ON THE GROWTH OF FUNCTIONS HAVING POLES OR ZEROS ON THE POSITIVE REAL AXIS

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1. Introduction. Let Γ denote the boundary of a region Δ containing a right half of the real axis, and $f(z)$ a function holomorphic in Δ except possibly for simple poles on the positive real axis. Through the residue theorem, the integral

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)e^{-zs}\,dz$$

makes correspond to $f(z)$ a Dirichlet series whose coefficients are the residues of $f(z)$ at its poles. It is the purpose of this paper to exhibit some immediate applications, using this familiar device, of the theory of asymptotic Dirichlet series to a study of the growth of functions with poles of bounded order, or zeros of at least a certain order, on the positive real axis. Let the poles, or zeros, occur at points $z = \lambda_n$, where $0 < \lambda_n \uparrow \infty$. Theorems 2A and 2B, in §3, relate the growth of leading coefficients in the Laurent developments about the points $z = \lambda_n$ to the growth of $f(z)$ in Δ. Theorems 3A and 3B, in §4, apply to functions of exponential type in the right half-plane outside Δ, satisfying much weaker growth conditions in Δ. Theorem 3A may be thought of as stating that such a function, with a specified rapidity of decrease on the imaginary axis, is holomorphic if it does not have poles of too great order; while Theorems 3A and 3B together may be regarded as saying that if such a function has "zeros" of minimum order too great for its growth on the imaginary axis, poles counting as negative zeros, then it vanishes identically. (The emphasis on this aspect of Theorem 3A and this interpretation of the uniqueness conclusions of Theorems 3A and 3B were suggested by the referee.)

2. Definitions, the fundamental inequality, and properties of the function $C(z)$. Let $\{\lambda_n\}$ $(n = 1, 2, \cdots)$ be a sequence of positive numbers, strictly increasing without bound $(0 < \lambda_n \uparrow \infty)$, such that

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D^* < \infty. \tag{1}$$

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Let

$$\Lambda_n^* = \prod_{k \neq n} \left| \frac{\lambda_k^2}{\lambda_k^2 - \lambda_n^2} \right|. \tag{2}$$

Let Δ denote a region in the complex z-plane $(z = x + iy)$ bounded by

$$y = g^+(x),\ y = -g^-(x),\ \text{and}\ x = a\ (0 \leq a < \lambda_1),$$

where $g^+(x)$ and $g^-(x)$ are defined for $x \geq a$, positive for $x > a$, continuous, bounded, and satisfy Lipschitz conditions: there exists a constant K such that

$$\begin{aligned} |g^+(x + \Delta x) - g^+(x)| &< K|\Delta x|, \\ |g^-(x + \Delta x) - g^-(x)| &< K|\Delta x|, \end{aligned} \tag{3}$$

for $x > a$, $x + \Delta x > a$. Denote by Γ the complete boundary of Δ, by Γ^+ the part of Γ in the upper half-plane, and by Γ^- the part of Γ in the lower half-plane.

Let $\{M_n\}$ $(n = 1, 2, \cdots)$ be a sequence of positive constants, and set

$$p(\sigma) = \sup_{n \geq 1} (\lambda_n \sigma - \log M_n).$$

Following the notation introduced by Mandelbrojt [6], we shall say that the sequence $\{M_n\}$ has the *uniqueness property* (U) if, for some positive number η,

$$\int^{\infty} p(\sigma) e^{-\eta\sigma}\, d\sigma = \infty. \tag{U}$$

(The absence of the lower limit indicates that the relation is to hold provided the lower limit is sufficiently large.) If, in particular,

$$\liminf_{n \to \infty} \frac{\log M_n}{\lambda_n} < \infty,$$

then $p(\sigma) \equiv \infty$ for σ sufficiently large, so that (U) holds. If

$$\lim_{n \to \infty} \cdot \frac{\log M_n}{\lambda_n} = \infty,$$

let the sequence $\{M_n^c\}$ $(n = 1, 2, \cdots)$ be defined so that $\log M_n^c$ is the greatest monotone, nondecreasing, convex function of n which is not greater than $\log M_n$.

Mandelbrojt has shown [6; cf. also 7, § 1.8, III-V] that condition (U) is equivalent to

$$(U)' \qquad \sum^{\infty} \left(\frac{M_n^c}{M_{n+1}^c} \right)^{\eta/(\lambda_{n+1}-\lambda_n)} [\lambda_{n+1} - \lambda_n] = \infty$$

for some positive η.

If n is a positive integer, the sums $\sum_{k=1}^{m} a_k \exp(-\lambda_k s)$ $(m \geq n)$ are said to represent a function $F(s)$ with *logarithmic precision* $p_n(\sigma)$ in a region containing points with arbitrarily large real part if for s in that region and for x sufficiently large,

$$(4) \qquad \inf_{m \geq n} \sup_{\sigma \geq x} \left| F(s) - \sum_{k=1}^{m} a_k \exp(-\lambda_k s) \right| \leq \exp(-p_n(x)).$$

Mandelbrojt has proved [7, § 3.7] a theorem relating the growth of the function $F(s)$ to the growth of the coefficients a_n. This theorem contains as a very special case the following theorem, sufficient for the needs of the present paper.

THEOREM 1. *Let the sequence* $\{\lambda_n\}$ *satisfy* (1). *Let* σ_0 and σ_1 *be real numbers, and* R *a positive number greater than* D^*, *such that* $F(s)$ $(s = \sigma + it)$ *is holomorphic in the half-plane* $\sigma \geq \sigma_0$, *and continuable analytically throughout the strip*

$$S : |t| \leq \pi R, \quad \sigma \geq \sigma_1 - R.$$

For each positive h, *let there exist a constant* A_h *such that the sums* $\sum_{k=1}^{m} a_k \exp(-\lambda_k s)$ $(m \geq n)$ *represent* $F(s)$ *in* $\sigma \geq \sigma_0$, $|t| \leq h$ *with logarithmic precision* $p_n(\sigma) - A_h$ *satisfying*

$$(5) \qquad \int^{\infty} p_n(\sigma) e^{-\eta\sigma} \, d\sigma = \infty$$

for some positive η.

Then there exists a constant B, *depending on* R, *such that*

$$(6) \qquad |a_n| \leq B \lambda_n \Lambda_n^* M(\sigma_1, R) \exp(\lambda_n \sigma_1),$$

where $M(\sigma_1, R)$ *denotes the maximum of* $|F(s)|$ *in the circle* $|s - \sigma_1| \leq R$.

Let $C(z)$ denote the entire function [1, p. 267] $\prod_{n=1}^{\infty}(1 - z^2/\lambda_n^2)$. The following well-known properties of $C(z)$ will be useful in extending to functions with poles of bounded order and to holomorphic functions the results obtained for functions with simple poles by the device described in the introductory paragraph.

I. *For every positive number* ϵ, *and for* z *sufficiently large* [1, p. 267],

$$|C(z)| < \exp[(D^* + \epsilon)|z|]. \tag{7}$$

II. *If*

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

and $\epsilon(r)$ *is a nonincreasing, positive function of* r *such that*

$$\inf_{r > 0} \frac{\log \epsilon(r)}{r} > -\infty, \tag{8}$$

then there exists a constant $K > 0$ *such that*

$$|C(z)| > \exp(-K|z|) \tag{9}$$

for z *exterior to every circle centered at a point* $z = \pm\lambda_n$ *and having radius* $\epsilon(\lambda_n)$.

Proof of II. This property of $C(z)$ follows easily from inequalities of Mandelbrojt on the numbers Λ_n^* [7, § 3.3, II]. If

$$\inf_n (\lambda_{n+1} - \lambda_n) \geq h \quad \text{and} \quad \sup_n \frac{n}{\lambda_n} \leq C,$$

then there exists a constant K_1 depending on C and on h but otherwise independent of the particular sequence $\{\lambda_n\}$, such that

$$\log \Lambda_n^* = \log \prod_{k \neq n} \left| \frac{\lambda_k^2}{\lambda_k^2 - \lambda_n^2} \right| < K_1 \lambda_n;$$

hence

$$\log \prod_{k \neq n} \left| 1 - \frac{\lambda_n^2}{\lambda_k^2} \right| > -K_1 \lambda_n.$$

Let $z = re^{i\theta} = x + iy$, and suppose (without loss of generality, since $C(z)$ is even in z) that z lies in the half-plane $x \geq 0$, exterior to each circle, centered at a point λ_k and of radius $\epsilon(\lambda_k)$:

$$|z - \lambda_k| > \epsilon(\lambda_k) \qquad (k = 1, 2, \cdots).$$

Fix z, and, if $r \geq \lambda_1$, choose n so that $\lambda_n \leq r < \lambda_{n+1}$. Then

$$\prod_{k \neq n} \left| 1 - \frac{z^2}{\lambda_k^2} \right| \geq \prod_{k \neq n} \left| 1 - \frac{r^2}{\lambda_k^2} \right| \tag{10}$$

since this inequality holds for corresponding factors. Define a sequence $\{\lambda_k^*\}$:

$$\lambda_k^* = \lambda_k \text{ for } k < n, \ \lambda_n^* = r, \ \lambda_{n+j}^* = \lambda_{n+j+1} \qquad (j = 1, 2, \cdots).$$

For the sequence $\{\lambda_k^*\}$ we have also

$$\inf_k \left(\lambda_{k+1}^* - \lambda_k^*\right) \geq h \quad \text{and} \quad \sup_k \frac{k}{\lambda_k^*} \leq C,$$

so that

$$\prod_{k \neq n} \left| 1 - \frac{\lambda_n^{*2}}{\lambda_k^{*2}} \right| = \prod_{k \neq n,\, n+1} \left| 1 - \frac{r^2}{\lambda_k^2} \right| > \exp(-K_1 r). \tag{11}$$

From (10) and (11) it follows that

$$|C(z)| = \prod_{k=1}^{\infty} \left| 1 - \frac{z^2}{\lambda_k^2} \right| > \left| 1 - \frac{z^2}{\lambda_n^2} \right| \left| 1 - \frac{z^2}{\lambda_{n+1}^2} \right| \exp(-K_1 r).$$

If $\lambda_{n+1} \geq 2r$, then

$$\left| 1 - \frac{z^2}{\lambda_{n+1}^2} \right| \geq \left| 1 - \frac{r^2}{\lambda_{n+1}^2} \right| \geq \frac{3}{4},$$

whereas if $\lambda_{n+1} < 2r$, then

$$\left| 1 - \frac{z^2}{\lambda_{n+1}^2} \right| = \frac{|\lambda_{n+1}^2 - z^2|}{\lambda_{n+1}^2} \geq \frac{\epsilon(\lambda_{n+1})}{\lambda_{n+1}} > \frac{\epsilon(2r)}{2r}.$$

Also

$$\left| 1 - \frac{z^2}{\lambda_n^2} \right| = \frac{|\lambda_n^2 - z^2|}{\lambda_n^2} \geq \frac{\epsilon(\lambda_n)}{\lambda_n}.$$

Hence

$$|C(z)| \geq \frac{\epsilon(\lambda_n)}{\lambda_n} \exp(-K_1 r) \min\left[\frac{3}{4}, \frac{\epsilon(2r)}{2r}\right]. \tag{12}$$

If $r < \lambda_1$, we define

$$\lambda_1^* = r, \quad \lambda_k^* = \lambda_k \qquad (k = 2, 3, \cdots)$$

and find that

$$\prod_{k=2}^{\infty} \left| 1 - \left(\frac{\lambda_1^*}{\lambda_k^*}\right)^2 \right| = \prod_{k=2}^{\infty} \left| 1 - \left(\frac{r}{\lambda_k}\right)^2 \right| > \exp(-K_1 r),$$

so that

$$|C(z)| = \prod_{k=1}^{\infty} \left| 1 - \left(\frac{z}{\lambda_k}\right)^2 \right| > \frac{|z^2 - \lambda_1^2|}{\lambda_1^2} \exp(-K_1 r) \geq \frac{\epsilon(\lambda_1)}{\lambda_1} \exp(-K_1 r).$$

From (8) and (12) it then follows that there exists a constant K such that

$$|C(z)| > e^{-Kr} = e^{-K|z|}. \tag{9}$$

III. *If* $\lim_{n\to\infty} n/\lambda_n = D > 0$, *and* $\theta \neq n\pi$ *for every positive integer* n, *then* [1, p. 279]

$$\lim_{r\to\infty} [\log |C(re^{i\theta})|]/r = \pi D |\sin\theta|. \tag{13}$$

3. Growth theorems.

LEMMA 1. *Let $f(z)$ satisfy the following hypotheses:*

(14) *$f(z)$ is holomorphic in Δ except for simple poles with residues a_k at points $z = \lambda_k$ $(k = 1, 2, \cdots)$, and is continuous on the boundary Γ of Δ:*

(15) *there exists a constant α such that $f(z) = O(e^{\alpha x})$ as $z \longrightarrow \infty$ on Γ;*

(16) *there exist positive constants M_n, and rectifiable curves γ_n, of uniformly bounded length, joining Γ^- and Γ^+, such that $x \geq \lambda_n$ and $|f(z)| < M_n$ for z on γ_n $(n = 1, 2, \cdots)$.*

Then

(17) *the integral $(1/2\pi i)\int_\Gamma f(z)\, e^{-zs}\, dz$ $(s = \sigma + it)$ (Γ being traversed from $z = \infty$ along Γ^+ and out along Γ^- to $z = \infty$ again) converges for $\sigma > \alpha$ to a function $F(s)$ holomorphic in $\sigma > \alpha$;*

(18) *if*

$$\liminf_{n\to\infty} \frac{\log M_n}{\lambda_n} < \infty,$$

then there exists a sequence $\{n_k\}$ of positive integers such that for σ sufficiently large, $\sum_{j=1}^{n_k} a_j \exp(-\lambda_j s)$ converges to $F(s)$ as $k \longrightarrow \infty$; if, further,

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

then the series $\sum_1^\infty a_j \exp(-\lambda_j s)$ converges to $F(s)$ for σ sufficiently large;

(19) *if*

$$\lim_{n\to\infty} \frac{\log M_n}{\lambda_n} = \infty, \quad |t| \leq h, \quad \text{and} \quad \sigma \geq \alpha + \delta,$$

where h, δ are arbitrary positive numbers, then there exists a constant $K = K(h, \delta)$ such that

$$\left| F(s) - \sum_{k=1}^{n} a_k \exp(-\lambda_k s) \right| < K M_n \exp(-\lambda_n \sigma) \quad (n = 1, 2, \cdots).$$

Proof. Let Δ_n denote that one of the two regions bounded by Γ and by γ_n which contains $\lambda_1, \lambda_2, \cdots, \lambda_n$, and let Γ_n denote its boundary. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\Gamma_n} f(z)\, e^{-zs}\, dz = \sum_{k=1}^{n} a_k \exp(-\lambda_k s). \tag{20}$$

Denote by Γ_n^+ and Γ_n^- those parts of Γ^+ and Γ^- (cf. page 2) respectively which do *not* belong to Γ_n. By (15), there exists a positive constant C_1 such that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n^+} f(z) e^{-zs}\, dz\right| \leq C_1 \int_{\lambda_n}^{\infty} e^{\alpha x} e^{-(\sigma x - |t|k)}\, |dz|,$$

where k is a bound on $g^+(x)$ and on $g^-(x)$. The function $g^+(x)$ satisfies the Lipschitz condition (3), so that if $|t| \leq h$, there exists a positive constant $C_2 = C_2(h)$ such that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n^+} f(z)\, e^{-zs}\, dz\right| \leq C_2 \int_{\lambda_n}^{\infty} e^{-(\sigma - \alpha)x}\, dx.$$

Finally, if also $\sigma \geq \alpha + \delta$, there exists a positive constant $C_3 = C_3(h, \delta)$, such that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n^+} f(z)\, e^{-zs}\, dz\right| \leq C_3 \exp(\alpha \lambda_n) \exp(-\lambda_n \sigma). \tag{21}$$

It follows that

$$\frac{1}{2\pi i}\int_{\Gamma^+} f(z)\, e^{-zs}\, dz$$

converges to a function $F^+(s)$, holomorphic in $\sigma > \alpha$. Similarly, if $|t| \leq h$ and $\sigma \geq \alpha + \delta$, there exists a positive constant $C_4 = C_4(h, \delta)$, such that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n^-} f(z)\, e^{-zs}\, dz\right| \leq C_4 \exp(\alpha \lambda_n) \exp(-\lambda_n \sigma). \tag{22}$$

The integral

$$\frac{1}{2\pi i}\int_{\Gamma^-} f(z)\, e^{-zs}\, dz$$

therefore also converges, to a function $F^-(s)$ holomorphic in $\sigma > \alpha$, and we have $F(s) = F^+(s) + F^-(s)$; the proof of (17) is complete.

We have also

$$\left| \frac{1}{2\pi i} \int_{\gamma_n} f(z)\, e^{-zs}\, dz \right| \leq \frac{M_n}{2\pi} \int_{\gamma_n} e^{-(\sigma x - ty)} \, |dz|$$

$$\leq \frac{LM_n}{2\pi} \exp(k|t|) \exp(-\lambda_n \sigma),$$

where L is a uniform bound on the lengths of the curves γ_n, and k is a bound on $g^+(x)$ and on $g^-(x)$. Thus if $|t| \leq h$, there exists a positive constant $C_5 = C_5(h)$, such that

$$\left| \frac{1}{2\pi i} \int_{\gamma_n} f(z)\, e^{-zs}\, dz \right| \leq C_5 M_n \exp(-\lambda_n \sigma). \tag{23}$$

Suppose now that

$$\liminf_{n \to \infty} \frac{\log M_n}{\lambda_n} < \infty.$$

Then from (20), (21), (22), and (23) it follows that there exists a constant C_6 and a sequence $\{n_k\}$ of positive integers, such that

$$\lim_{k \to \infty} \sum_{j=1}^{n_k} a_j \exp(-\lambda_j s) = F(s)$$

for σ sufficiently large. If in addition

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

it follows from a result of Mandelbrojt [7, § 3.10, V] that $\sum_1^\infty a_k \exp(-\lambda_k s)$ converges to $F(s)$ for σ sufficiently large. This completes the proof of (18).

If, on the contrary,

$$\lim_{n \to \infty} \frac{\log M_n}{\lambda_n} = \infty,$$

then there exists a positive constant C_7, such that $\exp(\alpha\lambda_n) < C_7 M_n$ $(n = 1, 2, \cdots)$. It then follows from (20), (21), (22), and (23) that if $|t| \leq h$, and $\sigma \geq \alpha + \delta$, there exists a positive constant $K = K(h, \delta)$ such that

$$\left| F(s) - \sum_1^n a_k \exp(-\lambda_k s) \right| \leq K M_n \exp(-\lambda_n \sigma) \qquad (n = 1, 2, \cdots).$$

This completes the proof of (19) and of Lemma 1.

THEOREM 2. *Let $f(z)$ satisfy the hypotheses of Lemma* 1, *and let the sequence $\{M_n\}$ have the uniqueness property* (U). *Then for every positive number δ,*

$$(24) \qquad |a_n| = o(\lambda_n \Lambda_n^* \exp[(\alpha + \delta + D^*)\lambda_n]) \text{ as } n \longrightarrow \infty.$$

If further

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

then the series $\sum a_k \exp(-\lambda_k s)$ converges to $F(s)$ for σ sufficiently large, which implies that

$$(25) \qquad \limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n} < \infty.$$

Thus it is clear that if

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

and the residues a_k increase so rapidly that

$$\lim_{n \to \infty} \frac{\log |a_n|}{\lambda_n} = \infty,$$

then the bounds M_n of the function $f(z)$ on the curves γ_n must increase so rapidly that the sequence $\{M_n\}$ fails to satisfy the uniqueness condition (U), that is, so rapidly that for every positive number η, the series

$$\sum^{\infty} \left(\frac{M_n^c}{M_{n+1}^c} \right)^{\eta/(\lambda_{n+1} - \lambda_n)} [\lambda_{n+1} - \lambda_n]$$

converges.

Proof of Theorem 2. If

$$\liminf_{n\to\infty} \frac{\log M_n}{\lambda_n} < \infty,$$

it follows from equations (20), (21), (22), and (23) that if

$$0 \leq \delta' < \delta, \quad \sigma \geq \alpha + \delta', \quad \text{and} \quad |t| \leq h,$$

then there exist a constant $C = C(h, \delta')$, a constant b, and a sequence $\{n_k\}$ such that

$$\left| F(s) - \sum_{j=1}^{n_k} a_j \exp(-\lambda_j s) \right| < C \exp[-\lambda_{n_k}(\sigma - b)].$$

It follows then from the definition (4) that for each δ' and h, and each positive integer n, the sums $\sum_{j=1}^{m} a_j \exp(-\lambda_j s)$ $(m \geq n)$ represent $F(s)$ in $\sigma \geq \alpha + \delta'$, $|t| \leq h$, with infinite logarithmic precision. The inequality (6), with $\sigma_1 = \alpha + \delta' + D^*$, then yields (24). If

$$\lim_{n\to\infty} \frac{\log M_n}{\lambda_n} = \infty,$$

then (19) and Theorem 1 imply the conclusion (24). Mandelbrojt has shown [7, § 3.3, III] that if

$$\inf_{n\geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

then

$$\sup \frac{\log \Lambda_n^*}{\lambda_n} < \infty,$$

which, with (24), implies (25) (cf. [1, p. 4]), and completes the proof of Theorem 2.

THEOREM 2A. *Let p be a positive integer, and let $f(z)$ satisfy the hypotheses of Theorem* 2, *except that the poles at points $z = \lambda_n$ need not be simple,*

but none is of order greater than p. Let $a_{n,-p}$ denote the coefficient of the leading term $a_{n,-p}/(z-\lambda_n)^p$ in the Laurent development of $f(z)$ about $z=\lambda_n$ $(n=1, 2, \cdots)$. Let the curve γ_n referred to in hypothesis (16) of Lemma 1 lie entirely between $x=\lambda_n$ and $x=\lambda_n+d$ for some fixed positive constant d $(n=1, 2, \cdots)$. Then for every positive number δ,

$$|a_{n,-p}| = o(\lambda_n^p \Lambda_n^{*p} \exp[(\alpha+\delta+D^*)\lambda_n]) \quad \text{as} \quad n \longrightarrow \infty. \tag{26}$$

If further

$$\inf_{n \geq 1} (\lambda_{n+1}-\lambda_n) > 0,$$

then

$$\limsup_{n\to\infty} \frac{\log |a_{n,-p}|}{\lambda_n} < \infty. \tag{27}$$

The added hypothesis on the curves γ_n is not very stringent. In fact, since these curves have uniformly bounded length, and since γ_n lies to the right of the line $x=\lambda_n$, there exists such a constant d as is required, if only each of the curves γ_n has at least one point within a fixed distance independent of n from the corresponding point $z=\lambda_n$.

THEOREM 2B. *Let p be a positive integer, and let $f(z)$ satisfy the hypotheses of Theorem 2, except that $f(z)$ is holomorphic in Δ, with zeros at points $z=\lambda_n$ of order at least p. Let $a_{n,p}$ denote the coefficient of the leading term $a_{n,p}(z-\lambda_n)^p$ in the Taylor series development of f about $z=\lambda_n$ $(n=1, 2, \cdots)$. Let the boundary of Δ satisfy the further condition that*

$$\liminf_{r\to\infty} \frac{\log g^+(r)}{r} > -\infty, \text{ and } \liminf_{r\to\infty} \frac{\log g^-(r)}{r} > -\infty.$$

Let there exist a nonincreasing, positive function $\epsilon(r)$ of r such that

$$\inf_{r>0} \frac{\log \epsilon(r)}{r} > -\infty,$$

and such that the curves γ_n referred to in hypothesis (16) of Lemma 1 do not penetrate circles of radius $\epsilon(\lambda_n)$ about the points $z=\lambda_n$. Further, let there exist a positive constant d such that $|z| < \lambda_n + d$ for z on γ_n. Then for

every positive number δ,

$$(28) \qquad |a_{n,p}| = o(\lambda_n^{-p}\ \Lambda_n^{*-p}\ \exp[(\alpha + D^* + \delta)]\lambda_n) \quad as \quad n \longrightarrow \infty.$$

If further

$$\inf_{n \geq 1} \ (\lambda_{n+1} - \lambda_n) > 0,$$

then

$$(29) \qquad \limsup_{n \to \infty} \ \frac{\log |a_{n,p}|}{\lambda_n} \ < \infty.$$

Proof of Theorems 2A *and* 2B. If

$$f^*(z) = f(z)\,[C(z)]^{j-1},$$

where $j = p$ for Theorem 2A, and $j = -p$ for Theorem 2B, then $f^*(z)$ has simple poles at points $z = \lambda_n$. The coefficient of the leading term in the Taylor's expansion of $C(z)$ about the point $z = \lambda_n$ has absolute value $2/\lambda_n \Lambda_n^*$, so that the residue of $f^*(z)$ at $z = \lambda_n$ has absolute value

$$|a_n^*| = |a_{n,-j}|\left(\frac{2}{\lambda_n \Lambda_n^*}\right)^{j-1}.$$

It follows from properties I and II of the function $C(z)$ that $f^*(z)$ satisfies hypothesis (15) of Lemma 1, and that, since, for z on γ_n, $|z| - \lambda_n$ has a bound independent of n, there exist positive constants A and B such that

$$|f^*(z)| < M_n^* = A\ M_n\ \exp(B\lambda_n) \quad \text{for } z \text{ on } \gamma_n.$$

One readily verifies that if the sequence $\{M_n\}$ has the uniqueness property (U), so also does the sequence $\{M_n^*\}$. Indeed,

$$p^*(\sigma) = \sup_{n \geq 1} \ (\lambda_n \sigma - \log M_n^*)$$

$$= \sup_{n \geq 1} \ \{\lambda_n(\sigma - B) - \log M_n - \log A\} = p(\sigma - B) - \log A.$$

Thus the function $f^*(z)$ satisfies the hypotheses of Theorem 2. Conclusion (24) applied to the coefficients a_n^* yields (26) and (28). Further, the ratio

$(\log \Lambda_n^*)/\lambda_n$ is bounded below; for if $L = \sup_{k \geq 1} k/\lambda_k$, then

$$\frac{1}{\Lambda_n^*} = \prod_{k \neq n} \left| 1 - \frac{\lambda_n^2}{\lambda_k^2} \right| \leq \prod_{k=1}^{\infty} \left(1 + \frac{\lambda_n^2}{\lambda_k^2} \right) \leq \prod_{k=1}^{\infty} \left(1 + \frac{L^2 \lambda_n^2}{k^2} \right)$$

$$= \frac{\sin \pi i L \lambda_n}{\pi i L \lambda_n} \leq \frac{1}{\pi L \lambda_1} \exp(\pi L \lambda_n).$$

Therefore (25) implies (29). Also, as was seen in the proof of Theorem 2, if

$$\inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

then $(\log \Lambda_n^*)/\lambda_n$ is bounded above, so that (25) implies (27). This completes the proof of Theorems 2A and 2B.

4. Functions of exponential type in the right half-plane outside Δ. The following lemma gives conditions on $f(z)$ sufficient in order that the function $F(s)$ defined in Lemma 1 shall be continuable analytically throughout a horizontal strip.

Let D denote the complement in the half-plane $x > 0$ of the region Δ, and $\overline{D}$ its closure.

LEMMA 2. *Let $f(z)$ satisfy the hypotheses of Lemma 1, and also the following:*

(30) *$f(z)$ is holomorphic in D, continuous in $\overline{D}$;*

(31) $|f(z)| = O(e^{a|z|})$ *as* $|z| \longrightarrow \infty$ *in* D;

(32) *there exists a positive constant β_1 such that $|f(iy)| = O(\exp(-\beta_1 y))$ as $y \longrightarrow \infty$;*

(33) *there exists a positive constant β_2 such that $|f(iy)| = O(\exp(\beta_2 y))$ as $y \longrightarrow -\infty$.*

Then the function $F(s)$ defined in Lemma 1 is holomorphic in the strip $-\beta_2 < t < \beta_1$, and uniformly bounded in every closed interior strip.

Proof. Let K^+ denote the curve extending from $z = \infty$ along the y-axis to the origin, and along the x-axis to $x = a$, and K^- the curve traversed from $x = a$

along the x-axis to the origin, then along the lower half of the y-axis. Let

$$F_1^+(s) = \frac{1}{2\pi i}\int_{K^+} f(z)\,e^{-zs}\,dz,\quad F_1^-(s) = \frac{1}{2\pi i}\int_{K^-} f(z)\,e^{-zs}\,dz,$$

and

$$F_1(s) = F_1^+(s) + F_1^-(s) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} f(z)\,e^{-zs}\,dz.$$

By hypotheses (32) and (33), there exist constants A^+ and A^- such that

$$\left|\frac{1}{2\pi i}\int_{i\infty}^{0} f(z)\,e^{-zs}\,dz\right| \le A^+\int_0^\infty \exp(-\beta_1 y)\exp(ty)\,dy,$$

and

$$\left|\frac{1}{2\pi i}\int_{0}^{-i\infty} f(z)\,e^{-zs}\,dz\right| \le A^-\int_{-\infty}^{0} \exp(\beta_2 y)\exp(ty)\,dy.$$

The functions $F_1^+(s)$ and $F_1^-(s)$ are therefore holomorphic in the half-planes $t < \beta_1$ and $t > -\beta_2$, respectively, and their sum $F_1(s)$ is holomorphic in the strip $-\beta_2 < t < \beta_1$; and, for every positive number ϵ, $F_1(s)$ is uniformly bounded in the strip $-\beta_2 + \epsilon < t < \beta_1 - \epsilon$. Let C_r^+ denote that part in D and above the x-axis of the circle of radius r centered at the origin, traversed in a clockwise direction. Let G_r^+ denote the boundary of the region above the x-axis, bounded by the x-axis, Γ^+, C_r^+, and the positive y-axis. Define similarly C_r^- and G_r^- below the x-axis. The contours G_r^+ and G_r^- enclose parts of D, in which $f(z)$ is, by hypothesis, holomorphic, so that

$$(34)\qquad \frac{1}{2\pi i}\int_{G_r^+} f(z)\,e^{-zs}\,dz \equiv \frac{1}{2\pi i}\int_{G_r^-} f(z)\,e^{-zs}\,dz \equiv 0.$$

The integral

$$\frac{1}{2\pi i}\int_{\Gamma^+} f(z)\,e^{-zs}\,dz$$

defines the function $F^+(s)$, which was seen in Lemma 1 to be holomorphic in the half-plane $\sigma > \alpha$. From (34) and the definitions of $F^+(s)$ and $F_1^+(s)$ it follows that

$$2\pi i\,[F_1^+(s) - F^+(s)] = \int_{i\infty}^{ir} + \int_{C_r^+} - \int_{\Gamma_r^+} f(z)\,e^{-zs}\,dz,$$

where Γ_r^+ is traversed along Γ^+ from $z = \infty$ to the intersection of Γ^+ with C_r^+, and where the path of integration for the first integral lies on the y-axis. It has already been seen that the first and last integrals tend to 0 as $r \longrightarrow \infty$, for s in the quarter-plane $\sigma > \alpha$, $t < \beta_1$. To show $F_1^+(s) \equiv F(s)$, it is therefore sufficient to show that, for s on a continuum in the region $\sigma > \alpha$, $t < \beta_1$, the second integral,

$$\int_{C_r^+} f(z)\,e^{-zs}\,dz,$$

tends to 0 as $r \longrightarrow \infty$. By hypothesis (31), there exists a constant B such that

$$\left|\int_{C_r^+} f(z)\,e^{-zs}\,dz\right| \leq Be^{\alpha r}\left[\int_0^{\pi/4} + \int_{\pi/4}^{\pi/2} r\exp\{-r(\sigma\cos\theta - t\sin\theta)\}d\theta\right]$$

$$\leq Br\,e^{\alpha r}\left[e^{r|t|}\int_0^{\pi/4}\exp(-\sigma r\cos\theta)\,d\theta + \int_{\pi/4}^{\pi/2}\exp(rt\sin\theta)\,d\theta\right]$$

if $\sigma > 0$, and

$$\left|\int_{C_r^+} f(z)\,e^{-zs}\,dz\right| \leq \frac{\pi}{4}\,Br\,e^{\alpha r}\left[\exp(r|t|)\exp\left(\frac{-r\sigma}{\sqrt{2}}\right) + \exp\left(\frac{rt}{\sqrt{2}}\right)\right]$$

if $\sigma > 0$, $t < 0$. Thus if

$$t = -t_0 < -\alpha\sqrt{2}, \text{ and } \sigma > \sqrt{2}\,(\alpha + t_0),$$

then the integral approaches 0 as $r \longrightarrow \infty$. It follows that $F_1^+(s) \equiv F^+(s)$. Similarly, for s on a suitable horizontal line in the upper half-plane and for σ sufficiently large we have

$$\frac{1}{2\pi i}\int_{C_r^-} f(z)\,e^{-zs}\,dz \longrightarrow 0 \text{ as } r \longrightarrow \infty,$$

so that $F_1^-(s) \equiv F^-(s)$. Hence $F(s) \equiv F_1(s)$, and this completes the proof of Lemma 2.

THEOREM 3A. *Let $f(z)$ satisfy the hypotheses of Theorem* 2A *and in*

addition (30) – (33), *with* $\beta_1 > \pi p D^*$, $\beta_2 > \pi p D^*$. *Then* $a_{n,-k} = 0$, $1 \leq k \leq p$ ($n = 1, 2, \cdots$), *so that* $f(z)$ *is holomorphic in* $x > 0$. *If, further,*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D = D^* > 0,$$

then $f(z) \equiv 0$.

THEOREM 3B. *Let* $f(z)$ *satisfy the hypotheses of Theorem* 2B *and in addition* (30) – (33), *except that* $\beta_1 > -\pi p D^*$, $\beta_2 > -\pi p D^*$. *Suppose further that*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D = D^* > 0.$$

Then $f(z) \equiv 0$.

Proof of Theorems 3A *and* 3B. As in the proof of Theorems 2A and 2B, set

$$f^*(z) = f(z)\,[C(z)]^{j-1},$$

where $j = p$ for Theorem 3A and $j = -p$ for Theorem 3B. It has been shown in the proofs of Theorems 2, 2A, and 2B, that for each pair of positive numbers δ and h, the associated function $F^*(s)$ is represented in $\sigma \geq \alpha + \delta$, $|t| \leq h$ by the sums

$$\sum_{k=1}^{m} a_k^* \exp(-\lambda_k s) \qquad (m \geq n)$$

with a logarithmic precision satisfying (5). Moreover, by properties I and III of the function $C(z)$, since $\beta_1 > \pi j D^*$ and $\beta_2 > \pi j D^*$, the function $f^*(z)$ satisfies the hypotheses of Lemma 2, with $\beta_1^* > \pi D^*$ and $\beta_2^* > \pi D^*$. The conclusion of Lemma 2 then states that $F^*(s)$ can be continued analytically throughout a strip $|t| \leq \pi R$, for some $R > D^*$, in which it is uniformly bounded. It follows, then, from conclusion (6) of Theorem 1 ($-\sigma_1$ arbitrarily large) that $a_n^* = 0$, hence $a_{n,-p} = 0$ (Theorem 3A) or $a_{n,p} = 0$ (Theorem 3B) ($n = 1, 2, \cdots$). Successive applications of this result show that under the hypotheses of Theorem 3B, $f(z) \equiv 0$; this completes the proof of Theorem 3B. Successive applications to the function in Theorem 3A show that $a_{n,-k} = 0$, $1 \leq k \leq p$, so that $f(z)$ is holomorphic in $x > 0$. If then in addition

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0,$$

Theorem 3B applies, and $f(z) \equiv 0$. This completes the proof of Theorem 3A.

If $f(z)$ is holomorphic in $x \geq 0$, and $f(z) = O(e^{\alpha|z|})$ as $|z| \longrightarrow \infty$ for some $\alpha > 0$ and for $x > 0$, then $f(z)$ satisfies the hypotheses of Theorem 2 with constants $M_n = K \exp(\alpha \lambda_n)$ for some constant K; this sequence $\{M_n\}$ clearly has property (U). If f is such a function, and

$$p = 1, \quad \lambda_n = n, \quad \text{and} \quad \alpha = -\beta_1 = -\beta_2,$$

then Theorem 3B yields as a special case a theorem of Carlson (cf. [8, p. 186]). Theorem 3B contains also a more general result noted by Boas [2, p. 844] as a consequence of Carleman's theorem. Both assume that $f(z)$ has zeros at points $z = \lambda_n$, and that $f(z)$ is of exponential type in the entire half-plane $x > 0$. The hypotheses on the sequence $\{\lambda_n\}$ of the last-mentioned result are essentially the same as in Theorem 3B.

We observe that Theorems 3A and 3B may be interpreted as relating the growth of f on the y-axis to the maximum order of poles, or minimum order of zeros, of f at points λ_n sufficient to imply $f \equiv 0$. In this connection it is interesting to note that Theorem 3B contains also a result of Boas in which $f(z)$ is assumed to have zeros of order 2 at points $z = \lambda_n = 2n$. This latter result Boas generalized in a different direction [3]. Fuchs [4], retaining the hypotheses that $f(z)$ is of exponential type throughout the half-plane $x > 0$ and has zeros at points $z = \lambda_n$, used Carleman's theorem and properties of the function

$$H(z) = \prod_{k=1}^{\infty} (\lambda_k - z) \frac{\exp(2z/\lambda_k)}{\lambda_k + z}$$

to give conditions on the sequence $\{\lambda_n\}$ which are necessary and sufficient in order that the only function satisfying these hypotheses shall vanish identically. Professor S. Mandelbrojt has kindly called to the author's attention recent work of Kahane [5] who has further generalized Carlson's theorem.

At the suggestion of the referee, the special case of Theorem 3A corresponding to Carlson's theorem is stated as a corollary of Theorem 3A.

COROLLARY 3A. *If $f(z)$ is holomorphic in $x \geq 0$, except for poles of maximum order p at points $z = n$ $(n = 1, 2, \cdots)$ on the x axis, if $f(z) = O(e^{\alpha|z|})$*

in $x > 0$ *for some real number* α, *and* $f(iy) = O(e^{-\beta|y|})$ *as* $|y| \longrightarrow \infty$ *for some positive number* $\beta > \pi p$, *then* $f(z) \equiv 0$.

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