THE NEUMANN PROBLEM FOR THE HEAT EQUATION

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1. Introduction. By the Neumann problem we mean the following boundary-value problem: to determine the solution $u(x, t)$ of the equation

$$(1.1) \qquad u_{xx}(x, t) - u_t(x, t) = 0$$

in the rectangle or semi-infinite strip $R^{(b,c)}$: $\{ b < x < c;\ a < t < T \leq \infty \}$, given $u(x, a)$ on $b < x < c$ and $u_x(b, t)$ and $u_x(c, t)$ on $a < t < T$. There is a formula in terms of the Green's function (essentially given by Doetsch in [2, p. 361]) which gives the answer to this problem if the closed rectangle is in the interior of a larger region in which $u(x, t)$ is a continuous solution of (1.1). This formula is as follows: let $d = c - b$, and let

$$F^{(b,c)}(x, t; y, s) = \frac{1}{2d}\left[\vartheta_3\left(\frac{x-y}{2d}, \frac{t-s}{d^2}\right) + \vartheta_3\left(\frac{x+y-2b}{2d}, \frac{t-s}{d^2}\right)\right]$$

where ϑ_3 is the Jacobi Theta function; then

$$(1.2) \quad u(x, t) = \int_b^c F^{(b,c)}(x, t; y, a)\, u(y, a)\, dy - \int_a^t F^{(b,c)}(x, t; b, s)\, u_x(b, s)\, ds$$
$$+ \int_a^t F^{(b,c)}(x, t; c, s)\, u_x(c, s)\, ds.$$

The purpose of this paper is to extend the use of formula (1.2) in the following manner: we will give conditions under which a solution of the heat equation can be written in the form (1.2) wherein $u(a, y)\, dy$, etc., are replaced by $dA(y)$ or by $a(y)\, dy$, where $A(y) \in$ BV (that is, of bounded variation) or $a(y) \in L$. And we will examine the senses in which these extensions of formula (1.2) solve the boundary-value problem; that is, the manner in which the solutions tend to the prescribed boundary data for approach to a boundary point. Furthermore, we will obtain criteria for the unique determination of the solutions of these generalized boundary-value problems.

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We will normalize our rectangle to be R: $\{0 < x < 1,\ 0 < t < T \leq \infty\}$, and for this region we will delete the superscripts from the Green's function and denote it simply by $F(x, t; y, s)$. And we will denote by H the class of solutions of (1.1) for which both u_{xx} and u_t are continuous.

It will be convenient to display here the formula (see [2, p. 307])

$$\vartheta_3\ (x/2,\ t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x+n)^2}{4t}\right], \tag{1.3}$$

from which it is clear that $F^{(b,c)}(x, t; y, s)$ is a uniformly continuous function of all six variables if d is bounded away from both zero and infinity, and if the point (x, t) is bounded away from the point (y, s).

It also follows easily from (1.3) that

$$F_x(x, t; 0, s) = -G_y(x, t; 0, s) \text{ and } F_x(x, t; 1, s) = -F_x(1-x, t; 0, s),$$

where

$$G(x, t; y, s) = \frac{1}{2}\left[\vartheta_3\left(\frac{x-y}{2},\ t-s\right) - \vartheta_3\left(\frac{x+y}{2},\ t-s\right)\right]$$

is the Green's function for the corresponding Dirichlet problem. (See [3; 4; 5; 6; 7].)

2. The Stieltjes integral representation. Our first main theorem gives the solution to one of the generalized boundary-value problems.

THEOREM 1. *For $u(x, t)$ to be representable in R by*

$$u(x, t) = \int_0^1 F(x, t; y, 0)\, dA(y) - \int_0^t F(x, t; 0, s)\, dB(s) \tag{2.1}$$

$$+ \int_0^t F(x, t; 1, s)\, dC(s),$$

where $A(y) \in \mathrm{BV}\ (0 \leq y \leq 1)$ and $B(s),\ C(s) \in \mathrm{BV}\ (0 \leq s \leq s_0)$ for every $s_0 < T$, it is necessary and sufficient that

(1) *$u(x, t) \in H$ in R,*

(2) *$\int_0^t |u_x(x, s)|\, ds < M_t$ uniformly for $0 < x \leq x_0$ and $x_1 \leq x < 1$ for some x_0, x_1, where M_t depends only on t,*

(3) $\int_0^1 |u(y, t)|\, dy \leq M$ *uniformly for* $0 < t \leq t_0$ *for some* t_0.

Proof. To prove the sufficiency, let $(x, t) \in R$. Then there exist $a, b, c > 0$ such that $u(x, t)$ is given by (1.2). But, by condition (3),

$$A_a(x) = \int_0^x u(y, a)\, dy \in \mathrm{BV}[0 \leq x \leq 1]$$

uniformly in a for $0 < a \leq t_0$. Hence the uniformity holds for any sequence of values of a tending to zero, and thus by the well-known convergence theorems of Helly and Bray (see, for example, [9, p. 29-31]) there exists a subsequence $\{a_n\}$ and a function $A(x) \in \mathrm{BV}\ (0 \leq x \leq 1)$, to which $A_{a_n}(x)$ converges substantially, such that

$$\lim_{n \to \infty} \int_b^c F^{(b,c)}(x, t; y, a_n)\, dA_{a_n}(y) = \int_b^c F^{(b,c)}(x, t; y, 0)\, dA(y).$$

Then (1.2) becomes

$$(2.2)\quad u(x, t) = \int_b^c F^{(b,c)}(x, t; y, 0)\, dA(y) - \int_0^t F^{(b,c)}(x, t; b, s)\, u_x(b, s)\, ds$$
$$+ \int_0^t F^{(b,c)}(x, t; c, s)\, u_x(c, s)\, ds,$$

where the existence of the two latter integrals is guaranteed by condition (2).

Furthermore,

$$B_b(t) = \int_0^t u_x(b, s)\, ds \quad \text{and} \quad C_c(t) = \int_0^t u_x(c, s)\, ds \in \mathrm{BV}[0 \leq t \leq t_0]$$

for every $t_0 < T$ uniformly for $0 < b \leq x_0$ and $x_1 \leq c < 1$. Hence the uniformity holds for any sequence of values of b tending to zero and of c tending to one. Hence there exist subsequences $\{b_n\}$ and $\{c_n\}$ and functions

$$B(t), C(t) \in \mathrm{BV}\ (0 \leq t \leq t_0)$$

such that

$$\lim_{n \to \infty} \int_0^t F^{(b_n,c_n)}(x, t; b_n, s)\, dB_{b_n}(s) = \int_0^t F(x, t; 0, s)\, dB(s)$$

and

$$\lim_{n\to\infty}\int_0^t F^{(b_n,c_n)}(x,\ t;\ c_n,\ s)\ dC_{c_n}(s) = \int_0^t F(x,\ t;\ 1,\ s)\ dC(s).$$

Hence $u(x,\ t)$ has the representation asserted.

We will later show that A, B, C are independent of the particular sequences of a, b, c used here (see Theorem 3).

To prove the necessity of condition (1) we must differentiate under the integral sign. The only difficulty encountered in this is the disposition of the terms which arise from the variable upper limit. If, however, one forms a difference quotient it is easy to see that the contribution arising from the variability of the upper limit must always vanish, due to the strong convergence to zero of the kernel as $s \longrightarrow t-0$.

To establish (2) we write

$$\begin{aligned} u_x(x,\ t) &= \int_0^1 F_x(x,\ t;\ y,\ 0)\ dA(y) - \int_0^t F_x(x,\ t;\ 0,\ s)\ dB(s) \\ &\qquad + \int_0^t F_x(x,\ t;\ 1,\ s)\ dC(s) \\ &= \int_0^1 F_x(x,\ t;\ y,\ 0)\ dA(y) + \int_0^t G_y(x,\ t;\ 0,\ s)\ dB(s) \\ &\qquad + \int_0^t G_y(1-x,\ t;\ 0,\ s)\ dC(s) \\ &= U_1(x,\ t) + U_2(x,\ t) + U_3(x,\ t). \end{aligned}$$

Now

$$|U_2(x,\ t)| \le \int_0^t G_y(x,\ t;\ 0,\ s)\,|dB(s)| = v_2(x,\ t)$$

and

$$|U_3(x,\ t)| \le \int_0^t G_y(1-x,\ t;\ 0,\ s)\,|dC(s)| = v_3(x,\ t),$$

where $v_2(x,\ t)$ and $v_3(x,\ t)$ are nonnegative solutions of (1.1). Then, by [3, p. 22-23] and [7, p. 373], $v_2(x,\ t)$ and $v_3(x,\ t)$ must satisfy condition (2). Hence so must $U_2(x,\ t)$ and $U_3(x,\ t)$.

To examine $U_1(x,\ t)$ we need to note that, by (1.3), for $0 < x,\ y < 1$,

$$F_x(x, t; y, 0) = -\frac{x-y}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x-y)^2}{4t}\right] - \frac{(x+y)^2}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x+y)^2}{4t}\right]$$
$$-\frac{(x+y-2)}{4\pi^{1/2}t^{3/2}}\exp\left[-\frac{(x+y-2)^2}{4t}\right] + \bar{u}_1(x, y, t),$$

where $\bar{u}_1$ is bounded, say $|\bar{u}_1| \leq B_1$. Then

$$\int_0^t |U_1(x, s)|\,ds \leq \int_0^t\int_0^1 \sum_{n=1}^{3} \frac{|a_n|}{4\pi^{1/2}s^{3/2}}\exp\left[-\frac{a_n^2}{4s}\right]|dA(y)|\,ds$$
$$+ tB_1 V_A(1),$$

where $a_1 = x-y$, $a_2 = x+y$, $a_3 = x+y-2$, and $V_A(1)$ is the variation of A. Then

$$\int_0^t |U_1(x, s)|\,ds \leq \frac{1}{4\pi^{1/2}}\int_0^1\int_0^t \sum_{n=1}^{3} \frac{|a_n|}{s^{3/2}}\exp\left[-\frac{a_n^2}{4s}\right]ds\,|dA(y)|$$
$$+ tB_1 V_A(1),$$

$$= \frac{1}{2\pi^{1/2}}\int_0^1 \sum_{n=1}^{3} \int_{a_n^2/4t}^{\infty} e^{-s}s^{-1/2}\,ds\,|dA(y)| + tB_1 V_A(1),$$

$$\leq \frac{3}{2\pi^{1/2}}\int_0^1\int_0^\infty e^{-s}s^{-1/2}\,ds\,|dA(y)| + tB_1 V_A(1),$$

$$= (3/2 + tB_1)\,V_A(1),$$

the change of order of integration being permissible by Fubini's theorem. Since $U_1(x, t)$, $U_2(x, t)$, and $U_3(x, t)$ separately satisfy condition (2), so must their sum, $u_x(x, t)$.

To verify condition (3) we write

$$u(x, t) = \int_0^1 - \int_0^t + \int_0^t = u_1(x, t) + u_2(x, t) + u_3(x, t),$$

and first consider

$$u_2(x, t) = -\int_0^t F(x, t; 0, s)\,dB(s).$$

But, by (1.3),

$$F(x, t; 0, s) = \pi^{-1/2}(t-s)^{-1/2} \exp\left[-\frac{x^2}{4(t-s)}\right] + \bar{u}_2(x, t, s),$$

where $\bar{u}_2$ is bounded, say by B_2. Then

$$\int_0^1 |u_2(x, t)|\, dx \le \int_0^t \pi^{-1/2}(t-s)^{-1/2} \int_0^1 \exp\left[-\frac{x^2}{4(t-s)}\right] dx\, |dB(s)| + B_2\, V_B(t),$$

$$\le \int_0^t \pi^{-1/2}(t-s)^{-1/2} \int_0^\infty \exp\left[-\frac{x^2}{4(t-s)}\right] dx\, |dB(s)| + B_2\, V_B(t),$$

$$= (1+B_2)\, V_B(t).$$

Similarly,

$$\int_0^1 |u_3(x, t)|\, dx \le (1+B_3)\, V_C(t).$$

We turn now to $u_1(x, t)$:

$$\int_0^1 |u_1(x, t)|\, dx \le \int_0^1 \int_0^1 F(x, t; y, 0)\, dx\, |dA(y)|.$$

But, again by (1.3),

$$F(x, t; y, 0) = \frac{1}{2}(\pi t)^{-1/2}\left\{\exp\left[-\frac{(x-y)^2}{4t}\right] + \exp\left[-\frac{(x+y)^2}{4t}\right] + \exp\left[-\frac{(x+y-2)^2}{4t}\right]\right\} + \bar{u}_4(x, y, t),$$

where $\bar{u}_4$ is bounded by, say, B_4. Then

$$\int_0^1 |u_1(x, t)|\, dx \le \frac{1}{2}(\pi t)^{-1/2} \int_0^1 \int_0^1 \left\{\exp\left[-\frac{(x-y)^2}{4t}\right] + \exp\left[-\frac{(x+y)^2}{4t}\right] + \exp\left[-\frac{(x+y-2)^2}{4t}\right]\right\} dx\, |dA(y)| + B_4\, V_A(1),$$

$$\le \frac{1}{2}(\pi t)^{-1/2} \int_0^1 \left\{\int_{-1}^1 \exp\left[-\frac{(x-y)^2}{4t}\right] dx + \int_0^1 \exp\left[-\frac{(x+y-2)^2}{4t}\right] dx\right\} |dA(y)| + B_4\, V_A(1),$$

$$\leq \frac{1}{2}(\pi t)^{-1/2} \int_0^1 2 \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4t}\right] dx\, |dA(y)| + B_4 V_A(1),$$

$$\leq V_A(1)(2+B_4)$$

Hence, for $0 < t \leq t_0$,

$$\int_0^1 |u(x, t)|\, dx \leq V_A(1)(2+B_4) + V_B(t_0)(1+B_2) + V_C(t_0)(1+B_3) = M$$

This completes the proof.

3. The behavior at the boundary. We are now prepared to examine in detail the behavior near the boundary of solutions of our generalized boundary value problem considered in section 2. The main result of the section is:

THEOREM 2. *If $u(x, t)$ is representable in R by (2.1), then*

$$\lim_{t \to 0+} u(x, t) = A'(x) \tag{1}$$

and

$$\lim_{x \to 0+} u_x(x, t) = B'(t-0); \quad \lim_{x \to 1-0} u_x(x, t) = C'(t-0) \tag{2}$$

$\left(\textit{where } B'(t-0) = \lim_{h \to 0+} \dfrac{B(t-0) - B(t-h)}{h}, \textit{ and similarly for } C'(t-0)\right)$,

wherever the derivatives in question exist.

Proof. If $u(x, t)$ is representable by (2.1), let

$$u(x, t) = \int_0^1 - \int_0^t + \int_0^t = u_1(x, t) + u_2(x, t) + u_3(x, t)$$

as before. Let I be any open interval whose closure is contained in $\{0 < x < 1\}$. Then for $x \in I$, $F(x, t; 0, s)$ and $F(x, t; 1, s)$ both converge uniformly to zero as $t \longrightarrow 0+$, as can be seen from (1.3). Then clearly $u_2(x, t)$, $u_3(x, t) \longrightarrow 0$ as $t \longrightarrow 0+$, for $x \in I$.

Also, for $x \in I$, by (1.3),

$$F(x, t; y, 0) = (4\pi t)^{-1/2} \exp\left[-\frac{(x-y)^2}{4t}\right] + o(1)$$

uniformly as $t \longrightarrow 0+$. Hence

$$u_1(x, t) = \int_0^1 (4\pi t)^{-1/2} \exp\left[-\frac{(x-y)^2}{4t}\right] dA(y) + o(1).$$

Then (see [3, p. 25-26 and 65-66] and [7, p. 393-394])

$$\lim_{t \to 0+} u_1(x, t) = A'(x)$$

wherever this derivative exists. Since any $x \in \{0 < x < 1\}$ can be caught in such an I, this establishes (1).

To verify conclusion (2) we write, as before,

$$u_x(x, t) = \int_0^1 F_x(x, t; y, 0)\, dA(y) + \int_0^t G_y(x, t; 0, s)\, dB(s)$$

$$+ \int_0^t G_y(1-x, t; 0, s)\, dC(s),$$

$$= U_1(x, t) + U_2(x, t) + U_3(x, t).$$

As $x \longrightarrow 0+$, $U_1(x, t)$ and $U_3(x, t)$ vanish since the kernels converge uniformly to zero, and as $x \longrightarrow 1-0$, $U_1(x, t)$ and $U_2(x, t)$ vanish for the same reason. Then by [5], $u_x(x, t)$ tends to $B'(t-0)$ or $C'(t-0)$ according as x tends to zero or one, whenever the derivatives exist.

We can now give criteria for the existence of boundary values of the function $u(x, t)$ itself on the sides $x = 0$, and $x = 1$.

COROLLARY 1. *If $u(x, t)$ is representable in R by (2.1), then $u(0+, t)$ exists if $B'(t-0)$ does.*

Proof. Let $0 < x_0 < 1$; then

$$u(x, t) = \int_{x_0}^{x} u_x(y, t)\, dy + u(x_0, t) \qquad (0 < x < 1),$$

and the integrand is bounded in $0 < x \le x_0$. Hence the integral exists for $x = 0$ and defines $u(0+, t)$.

We might also note in passing that for such t, the x difference quotient at the boundary also tends to $B'(t-0)$; for, by the mean value theorem,

$$\frac{u(h, t) - u(0+, t)}{h} = u_x(\bar{h}, t) \longrightarrow B'(t-0)$$

as $h \longrightarrow 0$.

From Theorem 1 we have:

COROLLARY 2. *For $u(0+, t)$ to exist it is sufficient that*

$$\int_0^{t-0} (t-s)^{-1/2} \, |dB(s)|$$

converge.

Proof. Define

$$f(t) = \int_0^1 F(0, t; y, 0)\, dA(y) + \int_0^t F(0, t, 1, s)\, dC(s)$$

$$+ 2\int_0^t \pi^{-1/2} (t-s)^{-1/2} \sum_{n=1}^{\infty} \exp\left[-\frac{n^2}{(t-s)}\right] dB(s)$$

$$- \pi^{-1/2} \int_0^{t-0} (t-s)^{-1/2}\, dB(s),$$

and consider

$$\limsup_{x \to 0+} |u(x, t) - f(t)|$$

$$\leq \pi^{-1/2} \limsup_{x \to 0+} \int_0^{t-0} (t-s)^{-1/2} \left\{1 - \exp\left[-\frac{x^2}{4(t-s)}\right]\right\} |dB(s)|.$$

Now given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\pi^{-1/2} \int_{t-\delta}^{t-0} (t-s)^{-1/2} \exp\left[-\frac{x^2}{4(t-s)}\right] |dB(s)|$$

$$\leq \pi^{-1/2} \int_{t-\delta}^{t-0} (t-s)^{-1/2}\, |dB(s)| \leq \epsilon,$$

so that

$$\limsup_{x \to 0+} |u(x, t) - f(t)|$$

$$\leq \pi^{-1/2} \limsup_{x \to 0+} \int_0^{t-\delta} (t-s)^{-1/2} \left\{1 - \exp\left[-\frac{x^2}{4(t-s)}\right]\right\} |dB(s)| + 2\epsilon = 2\epsilon.$$

Let $\epsilon \longrightarrow 0$ to get

$$\lim_{x \to 0+} u(x, t) = u(0+, t) = f(t),$$

which completes the proof.

We may also note that if $B(s)$ were monotone, then, since $\exp[-x^2/4(t-s)]$ converges to unity in a monotone way, we could invoke, the monotone convergence theorem to obtain the convergence of the integral as a necessary and sufficient condition for the existence of $u(0+, t)$.

Also with Theorem 2 at our disposal we can now prove:

THEOREM 3. *Let $u(x, t)$ be representable by* (2.1); *then the functions $A(x)$, $B(t)$, $C(t)$ are uniquely determined by $u(x, t)$, so that, at every point of continuity,*

$$A(x) = \lim_{a \to 0+} \int_0^x u(y, a)\, dy$$

and

$$B(t) = \lim_{b \to 0+} \int_0^t u_x(b, s)\, ds; \quad C(t) = \lim_{c \to 1-0} \int_0^t u_x(c, s)\, ds.$$

Proof. For suppose $B_1(t)$ and $B_2(t)$ arise from two distinct sequences. Then clearly if $B_3(t) = B_1(t) - B_2(t)$, we have

$$\int_0^t F(x, t; 0, s)\, dB_3(s) \equiv 0 \text{ in } R.$$

Hence, differentiating, we get

$$\int_0^t G_y(x, t; 0, s)\, dB_3(s) \equiv 0$$

We first show $B_3(s)$ is continuous: suppose it has a jump σ at t_0; then, for $t > t_0$,

$$0 = \int_0^t G_y(x, t; 0, s)\, dB_4(s) + \sigma\, G_y(x, t; 0, t_0),$$

where $B_4(s)$ is the boundary function remaining after the jump σ at t_0 is removed. Then

$$0 = \int_0^t \frac{x}{2\pi^{1/2}(t-s)^{3/2}} \exp\left[-\frac{x^2}{4(t-s)}\right] dB_4(s) + \frac{x\sigma}{2\pi^{1/2}(t-t_0)^3} \exp\left[-\frac{x^2}{4(t-t_0)}\right] + o(1).$$

Choose δ so small that

$$V_{B_4}(t_0+\delta) - V_{B_4}(t_0-\delta) < \frac{|\sigma|}{2},$$

set $t - t_0 = x^2/4$, and take x so small that $x^2/4 < \delta$. Then

$$0 = \int_{t_0-\delta}^{t_0+x^2/4} \frac{x}{2\pi^{1/2}(t_0-s+x^2/4)^{3/2}} \exp\left[-\frac{x^2}{4(t_0-s+x^2/4)}\right] dB_4(s) + \frac{4\sigma}{e\pi^{1/2}x^2} + o(1),$$

$$0 \geq \frac{4|\sigma|}{e\pi^{1/2}x^2} - \int_{t_0-\delta}^{t_0+x^2/4} \frac{x}{2\pi^{1/2}(t_0-s+x^2/4)^{3/2}} \exp\left[-\frac{x^2}{4(t_0-s+x^2/4)}\right] |dB_4(s)| + o(1).$$

The maximum of the integrand is at $s = t_0$, so that

$$0 \geq \frac{4|\sigma|}{e\pi^{1/2}x^2} - \frac{2|\sigma|}{e\pi^{1/2}x^2} + o(1) = \frac{2|\sigma|}{e\pi^{1/2}x^2} + o(1),$$

and we have a contradiction as $x \longrightarrow 0+$.

Similarly the jumps of $C(t)$ are determined.

Suppose $A_1(x)$ and $A_2(x)$ arise from two distinct sequences, and $A_3(x) = A_1(x) - A_2(x)$; then, as before,

$$0 \equiv \int_0^1 F(x, t; y, 0)\, dA_3(y) \text{ in } R.$$

And suppose it has a jump of σ at x_0; then, as before,

$$0 = \sigma(4\pi t)^{-1/2} + (4\pi t)^{-1/2} \int_0^1 \exp\left[-\frac{(y-x_0)^2}{4t}\right] dA_4(y) + o(1).$$

If δ is so small that

$$V_{A_4}(x_0+\delta) - V_{A_4}(x_0-\delta) \leq |\sigma|/2,$$

then

$$0 = \sigma(4\pi t)^{-1/2} + (4\pi t)^{-1/2} \int_{x_0-\delta}^{x_0+\delta} \exp\left[-\frac{(y-x_0)^2}{4t}\right] dA_4(y) + o(1),$$

$$0 \geq |\sigma|(4\pi t)^{-1/2} - |\sigma|(4\pi t)^{-1/2}/2 + o(1) = |\sigma|(4\pi t)^{-1/2}/2 + o(1),$$

and as $t \longrightarrow 0+$ we get a contradiction.

Then A_3, B_3, C_3 are continuous functions of bounded variation, and by Theorem 2 their derivatives are zero almost everywhere. Each of them must then have an infinite derivative on a nondenumerable set. (See e.g. [8, p. 128].) This then implies that lim $u(x, t)$ and lim $u_x(x, t)$ must become infinite on a nondenumerable set, which is a contradiction, and the functions A_3, B_3, C_3 are constants. Hence, since every sequence of a's, b's, or c's contains a subsequence for which $A_a(x)$, etc., converges to a common limit, the limit must also be attained for continuous approach. Thus the last statement of the theorem is established.

4. The Lebesgue integral representation. We are now in a position to establish:

THEOREM 4. *For $u(x, t)$ to be representable in R by*

$$u(x,t) = \int_0^1 F(x,t;y,0)\,a(y)\,dy - \int_0^t F(x,t;0,s)\,b(s)\,ds + \int_0^t F(x,t;1,s)\,c(s)\,ds, \tag{4.1}$$

where $a(y) \in L$ $(0 \leq y \leq 1)$ and $b(s)$, $c(s) \in L$ $(0 \leq s \leq s_0 < T \leq \infty)$ for every s_0, $(0 < s_0 < T)$, it is necessary and sufficient that

(1) $$u(x,t) \in H \text{ in } R,$$

(2) $$\lim_{y,y' \to 0+} \int_0^t |u_x(y,s) - u_x(y',s)|\,ds = 0$$

and

$$\lim_{y,y' \to 1-0} \int_0^t |u_x(y,s) - u_x(y',s)|\,ds = 0$$

for every t $(0 < t < T)$, and

$$(3) \qquad \lim_{s, s' \to 0} \int_0^1 |u(y, s) - u(y, s')|\, dy = 0.$$

Proof. For the sufficiency, let the closed finite interval $I \subset \{0 \leq s < T\}$ be prescribed, and let e be any measurable set in I. Given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, I)$ such that

$$\int_e |u_x(y, s) - u_x(y', s)|\, ds \leq \epsilon/2 \quad \text{for} \quad y, y' < \delta.$$

Then

$$\int_e |u_x(y, s)|\, ds \leq \int_e |u_x(y', s)|\, ds + \int_e |u_x(y, s) - u_x(y', s)|\, ds$$
$$\leq \int_e |u_x(y', s|\, ds + \epsilon/2.$$

Now keep y' fixed and take $m(e)$ so small that

$$\int_e |u_x(y', s)|\, ds \leq \epsilon/2,$$

so that, for $0 < y < \delta$,

$$\int_e |u_x(y, s)|\, ds \leq \epsilon$$

if $m(e)$ is sufficiently small. Hence $B_b(s)$ are uniformly absolutely continuous; consequently, so is $B(s)$, and $dB(s)$ can be replaced by $b(s)\, ds$, where $B'(s) = b(s)$ almost everywhere. Similarly $dC(s) = c(s) ds$ and $dA(y) = a(y)\, dy$.

The necessity of (1) follows by Theorem 1. To prove that of (2) we write

$$b(s) = b_1(s) - b_2(s),$$

where $b_1(s)$ and $b_2(s)$ are both nonnegative, say, for example,

$$b_1(s) = |b(s)| \text{ and } b_2(s) = |b(s)| - b(s).$$

Let

$$u^{(i)}(x, t) = -\int_0^t F(x, t; 0, s)\, b_i(s)\, ds \qquad (i = 1, 2).$$

Then

$$u_x^{(i)}(x, t) = \int_0^t G_y(x, t; 0, s)\, b_i(s)\, ds \qquad (i = 1, 2).$$

We know from Theorem 2 that

$$\lim_{x \to 0+} u_x^{(i)}(x, t) = b_i(t) \qquad (i = 1, 2),$$

almost everywhere, and, by Theorem 3,

$$\lim_{x \to 0+} \int_0^t u_x^{(i)}(x, s)\, ds = \int_0^t b_i(s)\, ds \qquad (i = 1, 2).$$

Since the $u_x^{(i)}(x, t)$ are nonnegative (see [**4**, Remark 1, p. 975]), we can say (see [**4**, p. 977])

$$\lim_{x \to 0+} \int_0^t |u^i(x, s) - b_i(s)|\, ds = 0 \qquad (i = 1, 2). \tag{4.2}$$

Now consider

$$\begin{aligned} \int_0^t |u_x(x, s) - b(s)|\, ds &\le \int_0^t |u_x^{(1)}(x, s) - b_1(s)|\, ds \\ &+ \int_0^t |u_x^{(2)}(x, s) - b_2(s)|\, ds + \int_0^t \left| \int_0^1 F_x(x, s; y, 0)\, a(y) \right| dy\, ds \\ &+ \int_0^t \left| \int_0^s G_y(1 - x, s; 0, \tau)\, c(\tau)\, d\tau \right| ds. \end{aligned} \tag{4.3}$$

As $x \longrightarrow 0+$, the first and second integrals on the right vanish by (4.2), and the fourth since $G_y(1 - x, s; 0, \tau)$ tends to zero uniformly in s and τ as $x \longrightarrow 0+$. To estimate the third we note

$$\begin{aligned} F_x(x, s; y, 0) = -\frac{1}{4\pi^{1/2} s^{3/2}} \Bigg\{ (x - y) \exp\left[-\frac{(x-y)^2}{4s}\right] \\ + (x + y) \exp\left[-\frac{(x+y)^2}{4s}\right] \Bigg\} + \bar{u}(x, y, s), \end{aligned}$$

where $\bar{u} = o(1)$ uniformly in y and s as $x \longrightarrow 0+$. Then

$$\begin{aligned} |F_x(x, s; y, 0)| \le \frac{1}{4\pi^{1/2} s^{3/2}} \Bigg\{ |x - y| \exp\left[-\frac{(x-y)^2}{4t}\right] \\ + (x + y) \exp\left[-\frac{(x+y)^2}{4t}\right] \Bigg\} + |\bar{u}|. \end{aligned}$$

But

$$\int_0^t s^{-3/2} \exp\left[-\frac{a^2}{4s}\right] ds = \frac{2}{|a|} \int_{a^2/4t}^{\infty} e^{-v} v^{-1/2}\, dv \le \frac{2\pi^{1/2}}{|a|} .$$

Hence

$$\int_0^t |F(x, s; y, 0)|\, ds \le 1 + o(1) \le 2$$

for x sufficiently small. Thus the third integral on the right side of (4.3) is dominated by

$$\int_0^1 |a(y)| \int_0^t |F_x(x, s; y, 0)|\, ds\, dy \le 2 \int_0^1 |a(y)|\, dy.$$

Then by the dominated convergence theorem we can pass to the limit under the integral sign, by which we get zero as a limit, since $F_x(x, s; y, 0)$ tends to zero. This proves

$$\lim_{x \to 0+} \int_0^t |u_x(x, s) - b(s)|\, ds,$$

from which condition (2) follows immediately.

Condition (3) follows similarly, but more easily.

5. Uniqueness. We now turn to the question of the extent to which the boundary data uniquely determine the solution of the boundary-value problem. We get one result as an immediately corollary of our Theorem 4.

COROLLARY 3. *If $u(x, t)$ is representable by (4.1) in R, and has zero boundary values almost everywhere for approach along the normal, then $u(x, t) \equiv 0$ in R.*

Proof. By Theorem 2, $a(y)$, $b(s)$, and $c(s)$ vanish almost everywhere.

The situation in the case of the Stieltjes representation is not so simple (see [**6**]): We can have a function representable in R by (2.1) which has boundary values identically zero for approach along the normal, yet which is itself not identically zero; for example, for $0 \le t_0$, let

$$u(x, t) = \begin{cases} 0 & (0 \le t \le t_0), \\ -F(x, t; 0, t_0) & (t_0 < t). \end{cases}$$

This is a nontrivial solution of the heat equation, representable by (2.1), for which

$$u(x, 0+) \equiv 0, \quad u_x(0+, t) \equiv 0, \quad u_x(1-0, t) \equiv 0.$$

However we can assert:

THEOREM 5. *Suppose $u(x, t)$ is representable in R by (2.1), that*

$$u(x, 0+) \equiv 0 \qquad (0 < x < 1),$$

and that $B(s)$ and $C(s)$ are monotone for $0 \leq s < T \leq \infty$. Let

$$\lim u_x(x, t) = 0 \quad \textit{as} \quad (x, t) \longrightarrow (0, s)$$

along a parabolic arc of the form $t - s = ax^2$, $(a > 0)$ and

$$\lim u_x(x, t) = 0 \quad \textit{as} \quad (x, t) \longrightarrow (1, s)$$

along a parabolic arc of the form $t - s = b(x-1)^2$ $(b > 0)$ for every s $(0 \leq s < T)$. Then $u(x, t) \equiv 0$ in R

Proof. Let x be fixed, $0 < x < 1$. Then, by (2.1) and (1.3),

$$u(x, t) = \int_0^1 F(x, t; y, 0)\, dA(y) + o(1) \quad \text{as} \quad t \longrightarrow 0+.$$

Choose $0 < \delta < (1/2) \min(x, 1-x)$, so that

$$\begin{aligned} u(x, t) &= \int_{x-\delta}^{x+\delta} (4\pi t)^{-1/2} \exp\left[-\frac{(y-x)^2}{4t}\right] dA(y) + o(1), \\ &= \int_{x-\delta}^{x+\delta} (4\pi t)^{-1/2} \exp\left[-\frac{(y-x)^2}{4t}\right] d[A(y) - A(x)] + o(1), \\ &= \int_{x-\delta}^{x+\delta} \frac{y-x}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{(y-x)^2}{4t}\right] [A(y) - A(x)]\, dy + o(1), \\ &= \int_{-\delta}^{\delta} \frac{z^2}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{z^2}{4t}\right] \frac{A(x+z) - A(x)}{z}\, dz + o(1). \end{aligned}$$

Then

$$\begin{aligned} u(x, t) &\geq \inf_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z} \int_{-\delta}^{\delta} \frac{z^2}{4\pi^{1/2} t^{3/2}} \exp\left[-\frac{z^2}{4t}\right] dz + o(1), \\ &= \inf_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z} \int_{-\delta/2t^{1/2}}^{\delta/2t^{1/2}} \frac{2}{\pi^{1/2}} \zeta^2 e^{-\zeta^2}\, d\zeta + o(1). \end{aligned}$$

Let $t \longrightarrow 0+$:

$$u(x, 0+) = 0 \geq \inf_{-\delta \leq z \leq \delta} \frac{A(x+z) - A(x)}{z}$$

Let $\delta \longrightarrow 0$:

$$0 \geq \underline{D}A(x).$$

Similarly,

$$0 \leq \overline{D}A(x)$$

for every x $(0 < x < 1)$. Now $A(x)$ is continuous, for if it had a jump it would violate one or the other of these conditions. Then by [**1**, p. 580], it must be both nonincreasing and nondecreasing, and hence constant.

Furthermore,

$$u_x(x, t) = \int_0^t G_y(x, t: 0, s)\,dB(s) + o(1) \quad \text{as} \quad (x, t) \longrightarrow (0, s).$$

Then, by [6], $B(s)$ is constant. Similarly one sees $C(s)$ is constant. This completes the proof.

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