M| > 0$) only for ellipsoids with center z. The efforts to prove this inequality, which has applications in Finsler spaces, led to the present investigation. The—because of Jensen's inequality—weaker estimate

$$(5) \qquad |M| \geq \frac{1}{n} \kappa_n^{-n/(n-1)} \int_{\Omega_n} |M(u)|^{n/(n-1)} \, d\omega_u^n,$$

with equality sign (if $|M| > 0$) only for the spheres with center z, was found previously by L. A. Santaló who communicated it to the author. It is also the special case $M_{n-1} = M$, $M_j = U_n$ for $j < n - 1$, of (3).

2. Let $M_1, \cdots, M_{n-1}$ be bounded Jordan measurable sets in E_n, $n > 3$, such that the intersection of M_j with any ν-dimensional linear subspace (which in the future will be indicated by L_ν) through a fixed given point z possesses a

ν-dimensional Jordan measure. Since the subscripts run sometimes from 1 to n and other times from 1 to $n-1$, we agree to use α, β for the former type, and j, k for the latter, and may then omit mentioning the range.

Let x_α be rectangular coordinates in E_n with z as origin. Take $n-1$ copies E_n^j of E_n with coordinates x_α^j, and let M_j' be the image of M_j in E_n^j; that is, $x^j \in M_j'$ if and only if the point x with $x_\alpha = x_\alpha^j$ lies in M_j. Then x_α^j may be considered as rectangular coordinates in the product space

$$E = E_n' \times \cdots \times E_n^{n-1} = \prod E_n^j ;$$

hence

$$(6) \qquad \prod |M_j| = \prod |M_j'| = \int_{\prod M_j'} dx_1^1 \cdots dx_n^1 \cdots dx_1^{n-1} \cdots dx_n^{n-1} .$$

In E we introduce new coordinates

$$\bar{x}_1^1, \cdots, \bar{x}_{n-1}^1, v_1, \cdots, \bar{x}_1^{n-1}, \cdots, \bar{x}_{n-1}^{n-1}, v_{n-1}$$

through the relations

$$(7) \qquad x_k^j = \bar{x}_k^j, \; x_n^j = v_1 \bar{x}_1^j + \cdots + v_{n-1} \bar{x}_{n-1}^j .$$

These equations fail to define v_j if $|x_k^j| = |\bar{x}_k^j| = 0$, i.e. if the points x^j in E_n are contained in an L_ν with $\nu < n-2$, or if the L_{n-2} spanned by the x^j is parallel to the x_n-axis. The geometric meaning of the right side of (2) shows that a special discussion of this case is superfluous.

To evaluate $\prod |(M_j')|$ in the new coordinates, observe that the first n rows in the $n(n-1)$-rowed Jacobian J of the transformation (7) are in blocks of $n \times n$ matrices:

$$\begin{matrix}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
v_1 & v_2 & \cdots & v_{n-1} & \bar{x}_1^1, & v_1 & v_2 & \cdots & v_{n-1} & \bar{x}_2^1, & v_1 & v_2 & \cdots & v_{n-1} & \bar{x}_{n-1}^1 ;
\end{matrix}$$

hence

(8) $$J = |\overline{x}_k^j|.$$

The unit normal u in E_n to the plane

$$x_n = x_1 v_1 + \cdots + x_{n-1} v_{n-1}$$

is, with $w = (1 + v_1^2 + \cdots + v_{n-1}^2)^{1/2}$, either

$$u_j = v_j w^{-1/2}, \; u_n = -w^{-1/2} \quad \text{or} \quad u_j = -v_j w^{-1/2}, \; u_n = w^{-1/2}.$$

Then $w^{-1} = |\cos\theta|$, where θ is the angle between u and the x_n-axis, so that $d\omega_u^n = w\, du_1 \cdots du_{n-1}$ is the area element of Ω_n. Here we disregard again planes parallel to the x_n-axis. Now

$$\left|\frac{\partial u_j}{\partial v_k}\right| = w^{-3(n-1)} \begin{vmatrix} w^2 - v_1^2 & -v_1 v_2 & \cdots & -v_1 v_{n-1} \\ -v_2 v_1 & w^2 - v_2^2 & \cdots & -v_2 v_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ -v_{n-1} v_1 & -v_{n-1} v_2 & \cdots & w^2 - v_{n-1}^2 \end{vmatrix}$$

Since all principal minors of the determinant $|-v_j v_k|$ of order greater than 1 vanish, it follows (compare [4, pp. 125, 126]) that

$$\left|\frac{\partial u_j}{\partial v_k}\right| = w^{-3(n-1)} \left(w^{2(n-1)} - w^{2(n-2)} \sum v_j^2\right) = w^{-n-1}$$

and

(9) $$d\omega_u^n = w^{1-n-1} dv_1 \cdots dv_{n-1} = |\cos^n\theta|\, dv_1 \cdots dv_{n-1}.$$

The volume element dV_{xj}^{n-1} of the hyperplane $x_n^j = \sum x_k^j v_k$ is

(10) $$dV_{xj}^{n-1} = dx_1^j \cdots dx_{n-1}^j \,|\sec\theta|.$$

If we now interpret the points $x^1, \cdots, x^{n-1}$ as lying in the same E_n, then (8) shows that $|J|/(n-1)!$ is the volume of the projection of the simplex with vertices $x^1, \cdots, x^{n-1}$, z on the plane $x_n = 0$. Since these points determine a hyperplane $H(u)$ with normal u through z,

$$(11) \qquad (n-1)!\ T(x^1, \cdots, x^{n-1}, z) = |J \sec \theta|.$$

Replacing x^j by p_j, we briefly summarize the results (9), (10), (11) as

$$(12) \quad dV^n_{p_1} \cdots dV^n_{p_{n-1}} = (n-1)!\ T(p_1, \cdots, p_{n-1}, z)\, dV^{n-1}_{p_1} \cdots dV^{n-1}_{p_{n-1}}\, d\omega^n_u.$$

After observing that in (2) by integrating over Ω_n every $M^j(u)$ is counted twice (once for u and once for $-u$), we see that *the relation* (2) *follows from* (12).

For brevity we introduce, for sets $M_1, \cdots, M_r$ in E_s with $r \leq s$, the notation

$$\tau_s(M_1, \cdots, M_r, z) = \int_{M_1} \cdots \int_{M_r} T(p_1, \cdots, p_r, z)\, dV^s_{p_1} \cdots dV^s_{p_s},$$

and may then write (2) in the form

$$(13) \quad |M_1| \cdots |M_{n-1}| = \frac{(n-1)!}{2} \int_{\Omega_n} \tau_{n-1}(M_1(u), \cdots, M_{n-1}(u), z)\, d\omega^n_u.$$

3. In order to obtain *expressions for* $|M_1| \cdots |M_r|$ *with* $r < n-1$, we replace successively $M_{n-1}, \cdots, M_{n-r+1}$ by the unit sphere U_n. The contribution of the latter sets to the right side of (13) can then be integrated out by using the following fact:

Let an L_μ, $0 < \mu < n-1$, through the center z of the unit sphere U_{n-1} in E_{n-1}, intersect U_{n-1} in U_μ. For any point q in U_{n-1}, denote by r the distance qz, and by ϕ the angle between the ray qz and the L_μ. Then

$$(14) \qquad \int_{U_{n-1}} r|\sin\phi|\, dV^{n-1}_q = \frac{\omega_{\nu-1}}{\omega_\nu} \cdot \kappa_n, \quad \nu = n - \mu,$$

where $\omega_\nu = \nu \cdot \kappa_\nu = 2\pi^{\nu/2}\, \Gamma^{-1}(\nu/2)$ is the area of the surface Ω_ν of U_ν, in particular $\omega_1 = 2$.

To prove (14), let the $L_{\nu-1}$ normal to the L_μ through q intersect U_μ in p, and U_{n-1} in the sphere $S_{\nu-1}$. If $\rho = pz$ then $S_{\nu-1}$ has radius $\sigma = (1-\rho^2)^{1/2}$. Then $s = pq = r\ |\sin\phi|$; hence

$$\int_{U_{n-1}} r\,|\sin\phi|\ dV^{n-1}_q = \int_{U_\mu} \left(\int_{S_{\nu-1}} s\ dV^{\nu-1}_q \right) dV^\mu_p,$$

If $d\omega^{\nu-1}_q$ denotes the area element of the $\Omega_{\nu-1}$ with center p in the $L_{\nu-1}$ at the point $\bar{q}$ of the ray pq, then

$$\int_{S_{\nu-1}} s\, dV_q^{\nu-1} = \int_{\Omega_{\nu-1}} \int_0^{\sigma} s\, s^{\nu-2}\, ds\, d\omega_{\bar{q}}^{\nu-1} = \frac{\sigma^\nu}{\nu}\, \omega_{\nu-1} = (1-\rho^2)^{\nu/2} \frac{\omega_{\nu-1}}{\nu}.$$

Therefore, with a similar notation,

$$\int_{U_{n-1}} r\,|\sin\phi|\, dV_q^{n-1} = \frac{\omega_{\nu-1}}{\nu} \int_{\Omega_\mu} \int_0^1 (1-\rho^2)^{\nu/2}\, \rho^{\mu-1}\, d\rho\, d\omega_{\bar{p}}^{\mu}$$

$$= \frac{\omega_{\nu-1}\,\omega_\mu}{\nu} \frac{\Gamma(\mu/2)\,\Gamma(\nu/2+1)}{2\Gamma(\mu/2+\nu/2+1)} = \frac{\omega_{\nu-1}\,\omega_\mu}{\nu} \frac{\Gamma(\mu/2)\nu/2\,\Gamma(\nu/2)}{2\Gamma(n/2+1)} = \frac{\omega_{\nu-1}}{\omega_\nu} \kappa_n.$$

Returning to (13), we replace M_{n-1} by U_n. Then $M_{n-1}(u)$ becomes the $(n-1)$-dimensional unit sphere $U_n(u)$ in the hyperplane with normal u. If ϕ is the angle between the ray zp_{n-1} and the L_{n-2} spanned by $p_1, \cdots, p_{n-2}, z$, then, with $r = zp_{n-1}$,

$$T(p_1, \cdots, p_{n-1}, z) = (n-1)^{-1}\, r\,|\sin\phi|\; T(p_1, \cdots, p_{n-2}, z).$$

Hence, carrying out the integration over $U_n(u)$, by (14) we obtain

$$|M_1| \cdots |M_{n-2}| \cdot \kappa_n$$

$$= \frac{1}{2}(n-2)!\, \frac{\omega_1}{\omega_2}\, \kappa_n \int_{\Omega_n} \tau_{n-1}(M_1(u), \cdots, M_{n-2}(u), z)\, d\omega_u^n$$

or

(15) $|M_1| \cdots |M_{n-2}|$

$$= \frac{(n-2)!}{2\pi} \int_{\Omega_n} \tau_{n-1}(M_1(u), \cdots, M_{n-2}(u), z)\, d\omega_u^n.$$

If now M_{n-2} is replaced by U_n, then because of (14) the factor

$$\frac{1}{n-2}\, \frac{\omega_2}{\omega_3}\, \kappa_n$$

is introduced on the right. Continuing in this manner leads to *the general relation*

$$(16) \qquad |M_1| \cdots |M_{n-r}| = \frac{(n-r)!}{\omega_r} \int_{\Omega_n} \tau_{n-1} (M_1(u), \cdots, M_{n-r}(u), z)\, d\omega_u^n .$$

The integrand occurs in many $H(u)$, and it would be more natural to replace the integration over Ω_n by an integration over all L_{n-r}. The results of integral geometry [5] lead to such a reduction for general r; however, we restrict our attention here to the two simplest cases, where no new formulas of the type (12) are required.

It is clear that *the last formula in the sequence* (16),

$$|M_1| = \frac{1}{\omega_n} \int_{\Omega_n} \tau_{n-1} (M_1(u), z)\, d\omega_u^n ,$$

must be essentially identical with (1). Indeed, if M_1 can be represented in the form $0 \leq \rho \leq r(u)$, $u \in \Omega_n$, and we write the induced representation of $M_1(u)$ in the form

$$0 \leq \rho \leq r(v), \quad v \in \Omega_{n-2}(u) = H(u) \cap \Omega_n ,$$

then, with $pz = \rho$, we have

$$\tau(M_1(u), z) = \int_{M_1(u)} \rho\, dV_p^{n-1} = \int_{\Omega_{n-1}(u)} \int_0^{r(v)} \rho\, \rho^{n-2}\, d\omega_v^{n-1}$$

$$= \frac{1}{n} \int_{\Omega_{n-1}(u)} r^n(v)\, d\omega_v^{n-1}$$

and

$$|M_1| = \frac{1}{n \cdot \omega_{n-2}} \int_{\Omega_n} \left(\int_{\Omega_{n-1}(u)} r^n(v)\, d\omega_v^{n-2} \right) d\omega_u^n .$$

Now according to the results on cinematic measure on the sphere (see [5], for $n = 3$ already [3]), integrating over the v-normal to u first, and then over u, leads to the same result as integrating over the $H(w)$ that contain v, that is, those for which w is normal to v, and then over v. The first of the latter two integrations yields $\omega_{n-2} r^n(v)$, and (1) follows.

As second example, we indicate briefly the reduction of (15). Denoting by L^p_{n-2} the L_{n-2} spanned by $p_1, \cdots, p_{n-2}, z$, and by M^p_i the intersection of M_i with L^p_{n-2}, we obtain from (12) that if L^p_{n-2} lies in $H(u)$ and has there the normal v, then

$$\tau_{n-1}(M_1(u), \cdots, M_{n-2}(u), z)$$

$$= \int_{M_1(u)} \cdots \int_{M_{n-2}(u)} T(p_1, \cdots, p_{n-2}, z)\, dV^{n-1}_{p_1} \cdots dV^{n-1}_{p_{n-2}}$$

$$= \frac{(n-2)!}{2} \int_{\Omega_{n-1}(u)} \int_{M^p_1} \cdots \int_{M^p_{n-2}} T^2(p_1, \cdots, p_{n-2}, z)\, dV^{n-2}_{p_1} \cdots dV^{n-2}_{p_{n-2}}\, d\omega^{n-1}_v .$$

Substituting this in (15) leads besides the integrations over the M^p_i to integrations over $\Omega_{n-1}(u)$ and Ω_n. Similarly as in the preceding case, these latter two may be reduced to one integration by using the cinematic measure dL^p_{n-2} on Ω_n of the Ω_{n-2} in which L^p_{n-2} intersects Ω_n (compare [5]). The result (given without verification because it will not be used) is

$$(17) \quad |M_1| \cdots |M_{n-2}|$$

$$= \frac{[(n-2)!]^2}{2} \int_{\Omega_n} \int_{M^p_1} \cdots \int_{M^p_{n-2}} T^2(p_1, \cdots, p_{n-2}, z)\, dV^{v-2}_{p_1} \cdots dV^{n-2}_{p_{n-2}}\, dL^p_{n-2} .$$

4. To obtain the estimate (3) we use Steiner's symmetrization in the form suggested by Blaschke's treatment of Sylvester's Problem (compare [1, § 24]). In the following the subscripts i, h run from 1 to m.

Let $M_1, \cdots, M_m$ be convex bodies with interior points in E_m. In an arbitrary system of rectangular coordinates with origin z, symmetrize each M_i with respect to the $(x_1, \cdots, x_{m-1})$-plane P; that is, slide a segment in which a line L_1 parallel to the x_m-axis intersects M_i along L_1 such that its center falls on P. Call $\overline{M}_i$ the image of M_i under this transformation, and $\overline{p}_i$ the image in $\overline{M}_i$ of a given point p_i in M_i. The mapping preserves volume, $dV^m_{\overline{p}_i} = dV^m_{p_i}$. We are going to show that $\tau_m(M_1, \cdots, M_m, z)$ does not increase.

If $p_i \in M_i$, denote by p'_i the point symmetric to p_i with respect to the center of that chord of M_i parallel to the x_m-axis which goes through p_i. If $p^1_i, \cdots, p^m_i$ are the coordinates of p_i, then with $\eta = 1/m\,!$

$$\pm T(p_1, \cdots, p_m, z) = \eta\, |p^h_i|, \quad \pm T(p'_1, \cdots, p'_m, z) = \eta\, |p'^h_i| .$$

The images $\overline{p}_i$ of p_i and $\overline{p}_i'$ of p_i' satisfy the relation

$$|\overline{p}_i^h| = -|\overline{p}_i'^h|;$$

hence

$$2T(\overline{p}_1, \cdots, \overline{p}_m, z) = 2T(\overline{p}_1', \cdots, \overline{p}_m', z) = \eta \Big| |\overline{p}_i^h| - |\overline{p}_i'^h| \Big|.$$

But

$$p_i^h = p_i'^h = \overline{p}_i^h = \overline{p}_i'^h \quad \text{for } 1 \leq h \leq m-1, \text{ and } p_i^m - p_i'^m = \overline{p}_i^m - \overline{p}_i'^m .$$

so that

$$|\overline{p}_i^h| - |\overline{p}_i'^h| = |p_i^h| - |p_i'^h|;$$

hence

$$(18) \qquad T(p_1, \cdots, p_m, z) + T(p_1', \cdots, p_m', z) \geq 2T(\overline{p}_1, \cdots, \overline{p}_m, z).$$

Since

$$\begin{aligned} \tau_m(M_1, \cdots, M_m, z) &= \int_{M_1} \cdots \int_{M_m} T(p_1, \cdots, p_m, z)\, dV^m_{p_1} \cdots dV^m_{p_m} \\ &= \int_{M_1} \cdots \int_{M_m} T(p_1', \cdots, p_m', z)\, dV^m_{p_1'} \cdots dV^m_{p_m'}, \end{aligned}$$

we conclude from (18) and

$$dV^m_{\overline{p}_i} = dV^m_{p_i}, \quad dV^m_{\overline{p}_i'} = dV^m_{p_i'}$$

that

$$(19) \qquad \tau_m(M_1, \cdots, M_m, z) \geq \tau_m(\overline{M}_1, \cdots, \overline{M}_m, z).$$

To discuss the equality sign, consider points p_i in M_i which are centers of chords parallel to the x_n-axis. Then $p_i = p_i'$, and the points $\overline{p}_i = \overline{p}_i'$ lie in P, so that the right side of (18) vanishes. Therefore the equality sign can hold in (18) only when the points $p_1, \cdots, p_m, z$ are coplanar. Choosing $p_1, \cdots, p_{i-1}$,

$p_{i+1}, \cdots, p_m$ such that they and z do not lie in an L_{m-2} (the M_i have interior points!) we see that all centers of chords of M_i parallel to the x_m-axis must lie in the L_{m-1} spanned by $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_m, z$. Moreover, this same L_{m-1} must contain the centers of the chords parallel to the x_m-axis of all the other M_n. Thus we have proved:

(20) If $M_1, \cdots, M_m$ are convex bodies in E_m with interior points, then simultaneous symmetrization of the M_i with respect to any plane P through z decreases $\tau_m(M_1, \cdots, M_m, z)$ unless z and the centers of the chords perpendicular to P of all M_i are coplanar.

For given positive values $|M_1|, \cdots, |M_m|$, the expression $\tau_m(M_1, \cdots, M_m, z)$ can therefore be minimal only if the centers of every family of parallel chords of the different M_j lie on the same plane through z. This implies, first, that *each* M_j *is an ellipsoid with center* z,[1] *and then that all these ellipsoids are homothetic.*

That *the minimum is actually reached* in this case is proved by the following standard argument (see [1, § 24]). Using a suitable sequence P_ν of planes through z, and symmetrizing $M_1, \cdots, M_m$ successively in $P_1, P_2, \cdots$, yields a sequence $M_1^\nu, \cdots, M_m^\nu$ of convex bodies which tend to spheres $S_1, \cdots, S_m$ with center z and, of course, $|S_i| = |M_i^\nu| = |M_i|$ (compare [2, § 41]).

The functional $\tau_m(M_1, \cdots, M_m, z)$ is monotone [that is, $M_i' \subset M_i$ implies $\tau_m(M_1', \cdots, M_m', z) \leq \tau_m(M_1, \cdots, M_m, z)$] and positive homogeneous:

$$\tau_m(\lambda M_1, \cdots, \lambda M_m, z) = \lambda^{m(m+1)} \tau_m(M_1, \cdots, M_m, z) \text{ for } \lambda > 0.$$

For a given $\epsilon > 0$, choose $N(\epsilon) > 0$ such that $S_i \subset (1+\epsilon)M_i^\nu$ for $\nu > N(\epsilon)$ and all i. Then for $\nu > N(\epsilon)$, because of (20) and the two mentioned properties, we have

$$\tau_m(S_1, \cdots, S_m, z) \leq (1+\epsilon)^{m(m+1)} \tau_m(M_1^\nu, \cdots, M_m^\nu, z)$$

$$\leq (1+\epsilon)^{m(m+1)} \tau_m(M_1, \cdots, M_m, z),$$

which proves $\tau_m(S_1, \cdots, S_m, z) \leq \tau_m(M_1, \cdots, M_m, z)$ and hence the mini-

[1] The proofs found in the literature all refer to the cases $m = 2, 3$; for references [2, § 70]. However, the extension to arbitrary m is immediate. A particularly simple proof, which works for all m and is not found in the literature, is obtained by using Loewner's result, that there is exactly one ellipsoid which has a given center, contains a given convex body, and has minimal volume.

mum property for homothetic ellipsoids with center z.

To evaluate $\tau_m(S_1, \cdots, S_m, z)$, denote the radius of S_i by r_i. Then the results of section 3 show that

$$\tau_m(S_1, \cdots, S_m, z) = \frac{1}{m} r_m^{m+1} \frac{\omega_1}{\omega_2} \kappa_{m+1} \tau_m(S_1, \cdots, S_{m-1}, z)$$

$$= \frac{1}{m(m-1)} \frac{\omega_1}{\omega_3} r_m^{m+1} r_{m+1}^{m+1} \kappa_{m+1}^2 \tau_m(S_1, \cdots, S_{m-2}, z) = \cdots$$

$$= \frac{1}{m!} \prod r_i^{m+1} \frac{\omega_1}{\omega_{m+1}} \kappa_{m+1}^m = \frac{1}{m!} \frac{2}{\omega_{m+1}} \frac{\kappa_{m+1}^m}{\kappa_m^{m+1}} \prod |S_i|^{(m+1)/m}.$$

Therefore we have:

(21) *If $M_1, \cdots, M_m$ are convex bodies in E_m with interior points, then*

$$\tau_m(M_1, \cdots, M_m, z) \geq \frac{2}{(m+1)!} \frac{\kappa_{m+1}^{m-1}}{\kappa_m^{m+1}} \prod |M_i|^{(m+1)/m},$$

and the equality sign holds only for homothetic ellipsoids with center z.

Applying this result to (13) yields the inequalities (3) and (4) with the conditions for the equality sign. The latter result may also be formulated as follows:

Among all convex bodies M with a given volume, the ellipsoids with center z (and only these) maximize $\int_{\Omega_n} |M(u)|^n d\omega_u^n$.

To ask for the minimum is senseless since for any convex body M the integral $\int_{\Omega_n} |M(u)|^n d\omega_u^n$ will tend to zero when M moves to infinity. However, it is a meaningful, but unsolved, problem to find the minimum of this integral for all convex bodies with a given volume and center z. This is equivalent to the problem of finding the smallest constant K such that for any convex M with center z the inequality

$$K \int_{\Omega_n} |M(u)|^n d\omega_u^n \geq |M|^{n-1}$$

holds. The existence of K follows readily from (2).

Finally (5) shows:

Among all convex bodies with center z the spheres (and only these) yield the maximum of

$$\min_u |M(u)|^n |M|^{1-n}.$$

The corresponding minimum maximum problem seems quite difficult.

References

1. W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. II, Berlin, 1923, and New York, 1949.

2. T. Bonnesen und W. Fenchel, *Theorie der konvexen Körper*, Berlin, 1934, and New York, 1948.

3. P. Funk, *Über Flächen mit lauter geschlossenen geodätischen Linien*, Math. Ann. 74 (1913), 278-300.

4. G. Kowalewski, *Einführung in die Determinantentheorie*, Leipzig, 1909.

5. B. Petkantschin, *Integralgeometrie 6. Zusammenhänge zwischen den Dichten der linearen Unterräume im n-dimensionalen Raum*, Abh. Math. Sem. Univ. Hamburg **11** (1936), 249-310.

University of Southern California