AN ISOPERIMETRIC MINIMAX

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Introduction. In the preceding paper J. W. Green considers for a given convex body K in the euclidean plane the minimum of the isoperimetric ratio r (ratio of squared perimeter l^2 to area a) taken over all affine transforms k of K. He then investigates the maximum value taken over all K of this minimum ratio, shows by variational methods that such a maximum is attained by some polygon of five or fewer sides, and conjectures that it is, in fact, attained by a triangle with $12\sqrt{3}$, the isoperimetric ratio of an equilateral triangle, as the minimax ratio. I shall prove this conjecture directly by refining an estimation used by Green, the precise statement of results being as follows:

I. *Let K be an nontriangular plane convex body; there then exists an affine transform k of K with $r(k) < 12\sqrt{3}$.*

II. *Let T be a nonequilateral triangle; then $r(T) > 12\sqrt{3}$.*

Before taking up the proof of these results we dispose of a lemma.

III. *Let k be a possibly degenerate convex body with $s \subset k \subset t$, wherein t is an equilateral triangle, and s a side of t; there then exists a number x with $0 \leqq x \leqq 1$ such that*

$$l(k) \leqq (2/3 + x/3)\ l(t)$$

$$a(k) \geqq x\ a(t),$$

simultaneous equality occurring if and only if either $x = 0$, $k = s$ or $x = 1$, $k = t$.

Proof of III. Let p be that supporting strip of k parallel to the line-segment s; and let x be the ratio of the width of p to the width or altitude of t. Thus $0 \leqq x \leqq 1$, with $x = 0$ or $x = 1$ according as $k = s$ or $k = t$. Choose a point at which k touches the side of p opposite s, and define k_* to be the triangle with this point as apex and s as base. Define k^* to be the trapezoid formed by intersection of p and t. Clearly $s \subset k_* \subset k \subset k^* \subset t$; and $k_* = k = k^*$ if and only if $k = s$ or $k = t$.

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Since $k \supset k_*$, it follows that $a(k) \geqq a(k_*)$, with equality if and only if $k = k_*$. And since $k \subset k^*$, it follows that $l(k) \leqq l(k^*)$ with equality if and only if $k = k^*$. These inequalities become, upon the easy computation of $a(k_*)$ and $l(k^*)$, the asserted inequalities of III.

Proof of I. Let K be the given nontriangular convex body. Since the area functional is continuous, it easily follows from a compactness argument that a triangle T of maximal area can be inscribed in K. Let the three sides of T be labelled S_i ($i = 1, 2, 3$), and let V_i be that vertex of T opposite S_i. Because the area of T is maximal, the line L_i through V_i and parallel to S_i is a line of support of K. The triangle formed by the three lines L_i then circumscribes K and also T; it is composed of four nonoverlapping congruent triangles T and T_i, where T_i is labelled so as to have S_i as a side. That part K_i of K in T_i is a possibly degenerate convex body with $S_i \subset K_i \subset T_i$. Now any triangle can be affinely transformed into any other triangle. In particular, T can be affinely transformed into an equilateral triangle t, with T_i going into t_i, S_i into s_i, K_i into k_i, and K into k. Therefore $s_i \subset k_i \subset t_i$, and t_i is congruent to t. According to III, ratios x_i exist giving inequalities on $l(k_i)$ and $a(k_i)$. Furthermore, since K and hence k is nontriangular, not all $x_i = 0$ and not all $x_i = 1$. Therefore $0 < x < 1$, where $x = \sum x_i/3$. Evidently k is composed of the four nonoverlapping sets t and k_i in such a way that

$$l(k) = \sum l(k_i) - l(t) \leqq (1 + x)\ l(t),$$

$$a(k) = \sum a(k_i) + a(t) \geqq (1 + 3x)\ a(t),$$

whereupon

$$r(k) \leqq \frac{(1+x)^2}{1+3x}\ r(t) = \left[1 - \frac{x(1-x)}{1+3x}\right]\ 12\sqrt{3} < 12\sqrt{3},$$

as was to be shown.

Proof of II. Through II is merely a matter of trigonometry, and very likely can be verified by exhibiting a neat but perhaps unperspicuous trigonometric identity, I shall here prove it by the sort of methods used above.

Let T be a nonequilateral triangle. Define S_i, V_i, L_i as above. Since T is nonequilateral, some two of its sides, say S_1 and S_2, are unequal. Let v_3 be that point on the line L_3, regarded as a linear mirror, at which $v_1 = V_1$ is reflected when viewed from $v_2 = V_2$; and let t be the so symmetrized isosceles triangle with vertices v_i and sides s_i. Then the path $s_1 s_2$ is shorter than $S_1 S_2$, so $l(t) < l(T)$; and, since both triangles have the same base and altitude, $a(t) = a(T)$. Therefore $r(t) < r(T)$. Consequently if the minimum isoperimetric ratio among triangles is attained, it is attained by an equilateral triangle only; whereupon it would follow that $r(T) > 12\sqrt{3}$, as was to be shown. Now all possible triangle isoperimetric ratios are realized by triangles of fixed perimeter containing a fixed point. By a compactness argument, some such triangle achieves a maximum area and hence a minimum isoperimetric ratio. This completes the proof.

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