A METHOD OF GENERAL LINEAR FRAMES IN RIEMANIAN GEOMETRY, I

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1. Introduction. In this paper we shall derive the basic quantities of Riemannian geometry, such as parallelism, curvature tensors, and so on, from a consideration of all linear frames in the various tangent spaces. This procedure has the advantage of subsuming both the classical approach through local coordinate frames and the more modern approach through orthonormal frames. The exact connection between these methods is thus made quite explicit.

The principal machinery used here is the exterior differential calculus of É. Cartan. (See [**1**, p. 201-208; **2**, p. 33-44; **3**, p. 4-6; **4**, p. 146-152; **7**, p. 3-10].) We shall follow the notation of Chern [**3**] with exceptions that we shall note in the course of the paper. It is important to keep in mind the following specific points of this calculus.

On a differentiable manifold of dimension n one has associated with each $p = 0, 1, 2, \cdots$ the linear space of exterior differential forms of degree p (p-forms). The coefficients form the ring of differentiable functions on the manifold. The 0-forms are simply the functions themselves, and the only p-form with $p > n$ is the form 0. Locally, if $u^1, \cdots, u^n$ is a local coordinate system then a one-form ω may be written

$$\omega = \sum f_i(u)\, du^i; \tag{1.1}$$

and, more generally, a p-form ω may be written

$$\omega = \sum_{(1 \le i_1 < \cdots < i_p \le n)} f_{i_1 \cdots i_p}(u)\, du^{i_1} \cdots du^{i_p} \tag{1.2}$$

$$= \frac{1}{p!} \sum f_{i_1 \cdots i_p}(u)\, du^{i_1} \cdots du^{i_p} \quad \text{with the } f_{(i)} \text{ skew-symmetric.}$$

Received June 30, 1952.
Pacific J. Math. 3 (1953), 551-565

If ω is a p-form and η a q-form, then

$$(1.3) \qquad \omega\eta = (-1)^{pq}\,\eta\omega$$

is the exterior product of ω and η, and is a $(p+q)$-form.

The operation d of exterior differentiation is intrinsically characterized by the following properties:

(A) d sends a p-form ω into a $(p+1)$-form $d\omega$;

(B) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$;

(C) $d(d\omega) = 0$;

(D) $df = \sum(\partial f/\partial u^i)du^i$, where f is a 0-form (function) and (u) is a local coordinate system;

(E) $d(\omega\eta) = d\omega\,\eta + (-1)^p\,\omega\, d\eta$, where ω is a p-form.

We shall also use matrices whose elements are differential forms. If A is such a matrix, ${}^tA$ will denote its transpose and dA will denote the matrix whose elements are obtained by applying d termwise to the elements of A. If A and B are square matrices of p-forms and q-forms respectively, then it follows from (1.3) that

$$(1.4) \qquad {}^t(AB) = (-1)^{pq}\,{}^tB\,{}^tA.$$

If A is a nonsingular matrix of functions (0-forms), then

$$(1.5) \qquad d(A^{-1}) = -A^{-1}\,dA\,A^{-1}.$$

This is the case because $AA^{-1} = I$, the identity matrix, hence

$$dA\,A^{-1} + AdA^{-1} = 0.$$

2. **Linear frames.** We shall now define the objects of this investigation. We begin with a differentiable manifold $\mathfrak{R}$ of dimension n and class C^∞. (See [**8**, p. 20].) On such a manifold one may form the space $C(\mathfrak{R})$ of all infinitely differentiable real-valued functions on $\mathfrak{R}$. If P is a point of $\mathfrak{R}$, a *tangent vector* at P is an operator $\mathbf{v}$ on $C(\mathfrak{R})$ to the reals satisfying

(A) $\mathbf{v}(f+g) = \mathbf{v}(f) + \mathbf{v}(g)$,

(B) $\mathbf{v}(fg) = f(P)\,\mathbf{v}(g) + g(P)\,\mathbf{v}(f)$, for all f, g in $C(\mathfrak{R})$.

It is well known [**5**, p. 76-78; **6**] that the set of all tangent vectors at P forms a linear space of dimension n under the usual operations of addition and scalar multiplication of operators. If $\mathfrak{U}$ is a coordinate neighborhood on $\mathfrak{R}$ with a local coordinate system $u^1, \cdots, u^n$, then the operators

$$\mathbf{e}_1 = \frac{\partial}{\partial u^1}, \cdots, \mathbf{e}_n = \frac{\partial}{\partial u^n} \tag{2.1}$$

may be considered as tangent vectors at each point P of $\mathfrak{U}$, where if f is in $C(\mathfrak{R})$ we have

$$\mathbf{e}_i(f) = (\partial f/\partial u^i)_P. \tag{2.2}$$

The vectors of (2.1) in fact form a basis for the tangent space of each P in $\mathfrak{U}$.

A *vector field* (*vector*, for short) is an assignment of a tangent vector $\mathbf{v}_P$ at P to each point P of $\mathfrak{R}$ [5, p. 82-83]. In terms of the basis (2.1), one may write a given vector field $\mathbf{v}$ on $\mathfrak{U}$ as follows:

$$\mathbf{v} = \lambda^i \mathbf{e}_i, \qquad \text{with } \lambda^i = \lambda^i(u^1, \cdots, u^n). \tag{2.3}$$

Here we use the Einstein summation convention, as we shall do in what follows. The vector field $\mathbf{v}$ is *infinitely differentiable* if each of the coordinate functions λ^i of the variables u^j is so. In the future we shall deal only with this kind of vector so that "vector field" or "vector" will always mean infinitely differentiable vector. It is important to note that that this definition is independent of the particular local coordinate system we have choosen, since a change in local coordinates merely effects a nonsingular linear transformation with C^∞ coefficients on the λ^i, in accordance with the usual tensor rules.

By a *linear frame* we shall mean a set $\mathbf{e}_1, \cdots, \mathbf{e}_n$ of vectors which form a basis for the tangent space at each point P of a given coordinate neighborhood $\mathfrak{U}$. One may visualize this as a choice of oblique coordinates in each of the tangent spaces at the various points of $\mathfrak{U}$ in such a way that the coordinate axes and units vary smoothly in moving from point to point. The vectors of (2.1) form an example of a linear frame, and we shall call such a frame a *coordinate frame* to indicate that it is derived from a local coordinate system.

The manifold $\mathfrak{R}$ is called a *Riemannian space* if it carries the following additional structure. For each P in $\mathfrak{R}$ one is given an inner product in the tangent space at P, making this space into a euclidean space. This assignment of inner products to the various tangent spaces must be infinitely differentiable in the following sense. If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors on $\mathfrak{R}$, then $\mathbf{v}\cdot\mathbf{w}$, the inner product of $\mathbf{v}$ and $\mathbf{w}$, which clearly is a point function on $\mathfrak{R}$, must be of class C^∞. This implies (and is equivalent to) the following. If $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is the coordinate frame of (2.1), then

$$\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik}(u^1, \cdots, u^n), \tag{2.4}$$

where the functions g_{ik} are C^∞ functions on $\mathfrak{U}$. In this case one custimarily writes

$$ds^2 = g_{ik}\{du^i du^k\}, \tag{2.5}$$

where $\{\ \}$ denotes the ordinary tensor product of the differentials, as distinguished from the exterior product.

An *orthonormal frame* $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a frame satisfying

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}, \qquad \text{the Kronecker } \delta. \tag{2.6}$$

If $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a frame on the space $\mathfrak{R}$, then there is uniquelly determined a (dual) basis $\sigma^1, \cdots, \sigma^n$ of the space of differential forms of degree one. This is the case because the algebraic dual of the space of tangent vectors at a point is precisely the space of 1-forms at that point. (Cf. [5, p. 81].) As is customary, we shall formally write

$$dP = \sigma^i \mathbf{e}_i, \tag{2.7}$$

and think of this *displacement vector dP* as a tangent vector whose components are differentials. (See [**1**, p. 34, 52, 101; **3**, p. 10; **6**, Chapter 2].)

3. Existence of parallel displacement. We shall now generalize the development of [**3**, § 5]. We first of all select a linear frame $\mathbf{e}_1, \cdots, \mathbf{e}_n$, and have

$$dP = \sigma^i \mathbf{e}_i, \tag{3.1}$$

where P is the variable point of $\mathfrak{U}$ and the σ^i are one-forms on $\mathfrak{U}$. We set

$$\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik}, \tag{3.2}$$

which defines a positive definite symmetric matrix $\| g_{ik} \|$ of functions on $\mathfrak{U}$.

We next wish to define differential forms ω_i^j of degree 1 so that if we set

$$d\mathbf{e}_i = \omega_i^j \mathbf{e}_j, \tag{3.3}$$

then the equations

$$d(dP) = 0, \tag{3.4}$$

$$d\mathbf{e}_i \cdot \mathbf{e}_k + \mathbf{e}_i \cdot d\mathbf{e}_k = dg_{ik} \tag{3.5}$$

will be formally satisfied. The first yields

$$d(dP) = d(\sigma^i \mathbf{e}_i) = d\sigma^j \mathbf{e}_j - \sigma^i \omega_i^j \mathbf{e}_j = 0,$$

hence

$$d\sigma^j - \sigma^i \omega_i^j = 0. \tag{3.6}$$

The second equation becomes

$$\omega_i^j g_{jk} + \omega_k^l g_{il} = d g_{ik}. \tag{3.7}$$

THEOREM 3.1. *The equations* (3.4), (3.5) *define unique* 1-*forms* ω_i^j.

Proof. It is convenient to work with covariant components. We set

$$\omega_{ir} = \omega_i^j g_{jr}, \quad \eta_r = d\sigma^j g_{jr}, \tag{3.8}$$

and our equations become

$$\sigma^i \omega_{ir} = \eta_r, \tag{3.4'}$$

$$\omega_{ik} + \omega_{ki} = d g_{ik}. \tag{3.5'}$$

The one-forms $\sigma^1, \cdots, \sigma^n$ are linearly independent, and so we may write

$$\eta_r = \frac{1}{2} h_{rst} \sigma^s \sigma^t, \tag{3.9}$$

$$dg_{ik} = c_{ikl} \sigma^l, \tag{3.10}$$

where the h_{rst} and c_{ikl} are known functions on $\mathfrak{U}$ satisfying

$$h_{rst} + h_{rts} = 0, \quad c_{ikl} = c_{kil}. \tag{3.11}$$

We seek unknown functions Γ_{ik}^j such that

$$\omega_i^j = \Gamma_{ik}^j \sigma^k, \tag{3.12}$$

or

$$\omega_{ir} = \Gamma_{irk} \sigma^k \qquad \text{with } \Gamma_{irk} = \Gamma_{ik}^j g_{jr}. \tag{3.13}$$

We now have

$$\sigma^i \omega_{ir} = \sigma^i \Gamma_{irk} \sigma^k = \frac{1}{2} (\Gamma_{irk} - \Gamma_{kri}) \sigma^i \sigma^k,$$

and so our equations (3.4′), (3.5′) become

$$\Gamma_{irk} - \Gamma_{kri} = h_{rik}, \tag{3.4″}$$

$$\Gamma_{ikl} + \Gamma_{kil} = c_{ikl}. \tag{3.5″}$$

These equations have a unique solution. To prove this, we derive as a consequence of our equations the expression

$$\begin{aligned}\Gamma_{irk} &= h_{rik} + \Gamma_{kri} = h_{rik} + c_{kri} - \Gamma_{rki} \\ &= h_{rik} + c_{kri} - h_{kri} - \Gamma_{ikr} = h_{rik} + c_{kri} - h_{kri} - c_{ikr} + \Gamma_{kir} \\ &= h_{rik} + c_{kri} - h_{kri} - c_{ikr} + h_{ikr} + \Gamma_{rik} \\ &= h_{rik} + c_{kri} - h_{kri} - c_{ikr} + h_{ikr} + c_{rik} - \Gamma_{irk}.\end{aligned}$$

This implies that the only possible solution is given by

$$\Gamma_{irk} = \frac{1}{2}(h_{rik} + h_{ikr} - h_{kri}) + \frac{1}{2}(c_{rik} + c_{kri} - c_{ikr}). \tag{3.14}$$

Substitution of this expression into the original equations (3.4′), (3.5″) shows that this indeed is a solution.

The functions Γ_{irk} are the components of the Christoffel symbols of the first kind--with respect to a general frame rather than a coordinate frame as is usual. In case of a coordinate frame (2.1) we have

$$\sigma^i = du^i, \quad d\sigma^i = 0, \quad h_{rst} = 0;$$

only the terms in the c_{rik} appear in (3.14). Since in this case

$$dg_{ik} = c_{ikl}\, du^l,$$

we have

$$c_{ikl} = \frac{\partial g_{ik}}{\partial u^l},$$

and so (3.14) is precisely the formula of Cartan [1, p.37]. In case of an orthonormal frame, the g_{ik} are constant, hence the c_{rik} all vanish; only the terms in the h_{rik} appear in (3.14). Thus formula (3.1) of [3] results. In view of these special cases and the right side of (3.14), it would appear that somehow a general frame can be decomposed into a coordinate frame and an orthonormal frame. This possibility seems worthy of further investigation.

We now can express our result in a convenient matrix form. We set

(3.15) $G = \|g_{ik}\|, \quad \mathbf{e} = {}^t(\mathbf{e}_1, \cdots, \mathbf{e}_n), \quad \sigma = (\sigma^1, \cdots, \sigma^n), \quad \Omega = \|\omega_i^k\|.$

We then have the vector equations

$$dP = \sigma \mathbf{e}, \quad d\mathbf{e} = \Omega \mathbf{e}, \quad \mathbf{e} \cdot {}^t\mathbf{e} = G, \tag{3.16}$$

and the form equations

$$d\sigma = \sigma \Omega, \quad dG = \Omega G + G\, {}^t\Omega. \tag{3.17}$$

It is perhaps well to keep in mind the relation in ordinary differentials

$$ds^2 = dP \cdot dP = g_{ik}\{\sigma^i \sigma^k\} = \{\sigma G\, {}^t\sigma\}. \tag{3.18}$$

Suppose that $\mathbf{X} = \lambda \mathbf{e}$ is a (contravariant) vector field on $\mathfrak{U}$, where $\lambda = (\lambda^1, \cdots, \lambda^n)$. We have

$$d\mathbf{X} = d\lambda \mathbf{e} + \lambda d\mathbf{e} = (d\lambda + \lambda\Omega)\, \mathbf{e}. \tag{3.19}$$

The vector field is said to be generated by parallel displacement along a subspace if the components of $d\mathbf{X}$ vanish on that subspace. Thus the condition is

$$d\lambda + \lambda\Omega = 0. \tag{3.20}$$

If $\mathbf{Y} = \mu \mathbf{e}$ is a second vector field, also generated by parallel displacement, so that

$$d\mu + \mu\Omega = 0,$$

then we have

$$\mathbf{X} \cdot \mathbf{Y} = \lambda G\, {}^t\mu,$$

hence

$$\begin{aligned} d(\mathbf{X} \cdot \mathbf{Y}) &= d\lambda\, G\, {}^t\mu + \lambda\, dG\, {}^t\mu + \lambda\, G\, {}^t d\mu \\ &= -\lambda\Omega\, G\, {}^t\mu + \lambda(\Omega G + G\, {}^t\Omega)\, {}^t\mu - \lambda\, G\, {}^t\Omega\, {}^t\mu = 0. \end{aligned}$$

This shows that parallel displacement is a euclidean transformation.

The differential forms given in (3.19) are often called the components of the *absolute differential* of the given field $\mathbf{X}$. (See [**1**, p. 38].) These are given explicitly by

$$(3.21) \qquad D\lambda = d\lambda + \lambda\Omega, \quad D\lambda = (D\lambda^1, \cdots, D\lambda^n).$$

If we express these forms in terms of the basis σ, we obtain the coefficients of the *covariant derivative* of λ:

$$(3.22) \qquad D\lambda^i = \lambda^i{}_{,j}\, \sigma^j, \quad \text{or} \quad D\lambda = \lambda, \sigma, \quad \text{where} \quad \lambda, = \|\lambda^i{}_{,j}\|.$$

One deals with covariant (form) fields and tensor fields similarly. Suppose for example that

$$\mathbf{T} = \lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j$$

is a contravariant tensor field of order two. Here

$$\mathbf{e}_i\, \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j$$

denotes the tensor product of the vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. We have

$$(3.23) \qquad d\mathbf{T} = d\lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j + \lambda^{ij}\, \omega_i^k\, \mathbf{e}_k\, \mathbf{e}_j + \lambda^{ij}\, \omega_j^l\, \mathbf{e}_i\, \mathbf{e}_l,$$

hence

$$(3.24) \qquad d\mathbf{T} = D\lambda^{ij}\, \mathbf{e}_i\, \mathbf{e}_j, \quad D\lambda^{ij} = d\lambda^{ij} + \lambda^{kj}\, \omega_k^i + \lambda^{il}\, \omega_l^j.$$

This again defines the covariant derivative

$$D\lambda^{ij} = \lambda^{ij}{}_{,k}\, \sigma^k.$$

4. Consequences; the curvature forms and the Bianchi identities. We begin with the basic relations (3.17). By differentiating the first of these, $d\sigma = \sigma\Omega$, we obtain

$$0 = d\sigma\Omega - \sigma d\Omega = \sigma(\Omega^2 - d\Omega).$$

Thus if we set

$$(4.1) \qquad \Theta = d\Omega - \Omega^2,$$

we obtain the relation

$$(4.2) \qquad \sigma\Theta = 0.$$

The elements of the matrix

$$\Theta = \|\theta_i^k\|$$

are two-forms, usually called the *curvature forms*. We set

$$\theta_i^k = \frac{1}{2} R^k_{ilm} \sigma^l \sigma^m \tag{4.3}$$

with

$$R^k_{ilm} + R^k_{iml} = 0,$$

defining the Riemann symbols of the second kind. The relation (4.2) may now be written

$$R^k_{ilm} \sigma^i \sigma^l \sigma^m = 0.$$

By expressing this 3-form in skew-symmetric canonical form, we obtain

$$R^k_{ilm} + R^k_{lmi} + R^k_{mil} = 0. \tag{4.4}$$

We next differentiate the relation (4.1) to obtain

$$d\Theta = -d\Omega\Omega + \Omega d\Omega = -(\Theta + \Omega^2)\Omega + \Omega(\Theta + \Omega^2).$$

This gives the *Bianchi relations*:

$$d\Theta = \Omega\Theta - \Theta\Omega. \tag{4.5}$$

It is easily shown that further differentiation of this relation yields nothing new.

Now let us work on the second relation,

$$dG = \Omega G + G\,{}^t\Omega,$$

of (3.17). This implies

$$\begin{aligned} 0 &= d\Omega G - \Omega dG + dG\,{}^t\Omega + G\,{}^t d\Omega \\ &= (\Theta + \Omega^2) G - \Omega(\Omega G + G\,{}^t\Omega) + (\Omega G + G\,{}^t\Omega)\,{}^t\Omega + G({}^t\Theta - ({}^t\Omega)^2); \end{aligned}$$

hence we have

$$\Theta G + G\,{}^t\Theta = 0. \tag{4.6}$$

One also verifies that differentiating this formula leads to nothing more. One now introduces the covariant components of Θ by setting

$$\theta_{ik} = \theta_i^j g_{jk}. \tag{4.7}$$

This implies

$$\theta_{ik} = \frac{1}{2} R_{iklm}\, \sigma^l\, \sigma^m \tag{4.8}$$

with

$$R_{iklm} = R^j_{ilm}\, g_{jk}.$$

These new symbols R are the Riemann symbols of the first kind (in the case of a coordinate frame) and are also called the components of the *covariant curvature tensor*. Their tensor nature will be verified in the next section. The relation (4.6) now has the simple expressions

$$\theta_{ik} + \theta_{ki} = 0, \quad R_{iklm} + R_{kilm} = 0. \tag{4.9}$$

We also have from the relations (4.3) and (4.4),

$$R_{iklm} + R_{ikml} = 0, \quad R_{iklm} + R_{lkmi} + R_{mkil} = 0. \tag{4.10}$$

On combining (4.9) with (4.10), one obtains in the usual way the symmetry relation

$$R_{iklm} = R_{lmik}. \tag{4.11}$$

5. Change of basis. Suppose that $\mathbf{e}^*$ is second frame on $\mathfrak{U}$. Then

$$\mathbf{e}^* = A\,\mathbf{e}, \tag{5.1}$$

where A is a nonsingular matrix of functions. For convenience we set $B = A^{-1}$, so that

$$dB = -B\,dA\,B.$$

The relation (5.1) implies

$$\sigma = \sigma^* A, \quad \text{or} \quad \sigma^* = \sigma B. \tag{5.2}$$

From the relation (3.16) we have

$$\mathbf{e} \cdot {}^t\mathbf{e} = G.$$

This implies

$$G^* = A\, G\, {}^tA.$$

Next we obtain the main transformation law:

THEOREM 5.1. *Under the change of basis* (5.1) *we have*

$$\Omega^* = A\Omega A^{-1} + dA\, A^{-1}. \tag{5.4}$$

Proof. According to Theorem 3.1, the matrix Ω^* is uniquely determined by the formulas

$$d\sigma^* = \sigma^* \Omega^*, \quad \Omega^* G^* + G^*\, {}^t\Omega^* = dG^*.$$

By differentiating (5.2), we obtain

$$d\sigma^* = d\sigma B - \sigma dB = \sigma \Omega B + \sigma B dA B = \sigma^*(A\Omega B + dAB),$$

which shows that the expression given in (5.4) satisfies the first of these conditions. The verification of the second condition is this:

$$\begin{aligned}(A\Omega B + dA\, B)\, AG\, {}^tA &+ AG\, {}^tA\, ({}^tB\, {}^t\Omega\, {}^tA + {}^tB\, {}^tdA) \\ &= A\Omega G\, {}^tA + dA\, G\, {}^tA + AG\, {}^t\Omega\, {}^tA + AG\, {}^tdA \\ &= dAG\, {}^tA + AdG\, {}^tA + AG\, {}^tdA = d(AG\, {}^tA) = dG^*.\end{aligned}$$

COROLLARY 1. *The curvature forms transform according to the law*

$$\Theta^* = A\,\Theta\, A^{-1} \tag{5.5}$$

Proof. We have

$$d\Omega^* = dA\Omega B + A\, d\Omega\, B + A\, d\Omega\, B + A\,\Omega B\, dA\, B + dA\, B\, dA\, B$$

and

$$\Omega^{*2} = A\,\Omega^2 B + A\Omega B\, dA\, B + dA\Omega B + dA\, B\, dA\, B,$$

hence

$$\Theta^* = d\Omega^* - \Omega^{*2} = A\, d\Omega\, B - A\Omega^2 B = A\Theta A^{-1}.$$

COROLLARY 2. *If* $\mathbf{X} = \lambda\, \mathbf{e} = \lambda^* \mathbf{e}^*$ *is a vector field on* $\mathfrak{U}$, *the following transformation laws hold*:

$$\lambda^* = \lambda A^{-1}, \quad D\lambda^* = D\lambda\, A^{-1}. \tag{5.6}$$

Proof. The first relation is simply the statement that λ satisfies the contravariant transformation law, and is obvious. The second relation is true because

$$D\lambda^* = d\lambda^* + \lambda^*\Omega^* = d\lambda\, B - \lambda B\, dA\, B + \lambda B(A\Omega\, B + dA\, B)$$
$$= d\lambda\, B + \lambda\Omega\, B = D\lambda\, B.$$

Corollary 1 asserts that the forms θ_i^j which compose the matrix Θ transform as a mixed tensor of order two. Theorem 5.1 gives the transformation law for the forms ω_i^j, and can easily be converted into a transformation law for the Christoffel symbols Γ_{ik}^j of §3. What is more important, however, is the assertion of Corollary 2, that the components $D\lambda^i$ of the absolute differential of X transform by the contravariant tensor rule. This proves incidentally that parallel displacement is intrinsic.

6. The volume element and Gaussian curvature. We set

$$\gamma = |G|^{1/2}\,\sigma^1, \cdots, \sigma^n. \tag{6.1}$$

Thus γ is a nonzero n-form on $\mathfrak{U}$. Here $|G|$ denotes the (positive) determinant of the positive-definite matrix G. It follows from equations (5.2) and (5.3) that under a change of frame we have

$$|G^*|^{1/2} = \epsilon_A\, |A| \cdot |G|^{1/2}, \qquad \sigma^1, \cdots, \sigma^n = |A|\, \sigma^{*1}, \cdots, \sigma^{*n}, \tag{6.2}$$

where

$$\epsilon_A = \operatorname{sgn} |A|.$$

Thus we have the transformation law satisfied by the *volume element* γ:

$$\gamma^* = \epsilon_A\, \gamma. \tag{6.3}$$

It is thus possible to define the volume of an orientable n-dimensional portion of $\mathfrak{R}$ by integrating γ over that portion.

We now borrow some information from the theory of skew-symmetric matrices. Let $S = ||x_{ij}||$ be a generic skew-symmetric matrix of even dimension $n = 2m$. Then there is a unique homogeneous polynomial $P(x_{ij})$ of degree m with the following properties:

(a) $|S| = [P(x_{ij})]^2$;

(b) if $S^* = AS\,{}^tA$, *where* A *is nonsingular, then*

$$P(x_{ij}^*) = |A|\, P(x_{ij});$$

(c) P *has value* 1 *for the specialization*

$$S = \begin{bmatrix} O_m & I_m \\ -I_m & O_m \end{bmatrix}$$

Now assume that our space $\mathfrak{R}$ has even dimension $n = 2m$. The matrix $H = \Theta G$ is skew-symmetric, by equations (4.7) and (4.9). Also the elements of H are 2-forms, and hence lie in the commutative ring generated by all forms of even degree. We set

$$\xi = -P(H)/|G|^{1/2}. \tag{6.4}$$

This form ξ is of degree n and is called the *Gaussian curvature form* [5]. When we combine (b) above with equation (6.2), we obtain the transformation law

$$\xi^* = \epsilon_A \xi. \tag{6.5}$$

Since γ is a nonzero n-form, and there is only one linearly independent n-form, we have

$$\xi = K\gamma, \tag{6.6}$$

where K is a function called the *Gaussian curvature.* We may combine (6.3) with (6.5) to obtain the intrinsic character of this quantity:

$$K^* = K. \tag{6.7}$$

7. A property of $|G|$. In this section we shall set

$$g = |G|.$$

The equation (5.3) then implies that

$$g^* = a^2 g,$$

where

$$a = |A|.$$

The following result is known [1, p. 44] for the classical case of a local coordinate frame.

THEOREM 7.1. *If*

$$S(\Omega) = \sum \omega_i^i$$

denotes the trace of the matrix Ω, *then*

$$\frac{1}{2}\frac{dg}{g} = S(\Omega). \tag{7.1}$$

The proof of this theorem will rest on the following known lemma.

LEMMA 7.1. *If A is a nonsingular matrix of functions, and*

$$a = |A|,$$

then

$$\frac{da}{a} = S(dA \cdot A^{-1}). \tag{7.2}$$

We shall include a short proof of this result for completeness. We set

$$C = \operatorname{cof} A, \quad B = A^{-1} = a^{-1} C.$$

Then

$$da = \sum \eta_i,$$

where η_i is the determinant formed from $|A|$ be replacing the i^{th} row of $|A|$ by the row $(da_{i1}, \cdots, da_{in})$. Thus

$$\eta_i = \sum_{j=1}^{n} (da_{ij})\, c_{ji}.$$

It follows that

$$da = \sum (da_{ij})\, c_{ji}, \quad \text{summed on } i \text{ and } j.$$

On the other hand,

$$S(dA \cdot A^{-1}) = a^{-1} S(dA \cdot C) = a^{-1} \sum (da_{ij})\, c_{ji} = a^{-1}\, da,$$

as asserted.

Proof of Theorem 7.1. We shall first show that the formula (7.1) is valid, provided that it is valid for a single moving frame. We have, under the change of frame (5.1),

$$dg^* = 2ag\ da + a^2\, dg;$$

hence

$$\frac{1}{2}\,\frac{dg^*}{g^*} = \frac{1}{2}\,\frac{dg}{g} + \frac{da}{a}.$$

Next, from equation (5.4) we have

$$S(\Omega^*) = S(\Omega) + S(dA \cdot A^{-1}).$$

It now follows from Lemma 7.1 that

$$S(\Omega^*) - \frac{1}{2}\,\frac{dg^*}{g^*} = S(\Omega) - \frac{1}{2}\,\frac{dg}{g}.$$

Finally, we note that for an orthonormal frame, Ω is skew-symmetric, hence $S(\Omega) = 0$, while $G = I$, $g = 1$, and so $g^{-1}\, dg = 0$.

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