LIPSCHITZ FUNCTIONS OF CONTINUOUS FUNCTIONS

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1. Introduction. The present paper was suggested by a note of W. S. Loud [2] in which the following theorem on functions of a real variable is proved.

THEOREM 1. *If α is a constant $(0 < \alpha < 1)$, there exist a continuous function $f(t)$ and a pair of positive constants K_1 and K_2 such that*

$$|f(t+h) - f(t)| < K_1 |h|^\alpha$$

for all t and all h, and such that

$$\limsup_{h \to 0} \frac{|f(t+h) - f(t)|}{|h|^\alpha} > K_2$$

for all t.

It is natural to examine the possibility of a variable exponent $\alpha(t)$ and to consider various definitions that associate with every continuous function $f(t)$ a "Lipschitz function" $\alpha(t; f)$. For a reasonable choice of the definition, Loud's result implies that every constant $\alpha(0 < \alpha < 1)$ is the Lipschitz function of some continuous function. The following sections offer two different definitions of Lipschitz functions, and deal with the problem of characterizing the functions that are Lipschitz functions of continuous functions.

2. The point Lipschitz function of a function. Let $f(t)$ be a continuous, real-valued function of t. Consider the quantity

$$Q(\alpha, t_0; f) = \limsup_{h \to 0} \frac{|f(t_0 + h) - f(t_0)|}{|h|^\alpha}.$$

If $Q(\alpha, t_0; f)$ is finite for $\alpha = \alpha'$, it is zero for all α less than α'; if Q is greater than zero for $\alpha = \alpha'$, it has the value $+\infty$ for all α greater than α'. Let $\alpha(t_0; f)$ denote the least upper bound of all α for which $Q(\alpha, t_0; f)$ is

finite. Then $\alpha(t; f)$ shall be called the *point Lipschitz function* of $f(t)$. A simple computation shows that

$$(1) \qquad \alpha(t; f) = \lim_{h \to 0} \inf \frac{\log |f(t + h) - f(t)|}{\log |h|} \quad ,$$

where the fraction on the right is to be interpreted as having the value $+ \infty$ whenever $f(t + h) = f(t)$.

THEOREM 2. *If $f(t)$ is continuous $(- \infty < t < \infty)$, then $\alpha(t; f)$ is the inferior limit of a sequence of continuous functions of t; if $\alpha(t; f) > 1$ throughout some open interval of the t-axis, then $\alpha(t; f) = \infty$ throughout the interval.*

Let $L(t, h; f)$ denote the fraction in the right member of (1), let ϵ denote a constant $(0 < \epsilon < 1/2)$, and let

$$(2) \qquad \alpha_{\epsilon}(t; f) = \min [1/\epsilon, \min_{\epsilon \leq |h| \leq 2\epsilon} L(t, h; f)].$$

Then $\alpha_{\epsilon}(t; f)$ is a continuous function of t, because of the restriction on h and the truncation of $L(t, h; f)$ imposed in the right member of (2). If ϵ is assigned the successive values $1/4$, $1/8$, $1/16$, $\cdots$, the first part of Theorem 2 follows from the formula (2).

For the second part of the theorem, consider an open interval I on the t-axis throughout which $\alpha(t; f) > 1$. Because $f'(t) = 0$ throughout I, $f(t)$ is constant in I, and the proof of the theorem is complete.

THEOREM 3. *Let $\{ \alpha_n(t) \}$ be a sequence of continuous functions $(0 \leq \alpha_n(t) \leq 1)$, and let*

$$\alpha(t) = \lim_{n \to \infty} \inf \alpha_n(t).$$

Then there exists a continuous function $f(t)$ such that $\alpha(t; f) = \alpha(t)$.

The theorem will be proved by a construction analogous to that used by Loud in his proof of Theorem 1. Let $g(t, s)$ be the continuous function which takes the value 0 when t is an even multiple of s, takes the value 1 when t is an odd multiple of s, and is linear between consecutive multiples of s. Let α be a constant between 0 and 1, and let $A > [2(1 - \alpha)]^{-1}$ be an integer. Loud proved that the series

(3)
$$\sum_{n=1}^{\infty} 2^{-2A\alpha n} g(t, 2^{-2An}),$$

converges to a function $f(t)$ which has the properties promised in Theorem 1. Roughly, the principal intuitive idea behind Loud's proof is that for every pair of values t and h at most one or two terms of the series (3) make a significant contribution to the difference $f(t+h) - f(t)$; the contribution is always small enough so that the first inequality in Theorem 1 is satisfied; and for every t there exist arbitrarily small values of h for which the contribution is so large that the second inequality in Theorem 1 is satisfied. The following three sections will be devoted to elaborations of Loud's method that lead to a proof of Theorem 3. We first summarize our construction of a continuous function $f(t)$ whose Lipschitz function $\alpha(t)$ is the inferior limit of a given sequence of continuous functions $\alpha_n(t)$. The construction is then described in full detail in the following two sections.

The function $f(t)$ will be written as an infinite series

$$f(t) = \sum_{m=1}^{\infty} G_m(t),$$

with G_m depending on the function $\alpha_m(t)$ alone. For every m, we divide the t-axis into intervals I over each of which the function $\alpha_m(t)$ lies within fixed bounds to be specified. The function $G_m(t)$ is defined separately in each interval. Over a fixed interval I, the graph Γ_m of the function $G_m(t)$ consists of rows of saw-teeth completely filling I. There is a row of relatively high and wide teeth in the central portion of the interval, flanked by two rows of somewhat lower and much narrower teeth, which are in turn flanked by two rows of still lower and narrower teeth, and so on. All the teeth of the central row have equal height and equal width, all the teeth of the two flanking rows have equal height and equal width, and so on. Toward the end-points of the interval I, the heights and the widths of the teeth of Γ_m approach zero. The function $G_m(t)$ is continuous for all t, is differentiable except at a countable number of points, and is not constant in any interval. The heights and the widths of the saw-teeth are so chosen that

$$\alpha(t; f) = \alpha(t)$$

for the function

$$f(t) = \sum_{m=1}^{\infty} G_m(t).$$

Details of the proof follow.

3. Classes of intervals. For each m, we denote by I_m a class of intervals to be constructed, with I_m depending on $I_1, I_2, \cdots, I_{m-1}$ as well as on the functional values of $\alpha_m(t)$. The class I_1 consists of finitely many or infinitely many disjoint, open intervals that meet the following three requirements: each point of the t-axis lies in the closure of one of the intervals; no point of the t-axis is a limit point of end-points of intervals of the class I_1; and throughout each of the intervals one of the conditions

$$0 \leq \alpha_1(t) \leq 3/4,$$

$$1/4 \leq \alpha_1(t) \leq 1$$

is satisfied.

When the classes $I_1, I_2, \cdots, I_{m-1}$ of intervals have been defined, we choose the intervals of the class I_m subject to the following four requirements: each point of the t-axis lies in the closure of one of the intervals of the class; no point of the t-axis is a limit point of end-points of intervals of the class; throughout each interval of the class, one of the conditions

$$0 \leq \alpha_m(t) \leq 3/2^{m+1},$$

$$1/2^{m+1} \leq \alpha_m(t) \leq 5/2^{m+1},$$

(4) $$3/2^{m+1} \leq \alpha_m(t) \leq 7/2^{m+1},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots ,$$

$$1 - 3/2^{m+1} \leq \alpha_m(t) \leq 1$$

is satisfied; and no end-point of an interval of the class I_m is an end-point of an interval of a previously defined class.

4. The saw-tooth functions. The function $f(t)$ to be constructed will be of the form

(5) $$f(t) = \sum_{m=1}^{\infty} G_m(t) = \sum_{n=1}^{\infty} g_n(t),$$

where

$$G_1(t) = g_1 + g_3 + g_6 + g_{10} + \cdots,$$

$$G_2(t) = g_2 + g_5 + g_9 + \cdots,$$

$$G_3(t) = g_4 + g_8 + \cdots,$$

$$G_4(t) = g_7 + \cdots,$$

$$\cdots \cdots.$$

The indices $n = 1, 3, 6, 10, \cdots$, are said to belong to $m = 1$, the indices $n = 2$, $5, 9, \cdots$, are said to belong to $m = 2$, and so on. The graphs of the functions $G_m(t)$ and $g_n(t)$ will be denoted by Γ_m and γ_n, respectively.

We summarize the construction of the functions $g_n(t)$. Corresponding to the function G_m we have selected a class I_m of intervals, and throughout each interval I of the class I_m the function $\alpha_m(t)$ satisfies one of 2^m inequalities specified in §3. We choose a class of exponents c in accordance with these inequalities, so that c is fixed throughout each interval I, and a decreasing sequence $\{s_n\}$ of positive numbers, where s_n depends only on $s_1, s_2, \cdots, s_{n-1}$ and on the value of m to which the index n belongs. The function $g_n(t)$ is continuous, with a graph γ_n consisting alternately of rows of equally high and equally wide saw-teeth and of segments of the t-axis. There are at most two subintervals of each interval I of the class I_m where $g_n(t)$ differs from zero, and in these subintervals it has the form

$$(6) \qquad\qquad g_n(t) = s_n^c \, g(t, s_n),$$

with the exponent c corresponding to the interval I. (The function $g(t, s)$ was defined in §2, immediately after the statement of Theorem 3.) The terms $g_n(t)$ of the series for $G_m(t)$ are so chosen that every point interior to an interval of the class I_m lies on the base of a tooth of the graph Γ_m of $G_m(t)$. It remains to describe the choice of the exponents c in (6), to select the sequence $\{s_n\}$, and to determine the position of the teeth in γ_n.

Let $g_n(t)$ be a term of $G_m(t)$, and let I be an interval of the class I_m. The graph γ_n of $g_n(t)$ shall contain rows of equal saw-teeth over at most two subintervals of I, and the exponent c in (6) determining the height of the teeth shall have the value $1/2^{m+1}$, $3/2^{m+1}, \cdots$, or $1 - 1/2^{m+1}$, according as the function $\alpha_m(t)$ satisfies in I the first, second, $\cdots$, or last of the conditions (4). It follows that the height s_n^c of a tooth in γ_n has one of finitely many values and depends only on the range of values taken by $\alpha_m(t)$ in the interval I where

the tooth appears, while the width $2s_n$ is the same for all teeth appearing any-
where in γ_n. It also follows that the derivative $g'_n(t)$ exists (except at denumer-
ably many points), and takes only finitely many values.

The number s_1 can be chosen arbitrarily, subject to the condition $0 < s_1 < 1$.
Once the numbers $s_1, s_2, \cdots, s_{n-1}$ are determined, s_n is chosen subject to the
following three conditions:

1) We require that s_{n-1} be an even multiple of s_n.

2) We require that the inequality

$$s_n^{2^{-m-1}} < \frac{s_{n-1}}{10}$$

be satisfied, where m refers to the function $G_m(t)$ of which $g_n(t)$ is a term.
If this requirement is met, the height of each tooth in γ_n is no greater than $1/20$
the base width of any tooth in γ_r $(r = 1, 2, \cdots, n-1)$, and every tooth in
γ_{n-1} is more than 10 times as high as every tooth in γ_n. It follows that the
series $\sum_{n=1}^{\infty} |g_n(t)|$ converges uniformly on the interval $(-\infty, \infty)$.

3) We require finally that the slopes of the sides of the lowest teeth in γ_n
be in absolute value greater than 10 times the sum of the greatest slopes that
can possibly occur in γ_r $(r = 1, 2, \cdots, n-1)$. Because the side of a tooth of
height s^c and width $2s$ has a slope numerically equal to $1/s^{1-c}$, this require-
ment is met provided s_n is chosen small enough.

We turn now to the disposition of the teeth in γ_n. Let I be any interval of the
class I_1. Then γ_1 shall have as many teeth in I as possible, subject to the
restriction that the distance from either end of I to any tooth of the graph shall
be greater than twice the height of the tooth.

Again, if I is an interval of the class I_1, and if γ_1 has no teeth in I, then
γ_3 shall have as many teeth in I as possible, subject again to the restriction
mentioned above. If γ_1 has beeth in I, then γ_3 shall have, in I, two rows of
teeth flanking the row of teeth of γ_1; again, the distance from either end of I
to any tooth in γ_3 shall be greater than twice the height of the tooth.

Next, if I is an interval of the class I_1, then γ_6 shall have teeth in the mid-
dle portion of I provided that γ_3 has no teeth in I and I is sufficiently long. If
γ_3 has teeth in I, then γ_6 shall have two rows of teeth: each of these rows shall
be adjacent to a previously constructed row, and shall extend as near as pos-
sible to the nearer end of I, subject to the condition that the remaining distance
be greater than twice the height of the teeth.

The construction of γ_n ($n = 10, 15, 21, \cdots$) proceeds according to the pattern that has been described. The construction of γ_n ($n = 2, 5, 9, \cdots$) is similar to the previous construction, with these modifications: the construction is carried out with reference to intervals of the class I_2, and with reference to the values of the function $\alpha_2(t)$; and the distance from each tooth to the end of the interval of the class I_2 in which it stands is required to be 2^2 times the height of the tooth (2^m times the height of the tooth in the construction of teeth belonging to the graph Γ_m). The construction of Γ_m is entirely independent of the construction of Γ_k ($k \neq m$). Further details are superfluous, and we must only prove that the function defined by (6) has the required properties.

5. Arithmetical estimates. First we show that if t_0 is a fixed point, $\alpha_0 = \alpha(t_0)$, and $\epsilon > 0$, then

$$(7) \qquad \limsup_{h \to 0} \frac{|f(t_0 + h) - f(t_0)|}{|h|^{\alpha_0 + \epsilon}} = \infty.$$

In other words, the point Lipschitz function of $f(t)$ has at t_0 a value not greater than α_0.

Let m be any integer such that t_0 is interior to an interval of the class I_m (at most one positive integer m fails to satisfy this requirement); then there exists an integer n such that $g_n(t)$ is a term in the series defining $G_m(t)$ and such that t_0 lies on the base of a tooth of the graph γ_n of $g_n(t)$. Therefore $g_n(t)$ is linear in a sufficiently small interval with t_0 as end-point; that is, there exists a number h, with

$$(8) \qquad \frac{1}{2} s_n \leq |h| < s_n,$$

such that the function $g_n(t)$ is linear in the interval joining the points $t = t_0$ and $t = t_0 + h$. For this number h we have further

$$|G_m(t_0 + h) - G_m(t_0)| = |g_n(t_0 + h) - g_n(t_0)| = |h| \, s_n^{c-1}$$

where

$$(9) \qquad c < \alpha_0 + 2^{-m} + 2^{-m-1}.$$

For all teeth that cover the segment joining t_0 and $t_0 + h$ and belong to graphs γ_r with $r < n$, the requirement 3) on $\{s_n\}$ implies that

$$s_n^{c-1} > 10 \sum_{r < n} \max |g_r'(t)|,$$

and therefore that

$$\sum_{r < n} |g_r(t_0 + h) - g_r(t_0)| < |h| \, s_n^{c-1}/10.$$

For the functions g_r with $r > n$, the requirement 2) on $\{s_n\}$ gives

$$\sum_{r > n} |g_r(t_0 + h) - g_r(t_0)| < s_n (1/10 + 1/100 + \cdots) < s_n/5.$$

From (8) and from the estimates obtained thus far it follows that

$$\frac{|f(t_0 + h) - f(t_0)|}{|h|^{\alpha_0 + \epsilon}} > \frac{|h| \, s_n^{c-1} - |h| \, s_n^{c-1}/10 - s_n/5}{|h|^{\alpha_0 + \epsilon}}$$

$$> s_n^c \, (9/20 - 1/5) \, s_n^{\alpha_0 + \epsilon} = \frac{1}{4} \, s_n^{c - \alpha_0 - \epsilon}.$$

By (9), the exponent in the last member is less than $-\epsilon/2$ if m is large enough, and therefore the relation (7) is established.

Secondly we must prove that, for every $\epsilon > 0$,

(10) $$\lim_{h \to 0} \sup \frac{|f(t_0 + h) - f(t_0)|}{|h|^{\alpha_0 - \epsilon}} = 0.$$

We choose an integer m_0 such that $2^{-m} + 2^{-m-1} < \epsilon$ for all $m > m_0$, and a positive quantity h_0 such that the interval

$$t_0 - h_0 \leq t \leq t_0 + h_0$$

contains no end-point of an interval of any of the classes $I_1, I_2, \cdots,$ or I_{m_0}, except possibly the point t_0 itself. Without restricting the generality of the proof, we suppose that $|h| < h_0$.

To establish (10), we make separate estimates of the variation of $G_m(t)$

for the following three cases: the point t_0 is an end-point of an interval of the class I_m; t_0 is not an end-point, and $m \leq m_0$; or t_0 is not an end-point, and $m > m_0$.

If t_0 is an end-point of an interval of the class I_m, then the disposition of the saw-teeth ensures that

(11) $$| G_m (t_0 + h) - G_m (t_0) | < | h | / 2^m .$$

For each $m \leq m_0$ for which t_0 is not an end-point of an interval of the class I_m, the quantity $G_m (t_0 + h) - G_m (t_0)$ can be written as the sum of finitely many terms of the form $g_n (t_0 + h) - g_n (t_0)$; the set of indices n that occur in this sum depends only on h_0, not on h. Since the corresponding derivatives $g_n'(t)$ are bounded, it follows that, for every $\alpha < 1$,

$$\lim_{h \to 0} \sum_{m \leq m_0} \frac{| G_m (t_0 + h) - G_m (t_0) |}{| h |^\alpha} = 0 .$$

For $m > m_0$ and $| h | < h_0$, two possibilities arise, for each index m: either t_0 and $t_0 + h$ both lie in the same interval of the class I_m, or they do not lie in the same interval. In the latter case, the inequality (11) holds, and it is therefore sufficient to discuss the former case.

Let k be the least integer such that $s_k \leq | h |$ and such that the term $g_k (t)$ occurs in the series defining a function $G_m (t)$ with $m > m_0$. The choice of the exponents in the definition (6) of $g_n (t)$, and the requirement 2) on the sequence $\{ s_n \}$, imply the inequality

$$\sum_{r \geq k} | g_r (t_0 + h) - g_r (t_0) | < 2 \, s_k^c \leq 2 \, | h |^c ,$$

and the quantity c is greater than $\alpha_0 - \epsilon$ because of our choice of m_0. Finally we estimate the contribution from those terms $g_r (t)$, occurring in the series that define functions $G_m (t)$ with $m > m_0$, for which $s_r > | h |$ while t_0 and $t_0 + h$ lie in the same interval of the class I_m. Let p be the greatest value of the index r for such terms. We find

$$| g_p (t_0 + h) - g_p (t_0) | < | h | \, s_p^{c-1} < | h |^c \, | s_p / h |^{c-1} < | h |^c ,$$

and the sum of the remaining terms of the same category is less than half of the last member. The inequality (10) is established, and the proof of Theorem 3 is

complete.

We observe that Theorem 3 no longer holds if the restriction $0 \leq \alpha_n(t) \leq 1$ is removed. For if $f(t)$ is a continuous function, the set of points where $\alpha(t; f) \leq 1$ cannot have isolated points. The complete characterization of the functions that are point Lipschitz functions of continuous functions appears to be difficult.

6. The local Lipschitz function of a function. Let $f(t)$ be a continuous, real-valued function of t, and let h be a variable taking positive values. Denote by $B(t_0, h; f)$ the least upper bound of all numbers β for which the quantity

$$\frac{|f(t'') - f(t')|}{(t'' - t')^\beta}$$

remains bounded as long as the variables t' and t'' satisfy the restriction $t_0 - h \leq t' < t'' \leq t_0 + h$. For each value t_0, $B(t_0, h; f)$ is a nonincreasing function of h. The quantity

$$(12) \qquad \qquad \beta(t) = \beta(t; f) = \lim_{h \to 0+} B(t, h; f)$$

shall be called the *local Lipschitz function* of $f(t)$.

It follows at once from the definition that

$$\beta(t; f) \leq \alpha(t; f),$$

for every continuous function $f(t)$. That equality does not always occur is seen from the following example. Consider the function

$$f(t) = t \sin 1/t \ (t \neq 0), \ f(0) = 0.$$

In every closed interval that does not contain $t = 0$, $f(t)$ has a bounded derivative, so that $\alpha(t; f) \geq 1$ for $t \neq 0$. Since $\alpha(0; f) = 1$, it follows that the point Lipschitz function of $f(t)$ is everywhere equal to 1 [except at the zeros of $f'(t)$, where $\alpha(t; f) = 2$]. On the other hand, the local Lipschitz function $\beta(t; f)$ has the value 1 everywhere except at $t = 0$, and $\beta(0; f) = 1/2$ (for details, see [1]). It follows that equations (1) and (12) do not define equivalent Lipschitz functions.

7. Characterization theorems. The following two theorems provide a characterization of bounded local Lipschitz functions of continuous functions.

THEOREM 4. *If $f(t)$ is continuous $(-\infty < t < \infty)$, then*

i) $\beta(t; f)$ *is lower semi-continuous;*

ii) *for each point t, either* $0 \leq \beta(t; f) \leq 1$, *or* $\beta(t; f) = \infty$;

iii) *the set of points t where* $\beta(t; f) \neq \infty$ *is a perfect set.*

Suppose that $\beta(t) = \beta(t; f)$ is the local Lipschitz function of a continuous function $f(t)$. If $\beta(t_0) > 1$, the point t_0 is interior to an interval in which $f(t)$ satisfies a Lipschitz condition with an exponent greater than 1. Then $f(t)$ is constant in an interval about t_0, so that $\beta(t_0; f) = \infty$. Part ii) of the theorem is proved.

The set of points where $0 \leq \beta(t; f) \leq 1$ cannot have isolated points, and the set of points where $\beta(t; f) = \infty$ is open. Therefore part iii) of the theorem is proved, as well as part i) for those points t where β is infinite. Finally, if $\beta(t_0) \leq 1$, for every $\epsilon > 0$ there exists an interval $|t - t_0| < \delta$ in which

$$\beta(t) > \beta(t_0) - \epsilon.$$

It follows that

$$\beta(t_0) \leq \liminf_{t \to t_0} \beta(t)$$

for all t_0, and the proof of the theorem is complete.

THEOREM 5. *Let* $\beta(t)$ $(0 \leq \beta(t) \leq 1)$ *be a lower semi-continuous function. Then there exists a strictly increasing continuous function* $F(t)$ *such that*

$$\beta(t; F) = \beta(t).$$

Let $\{I_r\}$ denote a sequence of closed intervals on the t-axis with the following property: for each point t_0 and for every $\epsilon > 0$, there exists an interval I_r of length less than ϵ, covering t_0. In each interval I_r we select a point t_r at which $\beta(t)$ assumes its minimum value in the interval. The function $F(t)$ will be chosen as an infinite series

$$F(t) = \sum_{r=1}^{\infty} f_r(t).$$

If the point t_r coincides with one of the points $t_1, t_2, \cdots, t_{r-1}$, we choose

$\dot{f}_r(t) \equiv 0$; otherwise we choose $f_r(t)$ as a strictly increasing function of t whose local Lipschitz function has the value $\beta(t_r)$ for $t = t_r$ and the value 1 everywhere else.

The term $f_r(t)$ is constructed from the function

$$f(t; \gamma) = \frac{|t|^\gamma \operatorname{sgn} t}{1 + |t|^\gamma}.$$

This function is strictly increasing; for

$$f'(t; \gamma) = \frac{|t|^{\gamma-1}}{(1 + |t|^\gamma)^2},$$

when $t \neq 0$, and the function is continuous at $t = 0$. Furthermore, the local Lipschitz function of $f(t; \gamma)$ has the value γ at $t = 0$, since

$$(13) \qquad\qquad |f(t''; \gamma) - f(t'; \gamma)| \leq 2|t'' - t'|^\gamma$$

for all distinct t' and t''; and it has the value 1 everywhere else, since $f'(t; \gamma)$ is continuous for $|t| > 0$. In order to adapt $f(t; \gamma)$ to our needs we require a sequence $\{\rho_r\}$ of positive numbers, chosen as follows. We set $\rho_1 = 1$. If t_r coincides with $t_1, t_2, \cdots,$ or t_{r-1}, the function $f_r(t)$ is identically zero, and no number ρ_r is needed. If all the quantities $|t_r - t_1|, |t_r - t_2|, \cdots, |t_r - t_{r-1}|$ $(r > 1)$ exceed 1, we set $\rho_r = 1$; otherwise we set ρ_r equal to the least of these quantities.

The nonzero terms $f_r(t)$ of the function $F(t)$ are given by the formula

$$(14) \qquad\qquad f_r(t) = 2^{-r} \rho_r f\left(\frac{t - t_r}{\rho_r} ; \beta(t_r)\right).$$

We prove first that

$$(15) \qquad\qquad \beta(t_0; F) \leq \beta(t_0),$$

for each point t_0. For every $h > 0$, the interval

$$t_0 - h < t < t_0 + h$$

has a subinterval I_r which contains the point t_0. Since

$$\beta(t_r) \leq \beta(t_0),$$

and since the function $F(t)$ does not satisfy in I_r a Lipschitz condition with an exponent greater than $\beta(t_r)$, the relation (15) is established.

Secondly, we show that

$$(16) \qquad \beta(t_0; F) \geq \beta(t_0) - \epsilon$$

for each point t_0 and every $\epsilon > 0$. Because $\beta(t)$ is lower semi-continuous,

$$\beta(t) > \beta(t_0) - \epsilon$$

in some interval $|t - t_0| < \delta$. We choose a pair of integers r_1 and r_2 ($r_2 > r_1$) such that

$$t_0 - \delta < t_{r_1} < t_0 < t_{r_2} < t_0 + \delta,$$

and denote by I the closed interval (t_{r_1}, t_{r_2}). We make separate estimates for those terms $f_r(t)$ for which t_r lies in I, for those terms with $r > r_2$ for which t_r is exterior to I, and for those terms with $r < r_2$ for which t_r is exterior to I.

By (13) and (14), the inequality

$$|f_r(t'') - f_r(t')| \leq 2^{1-r} |t'' - t'|^{\beta(t_r)}$$

holds for all distinct t' and t''. This inequality implies that the sum of all terms $f_r(t)$ for which t_r lies in I satisfies throughout I a Lipschitz condition with exponent $\beta(t_0) - \epsilon$. If $r > r_2$ and if t_r does not lie in I, then $f_r'(t) < 2^{r-1}$ in I, so that the sum of all terms $f_r(t)$ corresponding to such values of r has a bounded derivative in I, that is, satisfies throughout I a Lipschitz condition with exponent 1. Finally, let I' be a subinterval of I containing t_0 and sufficiently small to exclude all points t_r for which $r < r_2$, except those coinciding with t_0. The sum of the corresponding terms $f_r(t)$ also has a bounded derivative in I'. The inequality (16) is established, and the proof of Theorem 5 is complete.

References

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