METHODS OF SUMMATION

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1. Methods of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise § 1 and § 2 are unrelated. In this section we wish to consider some of the properties of the method (B^h), where we say that a series $\sum_{\nu=0}^{\infty} u_\nu$ is summable (B^h) when

$$B_n^h = \sum_{\nu=0}^{n} u_\nu \cos \frac{\pi}{2} \left(\frac{\nu}{n+h}\right) \longrightarrow S, \ n \longrightarrow \infty.$$

The method (B^h) has been the subject of recent papers by Agnew [**1**], Karamata [**5**, **6**], and Petersen [**7**]. It has been shown in the papers by Agnew and Petersen that for $h > 1/2$ the method (B^h) is equivalent to the arithmetic means of Cesaro (C), and in the paper by Agnew that for $0 < h < 1/2$ the method is equivalent to methods stronger than (C).

We shall now construct examples after a method of Hurwitz [**4**], to show that for $h < 0$ the method (B^h) sums a series not summable (C). Hence, since all series summable (C) are summable (B^h), we shall have proved that (B^h) is stronger than (C).

We shall first consider $-1 < h < 0$, so that all the coefficients in any row are positive except the nth coefficient $\cos \{\pi n/[2(n+h)]\}$. We choose $u_0 > 1$ and assume that the first $m-1$ terms of the series $\sum_{\nu=0}^{\infty} u_\nu$ are known. Then we select u_m so that

$$B_m^h = \sum_{\nu=0}^{m} u_\nu \cos \frac{\pi}{2} \left(\frac{\nu}{m+h}\right) = 0,$$

or

$$-u_m \cos \frac{\pi}{2} \left(\frac{m}{m+h}\right) = \sum_{\nu=0}^{m-1} u_\nu \cos \frac{\pi}{2} \left(\frac{\nu}{m+h}\right).$$

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All of the u_ν are positive; and since

$$\frac{u_m}{u_{m-2}} \geq \frac{\sin \frac{\pi}{2}\left(\frac{2+h}{m+h}\right)}{-\sin \frac{\pi}{2}\left(\frac{h}{m+h}\right)} \simeq -\left(\frac{2}{h}+1\right)$$

for $-1 < h < 0$, the u_ν do not satisfy $u_n = o(n)$, and hence $\sum_{\nu=0}^{\infty} u_\nu$ is not summable (C); see [3].

If $h \leq -1$, we consider

$$B_m^h = \sum_{\nu=0}^{m-1}\left[\cos \frac{\pi}{2}\left(\frac{\nu}{m+h}\right) - \cos \frac{\pi}{2}\left(\frac{\nu+1}{m+h}\right)\right] S_\nu + \cos \frac{\pi}{2}\left(\frac{m}{m+h}\right) S_m .$$

Here again we select positive increasing S_ν so that $B_\nu^h = 0$ for $\nu \leq m-1$. Under the assumption that $S_\nu \geq \nu$, $\nu \leq m-1$, we shall show that $S_m \geq m$. Observing that the first $m-1$ coefficients of the S_ν are positive, we have (setting $\pi/[2(m+h)] = \theta$):

$$\begin{aligned}
-\cos m\theta &\geq \sum_{\nu=0}^{m-1} [\cos \nu\theta - \cos(\nu+1)\theta]\nu \\
&= \sum_{\nu=0}^{m-1} \cos \nu\theta - (m-1)\cos m\theta \\
&= \Re \sum_{\nu=0}^{m-1} e^{i\nu\theta} - (m-1)\cos m\theta \\
&= \Re \frac{1-e^{im\theta}}{1-e^{i\theta}} - (m-1)\cos m\theta \\
&= \Re \frac{i(e^{-(i\theta)/2} - e^{i(m-1/2)\theta})}{2\sin \theta/2} - (m-1)\cos m\theta \\
&\geq \left(\frac{1}{2} - \frac{\pi}{2}h\right);
\end{aligned}$$

therefore,

$$S_m \geq \left(\frac{1}{2} - \frac{\pi}{2} h\right) \frac{m+h}{-h} \times \frac{2}{\pi} \geq qm, \; q > 1.$$

Hence the series constructed does not satisfy the condition $S_n = o(n)$, and is not summable (C).

2. A Nörlund method. The method defined by

$$\sigma_n = \left(1 - \frac{1}{n+3}\right) S_n + \frac{1}{n+3} S_{n+1}$$

has been used as an example in a recent paper by Agnew [2]. We shall treat this method in a manner similar to that in which the method

$$t_n = (1-a) S_{n-1} + a S_n$$

is treated in [7].

THEOREM. *If*

$$\sigma_n = \left[\left(1 - \frac{1}{n+3}\right) S_n + \frac{1}{n+3} S_{n+1}\right] \longrightarrow \sigma,$$

then

$$S_n = C \cdot (-1)^{n-1} (n+1)! + \sigma_n',$$

where σ_n' is convergent to σ and C is a constant.

Proof. Since (we may assume $S_0 = 0$)

$$\begin{aligned}
(n+2)\,\sigma_{n-1} &= (n+1)\,S_{n-1} + S_n \\
(n+1)\,\sigma_{n-2} &= n\,S_{n-2} + S_{n-1} \\
\cdot \qquad & \qquad \cdot \qquad \cdot \\
\cdot \qquad & \qquad \cdot \qquad \cdot \\
\cdot \qquad & \qquad \cdot \qquad \cdot \\
3\,\sigma_0 &= 2\,S_0 + S_1 ,
\end{aligned}$$

we have

$$S_n = (n+2)\sigma_{n-1} - (n+1)^2\sigma_{n-2} + n^2(n+1)\sigma_{n-3}$$

$$- (n-1)^2 n(n+1)\sigma_{n-4} + \cdots + (-1)^{n-2} 3^2 \cdot 4 \cdot 5 \cdot 6 \cdots (n+1)\sigma_0,$$

or

$$S_n = (-1)^{n-1}(n+1)!\left[(-1)^{n-1}\frac{n+2}{(n+1)!}\sigma_{n-1} + (-1)^{n-2}\frac{n+1}{n!}\sigma_{n-2} + \cdots\right.$$

$$\left. + (-1)^{\nu}\frac{\nu+3}{(\nu+2)!}\sigma_{\nu} + \cdots + \frac{3}{2}\sigma_0\right].$$

Let

$$(-1)^{\nu}\frac{\nu+3}{(\nu+2)!}\sigma_{\nu} = t_{\nu};$$

since $\sum_{\nu=0}^{\infty} t_{\nu}$ is absolutely convergent ($\sigma_{\nu} \longrightarrow \sigma$), we may write

$$t_0 + t_1 + \cdots + t_{n-1} = C - (t_n + t_{n+1} + \cdots)$$

$$= C - \frac{1}{(n+1)!}\left[\frac{n+3}{n+2}\,\frac{(n+2)!}{n+3}\,t_n + \frac{n+4}{(n+2)(n+3)}\,\frac{(n+3)!}{n+4}\,t_{n+1} + \cdots\right]$$

$$= C - \frac{(-1)^n}{(n+1)!}\left[\frac{n+3}{n+2}\sigma_n - \frac{n+4}{(n+2)(n+3)}\sigma_{n+1} + \cdots\right].$$

Then

$$S_n = (-1)^{n-1}(n+1)!\,[t_0 + t_1 + \cdots + t_{n-1}]$$

$$= (-1)^{n-1}\cdot C\cdot(n+1) + \left[\frac{n+3}{n+2}\sigma_n - \frac{n+4}{(n+2)(n+3)}\sigma_{n+1} + \cdots\right]$$

$$= (-1)^{n-1}\cdot C\cdot(n+1)! + \frac{n+3}{n+2}\sigma_n$$

$$- \frac{1}{n+2}\left[\frac{n+4}{n+3}\sigma_{n+1} - \frac{n+5}{(n+3)(n+4)}\sigma_{n+2} + \cdots\right]$$

$$= (-1)^{n-1}\cdot C\cdot(n+1)! + \frac{n+3}{n+2}\sigma_n - \frac{1}{n+2}\,O(1)$$

$$= (-1)^{n-1} \cdot C \cdot (n+1)! + \sigma_n + o(1).$$

This proves our assertion.

Obvious extensions can be made to the methods

$$\sigma_n = \left[\left(1 - \frac{1}{n+k}\right) S_n + \frac{1}{n+k} S_{n+1}\right],$$

or to iterations of these methods.

REFERENCES

1. R. P. Agnew, *Rogosinski-Bernstein trigonometric summability methods and modified arithmetic means*, Ann. of Math. (2) 56 (1952), 537-59.

2. ———, *Equivalence of methods for evaluation of sequences*, Proc. Amer. Math. Soc. 3 (1952), 550-56.

3. G. H. Hardy, *Divergent series*, Oxford, 1949.

4. W. A. Hurwitz, *Totally regular methods of summation*, Proc. London Math. Soc. (2) 26, (1926), 231-48.

5. J. Karamata, *Sur la sommabilité de S. Bernstein et quelques procédés de sommation qui s'y rattachent*, Mat. Sobornik, N.S. 21 (63) (1947), 13-24.

6. J. Karamata, *Über die Beziehung zwischen dem Bernsteinschen und Cesàroschen Limitierungsverfahren*, Math. Z. 52 (1949), 305.

7. G. M. Petersen, *A note on divergent series*, Canadian J. Math. 4 (1952), 445-54.

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