ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE HEAT EQUATION

W. FULKS

1. Introduction. Recently it has been shown independently by Hartman and Wintner [5] and by the present author [4] that if $u(x, t)$ has continuous derivatives u_{xx} and u_t, and is a nonnegative solution of the heat equation

$$(1) \qquad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle R: $\{0 < x < 1;\ 0 < t < k \leq \infty\}$, then $u(x, t)$ can be represented in the form

$$(2) \quad u(x, t) = \int_{0+}^{1-0} G(x, t; y, 0)\, dA(y) + \int_0^t G_y(x, t; 0, s)\, dB(s) - \int_0^t G_y(x, t; 1, s)\, dC(s),$$

where

$$(3) \qquad G(x, t; y, s) = \frac{1}{2}\left[\vartheta_3\left(\frac{x-y}{2}, t-s\right) - \vartheta_3\left(\frac{x+y}{2}, t-s\right)\right],$$

and where ϑ_3 is the Jacobi theta function. The integrals are Riemann-Stieltjes integrals with nondecreasing integrator functions, A, B, and C. The first integral may be improper but is absolutely convergent. It was further shown (see [5] and [3]) that

$$(4) \qquad u(x, 0+) = A'(x)$$

and

$$(5) \qquad u(0+, t) = B'(t-0); \quad u(1-0, t) = C'(t-0)$$

at every point where the derivatives in question exist.

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2. **Theorem.** As to the question of the extent to which (4) and (5) uniquely determine $u(x, t)$, it is clear that they do not do so completely, for the singular solution $G_y(x, t; 0, 0)$, called a heat explosion by Doetsch [2], has normal boundary values identically zero on the three boundaries $x = 0$, $x = 1$, and $t = 0$ of R. Yet A, B, C, through formula (2), do uniquely determine u; hence one might expect that by proper choice of the path of approach to the boundary, zero boundary values would assure the vanishing of u. In particular, because of the central role played by G and G_y in the representation (2), one might expect those paths to be the curves along which these functions become unbounded. This leads us to the following:

THEOREM. *Suppose*

(a) $u(x, t)$ *is a nonnegative solution of* (1) *in* R;

(b) u_{xx} *and* u_t *are continuous in* R;

(c) $u(x, 0+) = 0$ $\qquad (0 < x < 1)$;

(d) *for every* s $(0 \le s < k)$, $\lim u(x, t) = 0$ *as* (x, t) *tends to* $(0, s)$ *along some parabolic arc of the form* $t - s = ax^2$, $a > 0$, *and* $\lim u(x, t) = 0$ *as* (x, t) *tends to* $(1, s)$ *along some parabolic arc of the form* $t - s = a(x-1)^2$, $a > 0$.

Then $u(x, t) \equiv 0$ *in* R.

3. **Proof.** As we remarked in the first sentence, conditions (a) and (b) permit representation of u in the form (2). From the formula

$$\vartheta_3(x/2, t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x+2n)^2}{4t}\right], \tag{6}$$

which can be found in [2], it is easily seen that for $0 < x < 1$ the two latter integrals in formula (2) $\longrightarrow 0$ as $t \longrightarrow 0+$. Furthermore,

$$\int_{0+}^{1-0} G(x, t; y, 0)\, dA(y) = \int_{0+}^{\delta} G(x, t; y, 0)\, dA(y)$$
$$+ \int_{\delta}^{1-\delta} G(x, t; y, 0)\, dA(y) + \int_{1-\delta}^{1-0} G(x, t; y, 0)\, dA(y),$$

where $\delta < (1/2) \min [x, 1-x]$ and is taken so small that, given $\epsilon > 0$,

$$\left|\int_{0+}^{\delta} G(x, t; y, 0)\, dA(y)\right| < \epsilon \text{ and } \left|\int_{1-\delta}^{1-0} G(x, t; y, 0)\, dA(y)\right| < \epsilon$$

uniformly in t, for $0 < t \le t_0$ for some t_0. Possibility to do this is ensured by [5, Lemma 2, p. 385]. Now

$$\int_{\delta}^{1-\delta} G(x, t; y, 0)\, dA(y) = \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y)$$

$$+ \int_{\delta}^{1-\delta} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y+2n)^2}{4t}\right] dA(y)$$

$$- \int_{\delta}^{1-\delta} \sum_{n=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x+y+2n)^2}{4t}\right] dA(y).$$

The two latter integrals are easily seen to vanish with t. Since also the left side of (2) $\longrightarrow 0$ as $t \longrightarrow 0$, it follows that, if $\delta' < \delta$,

$$\overline{\lim_{t\to 0+}} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y)$$

$$\leq \overline{\lim_{t\to 0+}} \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \leq 2\epsilon.$$

Let $\epsilon \longrightarrow 0$ and obtain

$$\lim_{t\to 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(y-x)^2}{4t}\right] dA(y) = 0.$$

By [6, Th. 7], we see that $A(y)$ is constant between δ', and $1-\delta'$. Let $\delta' \longrightarrow 0$. This ensures the vanishing of the first integral of (2).

Now let us turn to the boundary $x = 0$. Suppose that for some t_0 the boundary function $B(s)$ is not continuous. If σ is the jump (positive since $B(s)$ is increasing) in $B(s)$ at $s = t_0$, then for $t > t_0$, since $G_y(x, t; 0, s) \geq 0$ (see [5, p. 370]).

$$u(x, t) \geq \int_0^t G_y(x, t; 0, s)\, dB(s) \geq \sigma\, G_y(x, t; 0, t_0)$$

$$= \frac{1}{2}\, \sigma x \pi^{-1/2} (t-t_0)^{-3/2} \exp\left[\frac{-x^2}{4(t-t_0)}\right]$$

$$+ \frac{1}{2}\, \sigma \pi^{-1/2} (t-t_0)^{-3/2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (2n+x) \exp\left[\frac{-(2n+x)^2}{4(t-t_0)}\right].$$

Since $u(x, t) \longrightarrow 0$ as $(x, t) \longrightarrow (0, t_0)$ along $t - t_0 = ax^2$ for some $a > 0$, we have

$$u(x, t) \geq \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp\left[\frac{-1}{4a}\right]$$

$$+ \frac{1}{2} \sigma \pi^{-1/2} a^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2n + x}{x^3} \exp\left[\frac{-(2n+x)^2}{4ax^2}\right],$$

As $x \longrightarrow 0+$, the sum clearly $\longrightarrow 0$; but

$$\lim_{(x,t)\to(0,t_0)} u(x, t) = 0 \geq \lim_{x \to 0} \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp\left[\frac{-1}{4a}\right] = \infty .$$

This is a contradiction. Hence $\sigma = 0$, and $B(s)$ is continuous for $0 \leq s < k$.

Now let $t = t_0 + ax^2$. Then

$$u(x, t) \geq \int_{t_0}^{t_0 + ax^2/2} G_y(x, t; 0, s)\, dB(s)$$

$$= \int_{t_0}^{t_0 + ax^2/2} \frac{1}{2} x \pi^{-1/2} (t-s)^{-3/2} \exp\left[\frac{-x^2}{4(t-s)}\right] dB(s)$$

$$+ \int_{t_0}^{t_0 + ax^2/2} \frac{1}{2} \pi^{-1/2} (t-s)^{-3/2} Q(x, t; s)\, dB(s),$$

where

$$Q(x, t; s) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n + x) \exp\left[\frac{-(2n+x)^2}{4(t-s)}\right]$$

Clearly the latter integral vanishes with x, Since in the interval of integration we have

$$\exp\left[\frac{-x^2}{4(t-s)}\right] \geq \exp\left[\frac{-x^2}{4(ax^2/2)}\right] = \exp\left[\frac{-1}{2a}\right]$$

and

$$t - s \leq ax^2,$$

it follows that

$$u(x, t) \geq \frac{1}{2} \pi^{-1/2} a^{-3/2} x^{-2} \exp\left[\frac{-1}{2a}\right]\left[B\left(t_0 + \frac{ax^2}{2}\right) - B(t_0)\right] + o(1)$$

$$\geq K \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} + o(1),$$

where K is a positive constant. Letting $x \longrightarrow 0$, we obtain

$$0 \geq \overline{\lim_{x \to 0}} \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} = D^+[B(t_0)].$$

Hence, by [1, p.580], $B(s)$ is a monotone decreasing function. Since it is non-decreasing, it must be constant. Similarly it can be shown that $C(s)$ is constant. This completes the proof.

It seems probable that conditions (b), (c) and (d) would ensure the vanishing of $u(x, t)$ if it were represented by (2) with A, B, C of bounded variation, but the proof eludes the author.

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