DERIVATIVES OF INFINITE ORDER

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1. Introduction. The major purpose here is to reexamine, chiefly from the standpoint of summation by Borel's exponential means, a number of problems concerning the existence and form of

$$\lim_{n \to \infty} f^{(n)}(x),$$

for x a real variable in an interval. Several articles have been contributed on this topic [**5**, **6**, **11**, **16**], all of which take the limit process involved to be ordinary convergence. In one [**5**], however, Boas and Chandrasekharan point to the desirability of interpreting the limit process in a more general sense and state without proof that one of their results (the case $\alpha = 1$, $\lambda_n = 1$ for all n, of Theorem 4 below) can be established by their method for any (presumably linear) summation method T having the property that, as $n \longrightarrow \infty$,

(1) T-lim s_n exists and equals s implies T-lim s_{n-1} exists and equals s.

Borel's method of exponential means, like his integral method, possesses property (1) although, curiously, not its converse, as Hardy [cf. 9, pp. 183, 196] pointed out. Methods satisfying both (1) and its converse include ordinary convergence and the summation methods of Abel, Cesàro, Euler, Hölder, and, when regular (see below), Voronoi-Nörlund.

It is not clear from [**5**] just how their proof of the cited result (that $f^{(n)}(x) \longrightarrow g(x)$ dominatedly in (a, b) implies $g(x) = ke^x$) can really be carried over to *all* linear summation methods of type (1). Since the transform $\{F_m(x)\}$, m discrete or continuous, of the sequence $\{f^{(n)}(x)\}$ converges dominatedly, it follows that

$$\lim_{m \to \infty} \int_c^x F_m(t)\,dt = \int_c^x g(t)\,dt, \text{ uniformly for } c, x \text{ in } (a, b).$$

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But further argumentation is needed to justify interchanging (in the left member) the integral and whatever limit process may be involved in defining $F_m(x)$ in terms of $\{f^{(n)}(x)\}$, which would seem to be the next step in the proof. Where $F_m(x)$ is a *finite* linear combination of $f(x), \cdots, f^{(m)}(x)$, as in the Cesàro, Euler, Hölder, and Voronoi-Nörlund methods, this is trivial. In the Abel and Borel methods, for example, however, the transforms involve infinite series. The usual difficulties incident to an interchange of limits therefore intrude themselves at this point of the argument. Perhaps this difficulty can be overcome; but [5] does not suggest how.

In the case of Borel's exponential means these difficulties can be avoided and more complete results obtained otherwise by rather simple arguments which get to the heart of the problem more directly. Borel's exponential means provide a natural tool for working with the problems at hand; for, when applied to the sequence $\{f^{(n)}(x)\}$, they give rise to the Taylor expansion of $f(x)$. Repeated use can then be made of the property that the value to which the Taylor series of an analytic function converges is independent of the point around which the expansion is taken, since the hypotheses of most of the theorems below either assume or imply that $f(x)$ is analytic.

A sequence $\{s_n\}$, $n = 0, 1, 2, \cdots$, is said to be B_α*-summable* to the value s if

$$(2) \qquad \alpha \lim_{r\to\infty} e^{-r} \sum_{n=0}^{\infty} \frac{s_n}{(\alpha n)!} r^{\alpha n} = s .$$

When (2) is satisfied, it is also written as

$$(3) \qquad B_\alpha\text{-}\lim_{n\to\infty} s_n = s .$$

This method is *regular* (sometimes called *permanent*) in the sense that any sequence $\{s_n\}$ converging in the ordinary sense to a value s is also B_α-summable and to the same value s.

If $\alpha = 1$, the definition (2) describes summation by Borel's exponential means. B_1-summation is denoted simply as B-summation, and, when $\alpha = 1$, (3) is written B-lim $s_n = s$.

B_α-summation possesses property (1) when α is a positive integer, since B-summation does: Let B_α-lim $s_n = s$ and define t_k to be αs_n when $k = \alpha n$ and to be 0 otherwise. Then B-lim $t_k = s$ and, upon α applications of (1), B-lim $t_{k-\alpha} = s$. But this last is the same as asserting B_α-lim $s_{n-1} = s$, completing the proof.

2. Borel limits of the sequence of derivatives. We shall establish the following result.

THEOREM 1. *If $f(x)$ is analytic in the real interval (a, b), and if*

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x_0) = ke^{x_0}$$

for a single x_0 in (a, b), then

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x) = ke^{x}$$

for each x in (a, b). The convergence is uniform if the interval (a, b) is finite.

Proof. The function $f(x)$ can be represented by its Taylor series in (a, b), being analytic in that interval. Thus

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (t - x_0)^n \qquad \text{for } t, x_0 \text{ in } (a, b). \tag{4}$$

The power series has an infinite radius of convergence in t for x_0 in (a, b), since the existence of the Borel limit of $f^{(n)}(x_0)$ may be written (with $r=t-x_0$)

$$\lim_{t \to \infty} e^{-(t-x_0)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (t - x_0)^n = ke^{x_0}. \tag{5}$$

Thus $f(t)$, t in (a, b), possesses a unique analytic extension $\phi(t)$, and this function is an entire function. Thus (5) can be written as

$$\lim_{t \to \infty} e^{-t} \phi(t) = k. \tag{6}$$

Expanding $\phi(t)$ about an arbitrary point x in (a, b), multiplying both sides of (6) by e^x, and placing $r = t - x$ completes the proof of the theorem, except for the part dealing with uniform convergence.

To prove that the convergence is uniform when (a, b) is finite, let $\epsilon > 0$ be given and find t_0 (whose existence is assured by (6)) such that

$$|e^{-t} \phi(t) - k| < \epsilon \qquad \text{for } t > t_0.$$

Then

$$|e^{-(t-x)}\phi(t)-ke^{x}| < \epsilon e^{x} < \epsilon e^{b}$$

for $t > t_0$ and all x in (a, b), and

$$\left| e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (t-x)^n - ke^x \right| < \epsilon e^b$$

for $t > t_0$ and all x in (a, b).

Hence, putting $r = t - x$, we get

$$\left| e^{-r} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} r^n - ke^x \right| < \epsilon e^b$$

for $r > t_0 - a$ and all x in (a, b). This completes the proof.

An examination of this proof makes it clear that the point x_0 and the interval (a, b) do not have to be required to be real. What is essential is to have the quantity $t - x_0$ become positively infinite through real values, to conform to the definition of Borel summation. With this in mind, we can rephrase Theorem 1 in the following somewhat more general form:

THEOREM 1.′ *If $f(x + iy_0)$, regarded as a function of the real variable x, is analytic for $a < x < b$, y_0 fixed, and if*

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x_0 + iy_0) \textit{ exists and equals } ke^{x_0 + iy_0}$$

for a single x_0 in (a, b), then

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x + iy_0) \textit{ exists and equals } ke^{x + iy_0}$$

for each x in (a, b). The convergence is uniform if the interval (a, b) is finite.

This theorem enables one to pass from a fixed point $z_0 = x_0 + iy_0$ in the complex plane to any other point in a certain interval on the horizontal line passing through z_0. But what about points z not on this line? The proof of Theorem 1 is not adequate to cover this situation, since it must be shown that the limit in (6) exists and has the value k as $r = t - z$ becomes positively infinite through real values. (Here the complex value z replaces the real number x.) This is required by the very definition of Borel summation. In turn, moreover, this necessitates establishing that the limit (6) exists and equals k as t becomes infinite to the right, not only on the given horizontal line $y = y_0$, but also on other horizontal lines. This can be done in certain circumstances.

THEOREM 1′′. *Let $f(z)$ be analytic in S, a horizontal half-strip, quadrant, or half-plane, opening to the right:*

$$z = x + iy, \ x = a, \ c < y < d.$$

Let $f(z) = O(e^z)$ as z becomes infinite in S. Suppose that

$$B\text{-}\lim_{n\to\infty} f^{(n)}(z_0) = ke^{z_0}$$

for a single z_0 in S. Then

$$B\text{-}\lim_{n\to\infty} f^{(n)}(z) \text{ exists and equals } ke^z$$

for all z in S. If c and d are finite, then the convergence is uniform in $c + \delta \leq y \leq d - \delta$ for any positive δ. If S is a quadrant or half-plane, then the convergence is uniform in any half-strip in its interior.

Proof. In the preliminary discussion, it has been noted that only one issue needs be settled in order to extend the proof of Theorem 1 to this theorem as well: That is the existence and value of the limit in (6) as $t - z$, z an arbitrary point in S, becomes positively infinite through real values, where the imaginary parts of z and z_0 may be unequal. This limit, for z arbitrary in S, does exist and have the value k under the assumption made here that $f(z) = O(e^z)$ as $z \longrightarrow \infty$ in S. This follows from Montel's theorem [**15**, p. 170], after that theorem has been expressed in terms of the horizontal strips involved here, rather than the vertical strips used in [**15**]. The conclusion concerning uniformity is also a consequence of this formulation of Montel's theorem.

THEOREM 2. *If $f(x)$ belongs to a Denjoy-Carleman quasi-analytic class in the (open) interval (a, b) and if*

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x_0) = ke^{x_0}$$

for a single x_0 in the open interval (a, b), then $f(x)$ is analytic in (a, b) (and

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x) = ke^{x}$$

for all x, $a < x < b$).

Proof. It is sufficient to prove the first half of the conclusion, the analyticity of $f(x)$; the other half is then a consequence of Theorem 1.

As in the previous proof, the Borel summability of the sequence $\{f^{(n)}(x_0)\}$ implies that the right hand member of (4) has an infinite radius of convergence, and so defines an entire function $\phi(t)$. Expanding $\phi(t)$ in a Taylor series about the point x_0 in (a, b) shows that

$$\phi^{(n)}(x_0) = f^{(n)}(x_0) \qquad (n = 0, 1, 2, \cdots)\,.$$

The analyticity of $f(x)$ in (a, b) is a consequence of the following result of Bang [**1**, p. 84], as quoted in [**6**]: "... If $f(x)$ belongs to a quasianalytic class on $a < x < b$ and $g(x)$ is analytic, then $f^{(n)}(x_0) = g^{(n)}(x_0)$ for all n and $a < x_0 < b$ implies $f(x) \equiv g(x) \cdots$." This completes the proof.

The next theorem provides a simple set of necessary and sufficient conditions on the structure of $f(x)$ as well as on that of $g(x)$. That these conditions are not sufficient if convergence is used instead of Borel summation is shown by the example

$$f(x) = ke^{x} + \sin x\,.$$

The Borel limit of the sequence of derivatives exists and equals ke^{x} for all x, whereas the (convergence) limit of this sequence does not even exist. Analyticity is not assumed in the necessity part of the theorem, but is inferred as in Theorem 1 of [5].

THEOREM 3. *A set of necessary and sufficient conditions that*

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x) = g(x)$$

for each x in (a, b), where $g(x)$ is finite, is (i) *that $f(x)$ coincide in (a, b)*

with an entire function $\phi(x)$, *having the property that*

$$\phi(x) = ke^x + o(e^x),$$

as x *becomes infinite, and* (ii) *that*

$$g(x) = ke^x, \ x \text{ in } (a, b).$$

Proof of sufficiency. Here

$$\phi(t) = ke^t + o(e^t); \quad \phi(t) = f(t), \text{ for } t \text{ in } (a, b),$$

and $\phi(t)$ is an entire function. Then

$$e^{-(t-x)}\phi(t) = e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(t-x)^n, \ x \text{ in } (a, b).$$

By hypothesis,

$$\lim_{t\to\infty} e^{-(t-x)}\phi(t) = ke^x,$$

whence, with $r = t - x$,

$$\lim_{r\to\infty} e^{-r} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} r^n = ke^x,$$

completing the proof of sufficiency.

Necessity. Putting $r = t - x$, we can write the assumption of Borel summability as follows:

$$\lim_{t\to\infty} e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(t-x)^n = g(x) \qquad \text{for each } x \text{ in } (a, b).$$

This implies that the radius of convergence of the power series above is infinite for each x in (a, b). Hence $f(t)$ is analytic in (a, b), as a consequence of a theorem of Pringsheim [13] for which a complete proof was supplied first by Boas [4] and again later by Zahorski [17]. In fact, $f(t)$ has as analytic continuation an entire function, $\phi(t)$. Then

$$\lim_{t \to \infty} e^{-(t-x)} \phi(t) = g(x) \qquad \text{for each } x \text{ in } (a, b),$$

whence

$$\lim_{t \to \infty} e^{-t} \phi(t) = e^{-x} g(x) \qquad \text{for each } x \text{ in } (a, b).$$

The left side is independent of x since $\phi(t)$ is, and this is the case because the values of an analytic function do not depend on the point in the region of analyticity around which the function is expanded. Hence the right side must be a constant k. This completes the proof.

3. Subsequences of $\{f^{(n)}(x)\}$. For the proof of the theorem below, the following lemma is needed. The proof given first is due to Julian H. Blau.

LEMMA 1. *If a sequence of polynomials,* $\{P_n(x)\}$, *defined in the closed interval* $[c, d]$, *each of which is of degree at most* β, *has a limit* $h(x)$ *in* $[c, d]$, *then this limit is likewise a polynomial of degree at most* β.

Proof of lemma (by induction). Let each $P_n(x)$ be written as a polynomial in $x - c$.

(i) The lemma is obvious for $\beta = 0$.

(ii) Assume that the result is valid for all integers γ, $0 \leq \gamma < \beta$. Let $\{P_n(x)\}$ be a convergent sequence of polynomials of degree at most $\gamma + 1$. Then

$$P_n(x) - P_n(c) \longrightarrow h(x) - h(c).$$

The left side is divisible by $x - c$, giving a sequence $\{Q_n(x)\}$ of polynomials of degree at most γ, and

$$Q_n(x) = \frac{P_n(x) - P_n(c)}{x - c} \longrightarrow \frac{h(x) - h(c)}{x - c} \qquad (x \neq c).$$

From the induction hypothesis, the right member is a polynomial of degree at most γ. Hence $h(x)$ is a polynomial of degree at most $\gamma + 1$. This completes the induction.

The referee suggests the following alternative proof of the lemma: If $P_n(x)$ converges pointwise, so does $\Delta^{\beta+1} P_n(x)$; but these differences are all zero, and so $\Delta^{\beta+1} h(x) = 0$ (for all spans). It is well known that the polynomials of degree $\leq \beta$ are characterized among measurable functions by the property of having vanishing $(\beta + 1)$th differences; and $h(x)$ is even of the first Baire class.

He also comments that the lemma is well known, but that, like the author, he can think of no specific reference.

The case $\alpha = 1$, $\lambda_n = 1$ (all n) of Theorem 4 below is proved in the opening remarks of [5]. Theorem 3 of [5] is also included in Theorem 4 below, which gives somewhat more precise information than is formulated in the statement of Theorem 3 of [5], even for the case $\alpha = 1$, which is the case analyzed in Theorem 3 of [5]. The proof below is fashioned after that of the latter theorem.

THEOREM 4. *Let $\{\lambda_n\}$ be a given sequence of constants; let α be a fixed positive integer; and let*

$$\lim_{n\to\infty} \frac{f^{(\alpha n)}(x)}{\lambda_n} = g(x) \text{ dominatedly in } a \leq x \leq b. \tag{7}$$

Then the following statements are true for $a \leq x \leq b$.

(i) *If*

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 0,$$

then $g(x) = 0$ almost everywhere. If (7) holds uniformly, then $g(x) \equiv 0$.

(ii) *If*

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = L \neq 0,$$

L finite, then $Lg^{(\alpha)}(x) = g(x)$.

(iii) *If the sequence $\{\lambda_{n-1}/\lambda_n\}$ has an infinite limit-point, then $g(x) = P_{\alpha-1}(x)$, where $P_{\alpha-1}(x)$ is a polynomial whose degree does not exceed $\alpha - 1$.*

(iv) *If the sequence $\{\lambda_{n-1}/\lambda_n\}$ has at least two limit-points, of which at least one is finite, then $g(x) \equiv 0$.*

Proof. The common hypothesis gives the following extension of (3) of

[5] in all four cases, since the sequence obtained by integrating a dominatedly convergent sequence converges uniformly [10, p. 290, p. 304], whence successive termwise integrations are valid for x, c in $[a, b]$:

$$\lim_{n\to\infty}\left[\frac{\lambda_{n-1}}{\lambda_n}\left\{\frac{f^{(\alpha n-\alpha)}(x)}{\lambda_{n-1}}-\frac{f^{(\alpha n-\alpha)}(c)}{\lambda_{n-1}}\right\}\right.$$

$$\left.-\frac{1}{\lambda_n}\frac{f^{(\alpha n-1)}(c)}{(\alpha-1)!}(x-c)^{\alpha-1}-\cdots-\frac{f^{(\alpha n-\alpha+1)}(c)}{\lambda_n}(x-c)\right] \tag{8}$$

$$=\int_c^x\int_c^{x_\alpha}\cdots\int_c^{x_2} g(x_1)\,dx_1\cdots dx_\alpha.$$

Moreover,

$$\frac{f^{(\alpha n-\alpha)}(x)}{\lambda_{n-1}}\longrightarrow g(x),\quad \frac{f^{(\alpha n-\alpha)}(c)}{\lambda_{n-1}}\longrightarrow g(c),$$

since $s_n\longrightarrow s$ implies $s_{n-1}\longrightarrow s$.

To prove (i), note that the first term of the left member of (8) approaches zero. Then, from Lemma 1, the combined remaining terms have as their collective limit a polynomial $P_{\alpha-1}(x)$ whose degree does not exceed $\alpha-1$. Differentiating both sides of (8) $\alpha-1$ times, under these circumstances, shows that $\int_c^x g(t)\,dt$ is constant for all x in $[a, b]$, whence $g(x)=0$ almost everywhere, as asserted in the first part of (i). If (7) holds uniformly, then $g(x)$ is continuous and hence identically zero.

To prove (ii), note that (8) becomes, as above,

$$L\{g(x)-g(c)\}-P_{\alpha-1}(x)=\int_c^x\int_c^{x_\alpha}\cdots\int_c^{x_2} g(x_1)\,dx_1\cdots dx_\alpha.$$

Differentiating both sides α times with respect to x completes the proof of (ii).

To prove (iii), rewrite (8) by using λ_{n-1}/λ_n as a factor of all the terms within the brackets and not just of the terms in the braces. Then the (new) expression inside the brackets must approach zero (since the right member of (8) is finite) as n becomes infinite through a subsequence for which the corresponding λ_{n-1}/λ_n becomes infinite. Using Lemma 1 again shows that

$$g(x) - g(c) - P_{\alpha-1}(x) = 0;$$

and, of course, $g(c)$ can be absorbed in $P_{\alpha-1}(x)$, completing the proof of (iii).

To prove (iv), consider first the case in which there are exactly two limit-points, one of which is zero. The presence of the zero limit-point implies (by use of an appropriate subsequence of $\{\lambda_{n-1}/\lambda_n\}$ in the proof of (i)) that $g(x) = 0$ almost everywhere. The other limit-point may be finite or infinite. If finite, the same modification is introduced into the proof of (ii), showing $g(x)$ to be continuous. If infinite, (iii) applies directly, again showing $g(x)$ to be continuous. Hence, in this case, $g(x) \equiv 0$.

In the remaining ("general") case of (iv), there is a finite nonzero limit-point L, whence, modifying (ii) as above, we obtain

$$Lg^{(\alpha)}(x) = g(x) \tag{9}$$

and *either* another finite nonzero limit-point M, implying

$$Mg^{(\alpha)}(x) = g(x)$$

with $L \neq M$, *or* an infinite limit-point, in which eventuality $g(x)$ is a polynomial whose degree does not exceed $\alpha - 1$, from (iii). Comparing either of these alternatives for $g(x)$ with (9) shows that $g(x) = 0$.

This completes the proof of (iv) and of the theorem.

Theorem 4 (iv) does not exclude the possibility that

$$\liminf \left| \frac{\lambda_{n-1}}{\lambda_n} \right|$$

may be zero. For the case $\alpha = 1$, therefore, it overlaps—and partially generalizes—Theorem 3(i) of [5] in which it is assumed, instead of (7), that

$$\frac{f^{(n)}(x)}{\lambda_n} \longrightarrow g(x)$$

uniformly in $[a, b]$, as in Theorem 4(i) here, in order to infer that $g(x) \equiv 0$.

This casts further light on the significance of counter-examples connected with Theorem 3(i) of [5] (which is the case $\alpha = 1$ of Theorem 4(i) above).

One is due to Boas and Chandrasekharan [5], another to Bang [1], described also in the final paragraph of [6]. Each exhibits a sequence $\{f^{(n)}(x)/\lambda_n\}$ converging dominatedly to $g(x)$ in $[a, b]$ with $\lim (\lambda_{n-1}/\lambda_n) = 0$ and $g(x)$ *not* identically zero there, although, of course, it is zero almost everywhere. In their examples, in fact, $g(x)$ is zero except for a single point.

In addition to the examples due to these authors, Philip Davis has called attention to earlier constructions [2a; 3; **7**, pp. 38-42; **8**; **12**, p. 244; **14**] of functions differentiable infinitely often on an interval and analytic on that interval except for one or more interior points at which the successive derivatives increase arbitrarily rapidly. Taking λ_n to be the nth derivative at a singular point converts these constructions into examples of the phenomenon described above.

R. P. Boas, who transmitted Davis's information to the author, added a reference to another exposition [2b] of S. Bernstein's examples.

Theorem 4(iv) shows, *i. a.*, that it is impossible to construct similar counter-examples in which the condition on the λ_n's is weakened to

$$\liminf \left| \frac{\lambda_{n-1}}{\lambda_n} \right| = 0$$

with $\lim (\lambda_{n-1}/\lambda_n)$ nonexistent.

This last remark can be inferred also from a consideration of formula (3) of [5], which is valid for dominatedly convergent sequences and which reads as follows:

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} \left\{ \frac{f^{(n-1)}(x)}{\lambda_{n-1}} - \frac{f^{(n-1)}(c)}{\lambda_{n-1}} \right\} = \int_c^x g(t)\,dt,\ a < c < b.$$

Choose c to be a point such that $g(c) \neq 0$, x a point at which $g(x) = 0$. The right member is zero, since $g(x) = 0$ almost everywhere. Thus

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} \quad g(c) = 0,\ g(c) \neq 0,$$

whence

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 0.$$

When $\lambda_n = 1$ for all n, Theorem 4 (of which only part (ii) is now relevant) can be extended readily to certain summation methods. Consider the transformation

$$(10) \qquad T: t_r(x) = \sum_{n=0}^{\infty} c_n(r) s_n(x),$$

where r is continuous or discrete.

DEFINITION. The transformation T of (10) will be said to be of *dominated type* in the interval (a, b) with respect to a sequence of Lebesgue integrable functions $\{s_n(x)\}$, defined in (a, b), if the infinite series (10) taking the sequence $\{s_n(x)\}$ into $t_r(x)$ converges dominatedly (in the sense that all its partial sums are uniformly less, in absolute value, than a fixed Lebesgue integrable function) in (a, b) for each sufficiently large r.

Any row-finite or row-bounded matrix transformation is of dominated type with respect to all sequences of Lebesgue integrable functions. This includes all Hausdorff and Voronoi-Nörlund methods, in particular Cesàro's and Euler's. All regular (or even merely convergence-preserving) transformations given by (10) are of dominated type with respect to any sequence of Lebesgue integrable functions dominated as a whole by a single Lebesgue integrable function.

LEMMA 2. *Let T be a summation method of dominated type with respect to the sequence of Lebesgue integrable functions $\{s_n(x)\}$ in (a, b). Suppose that $\{s_n(x)\}$ is dominatedly T-summable in (a, b) to $s(x)$. Then*

$$(11) \qquad T\text{-}\lim \int_c^x s_n(t)\,dt = \int_c^x s(t)\,dt,$$

uniformly for c, x in (a, b).

Proof. The transformation T being of dominated type, it follows [**10**, pp. 290, 304] as in the justification of (8), that

$$\sum_{n=0}^{\infty} c_n(r) \int_c^x s_n(t)\,dt = \int_c^x \sum_{n=0}^{\infty} c_n(r) s_n(t)\,dt,$$

uniformly for c, x in (a, b), for each sufficiently large r. In turn, the right member approaches the right member of (11) uniformly for c, x in (a, b) as

$r \longrightarrow \infty$, since the integrand approaches $s(t)$ dominatedly. The left member is the T-transform of the integral of $s_n(t)$. Hence the lemma is established.

THEOREM 5. *Let T be a summation method satisfying* (1) *and of dominated type with respect to the sequence* $\{f^{(\alpha n)}(x)\}$, *x in* (a, b), *where α is a fixed positive integer. If*

$$T\text{-lim } f^{(\alpha n)}(x) = g(x),$$

dominatedly in (a, b), *as $n \longrightarrow \infty$, then $g(x)$ satisfies the differential equation* $g^{(\alpha)}(x) = g(x)$ *in* (a, b).

Proof. By α applications of Lemma 2 we obtain

$$\lim_{r \to \infty} \sum_{n=0}^{\infty} c_n(r)\, [f^{(\alpha n - \alpha)}(x) - f^{(\alpha n - \alpha)}(c)]$$

$$- \lim_{r \to \infty} \sum_{n=0}^{\infty} c_n(r) \left[f^{(\alpha n - \alpha + 1)}(c) \frac{x - c}{1!} + \cdots + f^{(\alpha n - 1)}(c) \frac{(x - c)^{\alpha - 1}}{(\alpha - 1)!} \right]$$

$$= \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1)\, dx_1 \cdots dx_\alpha$$

uniformly for c, x in (a, b). Lemma 2 actually gives the existence and value of the limit of the difference of the two sums, rather than the difference of the limits of the individual sums, as written above. However, once the existence of the first limit above is established, that of the second is immediate.

Writing $\alpha n - \alpha$ as $\alpha(n - 1)$, we see from (1) that the first limit exists and is $g(x) - g(c)$. Lemma 1, with $\beta = \alpha - 1$, shows that the second limit, whose existence is now assured, is a polynomial in $x - c$ of degree at most $\alpha - 1$, say $P_{\alpha - 1}(x - c)$, vanishing for $x = c$. Then

$$g(x) - g(c) - P_{\alpha-1}(x - c) = \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1)\, dx_1 \cdots dx_\alpha .$$

Continuity and then α-fold differentiability follow from this equation. Differentiating α times completes the proof.

Some open questions. If $\lim f^{(\alpha n)}(x)$, $n \longrightarrow \infty$, α a fixed positive integer, exists, and is finite for each x in (a, b), then must the convergence necessarily be dominated or perhaps even bounded or uniform? If this is not the case for general indefinitely differentiable functions, would it be true for $f(x)$ in a quasi-analytic class? If not then, what if $f(x)$ is analytic? If $\alpha = 1$, then the answer to the first (and hence to all) of these questions is affirmative. If the answer to any of these questions is affirmative for other α, it would then follow, from Theorem 4(ii), that the limit, $g(x)$, satisfies the differential equation $g^{(\alpha)}(x) = g(x)$. Similar questions can be framed for more general sequences of λ_n's.

REFERENCES

1. T. Bang, *Om quasi-analytiske Funktioner*, Copenhagen Thesis, 1946.

2a. S. Bernstein, *Sommation des séries de Taylor partout divergentes*, Appendix to R. D'Adhémar, *Leçons sur les principes de l'analyse*, vol. II, Paris, 1913, 272-275.

2b. S. Bernstein, *Summation of everywhere divergent Taylor series*, (in Russian), Сообщения Харьковского Математического Общества (Communications de la Société mathématique de Kharkow) (2) **13** (1913), 195-199. Reprinted in his *Собрание Сочинений* (*Collected Works*), vol. I, Moscow, 1952, 107-111.

3. A. Besikowitsch, *Über analytische Funktionen mit vorgeschriebenen Werten ihrer Ableitungen*, Math. Z. **21** (1924), 111-118.

4. R. P. Boas, Jr., *A theorem on analytic functions of a real variable*, Bull. Amer. Math. Soc. **41** (1935), 233-236.

5. R. P. Boas and K. Chandrasekharan, *Derivatives of infinite order*, Bull. Amer. Math. Soc. **54** (1948), 523-526; *Correction*, ibid., 1191.

6. R. P. Boas and K. Chandrasekharan, *Addendum: Derivatives of infinite order*, Proc. Amer. Math. Soc. **2** (1951), 422.

7. É. Borel, *Sur quelques points de la théorie des fonctions*, Ann. Sci. Ecole Norm. Sup. (3) **12** (1895), 9-55.

8. P. Franklin, *Functions of a complex variable with assigned derivatives at an infinite number of points, and an analogue of Mittag-Leffler's theorem*, Acta Math. **47** (1926), 371-385.

9. G. H. Hardy, *Divergent series*, Oxford, 1949.

10. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, 2nd edition, vol. II, Cambridge, 1926.

11. V. Ganapathy Iyer, *On singular functions*, J. Indian Math. Soc. N.S. **8** (1944), 94-108.

12. G. Pólya, *Eine einfache, mit funktionen-theoretischen Aufgaben verknüpfte, hinreichende Bedingung für die Auflösbarkeit eines Systems unendlich vieler linearer Gleichungen*, Comment. Math. Helv. **11** (1939), 234-252.

13. A. Pringsheim, *Zur Theorie der Taylor'schen Reihe und der analytischen Funktionen mit beschränktem Existenzbereich*, Math. Ann. **42** (1893), 153-184.

14. J. F. Ritt, *On the derivatives of a function at a point*, Ann. of Math. (2), **18** (1916), 18-23.

15. E. C. Titchmarsh, *The theory of functions*, 2nd edition, Oxford, 1939.

16. G. Vitali, *Sui limiti per* $n=\infty$ *delle derivate* n^{ma} *delle funzioni analitiche*, Rend. Circ. Mat. Palermo **14** (1900), 209-216.

17. Z. Zahorski, *Sur l'ensemble des points singuliers d'une fonction d'une variable réelle admettant les dérivées de tous les ordres*, Fund. Math. **34** (1947), 183-245; *Supplément*, ibid., **36** (1949), 319-320.

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