A GENERALIZATION OF AN INEQUALITY DUE TO BEURLING

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1. Introduction. In 1941, Arne Beurling gave a proof (unpublished) of the following result:

If $a_n \geq 0$, $b_n \geq 0$ *for* $n = 1, 2, \cdots$, *and*

$$\sum_{m=1}^{\infty} m a_m^2 < \infty, \quad \sum_{n=1}^{\infty} n b_n^2 < \infty,$$

then

$$(1) \qquad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \log (m+n) \leq K \left(\sum_{m=1}^{\infty} m a_m^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} n b_n^2 \right)^{1/2}$$

with $0 < K < 4e$.

If we set

$$\alpha(x) = \int_1^x t^{-1}\, dt,$$

then the inequality (1) is of the form

$$(2) \qquad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \alpha(m+n) \leq K(\alpha) \left[\sum_{m=1}^{\infty} a_m^2 / \alpha'(m) \right]^{1/2} \left[\sum_{n=1}^{\infty} b_n^2 / \alpha'(n) \right]^{1/2},$$

and it is the purpose of this note to generalize this latter inequality. As an example of the type of result to be obtained, we quote the integral analogue of (2):

THEOREM 1. *Let* $\alpha(x)$ *be nonnegative, nondecreasing, and locally absolutely continuous on the interval* $0 \leq x < \infty$. *If* $F(x) \geq 0$, $G(x) \geq 0$ *for* $0 \leq x < \infty$, *and*

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$$\int_0^\infty \frac{[F(x)]^p}{[\alpha'(x)]^{p-1}}\,dx < \infty,\quad \int_0^\infty \frac{[G(y)]^q}{[\alpha'(y)]^{q-1}}\,dy < \infty,$$

where $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$, *then*

$$(3)\quad \int_0^\infty \int_0^\infty \frac{F(x)G(y)}{\alpha(x+y)}\,dxdy$$

$$\leq K(\alpha)\left\{\int_0^\infty \frac{[F(x)]^p}{[\alpha'(x)]^{p-1}}\,dx\right\}^{1/p}\left\{\int_0^\infty \frac{[G(y)]^q}{[\alpha'(y)]^{q-1}}\,dy\right\}^{1/p}$$

with $0 < K(\alpha) \leq p + q$.

If $\alpha(x+y) \geq \alpha(x) + \alpha(y)$, *then* $K(\alpha) \leq \pi/\sin(\pi/p)$. *If* $\alpha(0) = 0$, $\alpha(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$, *and* $\alpha(x+y) \leq \alpha(x) + \alpha(y)$, *then* $K(\alpha) \geq \pi/\sin(\pi/p)$.

The author wishes to acknowledge that any novelty in the subject matter of this note is due entirely to Professor Beurling who suggested the very general Theorem 2 below.

2. The main result. This is:

THEOREM 2. *Let* $\alpha(x)$ *be nonnegative, nondecreasing, and continuous from the right for* $0 \leq x < \infty$. *Let* $f(x) \geq 0$, $g(x) \geq 0$ *for* $0 \leq x < \infty$. *Let* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *If*

$$\int_0^\infty [f(x)]^p\,d\alpha(x) < \infty,\quad \int_0^\infty [g(y)]^q\,d\alpha(y) < \infty,$$

then

$$(4)\quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)}\,d\alpha(x)d\alpha(y)$$

$$\leq K(\alpha)\left\{\int_0^\infty [f(x)]^p\,d\alpha(x)\right\}^{1/p}\left\{\int_0^\infty [g(y)]^q\,d\alpha(y)\right\}^{1/q}$$

with $0 < K(\alpha) \leq p + q$.

If $\alpha(x+y) \geq \alpha(x)+\alpha(y)$, *then*

$$K(\alpha) \leq \pi/\sin(\pi/p). \tag{5}$$

If $\alpha(0)=0$, $\alpha(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$, $\alpha(x)$ *is continuous for* $0 \leq x < \infty$, *and* $\alpha(x+y) \leq \alpha(x)+\alpha(y)$, *then*

$$K(\alpha) \geq \pi/\sin(\pi/p). \tag{6}$$

Proof. We have

$$I = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)}\, d\alpha(x)\, d\alpha(y)$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)} [\alpha(x)/\alpha(y)]^{1/pq} [\alpha(y)/\alpha(x)]^{1/pq} d\alpha(x)\, d\alpha(y)$$

$$\leq P^{1/p} Q^{1/q}.$$

by Hölder's Inequality [**1**, p. 11], where

$$P = \int_0^\infty \int_0^\infty \frac{[f(x)]^p}{\alpha(x+y)} [\alpha(x)/\alpha(y)]^{1/q} d\alpha(x)\, d\alpha(y),$$

$$Q = \int_0^\infty \int_0^\infty \frac{[g(y)]^q}{\alpha(x+y)} [\alpha(y)/\alpha(x)]^{1/p} d\alpha(x)\, d\alpha(y).$$

Since $\alpha(x)$ is nondecreasing, we have

$$\alpha(x+y) \geq \max[\alpha(x), \alpha(y)].$$

Consequently,

$$J = \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x+y)} d\alpha(y) \leq \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\max[\alpha(x), \alpha(y)]} d\alpha(y)$$

$$= [\alpha(x)]^{-1/p} \int_0^x [\alpha(y)]^{-1/q} d\alpha(y) + [\alpha(x)]^{1/q} \int_x^\infty [\alpha(y)]^{-1-1/q} d\alpha(y)$$

$$\leq p\{1-[\alpha(0)/\alpha(x)]^{1/p}\} + q\{1-[\alpha(x)/\alpha(\infty)]^{1/q}\} \leq p+q.$$

In a similar way, we find that

$$\int_0^\infty \frac{[\alpha(y)/\alpha(x)]^{1/p}}{\alpha(x+y)}\, d\alpha(x) \le p+q.$$

Thus,

$$I \le P^{1/p}Q^{1/q} \le (p+q)\left\{\int_0^\infty [f(x)]^p d\alpha(x)\right\}^{1/p}\left\{\int_0^\infty [g(y)]^q d\alpha(y)\right\}^{1/q},$$

and this implies (4).

If $\alpha(x+y) \ge \alpha(x)+\alpha(y)$, then we have

$$J = \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x+y)}\, d\alpha(y) \le \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x)+\alpha(y)}\, d\alpha(y)$$

$$\le \int_{\alpha(0)/\alpha(x)}^{\alpha(\infty)/\alpha(x)} t^{-1/q}(1+t)^{-1}\,dt \le \int_0^\infty t^{-1/q}(1+t)^{-1}\,dt = \pi/\sin(\pi/p),$$

and this implies (5).

If $\alpha(0)=0$, $\alpha(\infty)=\infty$, $\alpha(x)$ is continuous for $0 \le x < \infty$, and $\alpha(x+y) \le \alpha(x)+\alpha(y)$, then

$$I \ge \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x)+\alpha(y)}\, d\alpha(x)\,d\alpha(y) = \int_0^\infty \int_0^\infty \frac{F(s)G(t)}{s+t}\, ds\,dt = I_1,$$

where we have made the changes of variable $\alpha(x) = s$, $\alpha(y) = t$ and set $F(s) = f(x)$, $G(t) = g(y)$. By Hilbert's Inequality [**1**, 226],

$$I_1 \le [\pi/\sin(\pi/p)]\left\{\int_0^\infty [F(s)]^p ds\right\}^{1/p}\left\{\int_0^\infty [G(t)]^q dt\right\}^{1/q}$$

$$= [\pi/\sin(\pi/p)]\left\{\int_0^\infty [f(x)]^p d\alpha(x)\right\}^{1/p}\left\{\int_0^\infty [g(y)]^q d\alpha(y)\right\}^{1/q},$$

and the constant $\pi/\sin(\pi/p)$ is the best possible [**1**, p. 226]. This gives (6).

We note that the inequality (4), for $\alpha(x)$ continuous, could be obtained directly from Theorem 319 of [**1**, p. 229] as follows:

$$I \leq \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max[\alpha(x), \alpha(y)]} d\alpha(x) d\alpha(y) = \int_0^\infty \int_0^\infty \frac{F(s)G(t)}{\max[s,t]} dsdt$$

$$\leq k\left\{\int_0^\infty [F(s)]^p ds\right\}^{1/p}\left\{\int_0^\infty [G(t)]^q dt\right\}^{1/q}$$

$$= k\left\{\int_0^\infty [f(x)]^p d\alpha(x)\right\}^{1/p}\left\{\int_0^\infty [g(y)]^q d\alpha(y)\right\}^{1/q},$$

where

$$k = \int_0^\infty \frac{s^{-1/q}\, ds}{\max[s,1]} = p + q.$$

We have made the changes of variable $\alpha(x) = s$, $\alpha(y) = t$ and set

$$\begin{aligned} F(s) &= f(x) \quad \text{for } \alpha(0) \leq s \leq \alpha(\infty), \\ &= 0 \quad \text{otherwise}, \\ G(t) &= g(y) \quad \text{for } \alpha(0) \leq t \leq \alpha(\infty), \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

3. Corollaries. If we set $f(x) = F(x)/\alpha'(x)$, $g(y) = G(y)/\alpha'(y)$ in Theorem 2, we obtain Theorem 1.

As another application of Theorem 2, we deduce:

THEOREM 3. *Let the sequence* $\{\alpha_n\}_1^\infty$ *be nonnegative and nondecreasing for* $n = 1, 2, \cdots$, *and set* $\alpha_0 = 0$. *Let* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *If* $a_n \geq 0$, $b_n \geq 0$ *for* $n = 1, 2, \cdots$ *and*

$$\sum_{m=1}^\infty a_m^p/(\alpha_m - \alpha_{m-1})^{p-1} < \infty, \quad \sum_{n=1}^\infty b_n^q/(\alpha_n - \alpha_{n-1})^{q-1} < \infty,$$

then

$$(7) \quad \sum_{m=1}^\infty \sum_{n=1}^\infty a_m b_n/\alpha_{m+n}$$

$$\leq K(\alpha)\left\{\sum_{m=1}^\infty a_m^p/(\alpha_m - \alpha_{m-1})^{p-1}\right\}^{1/p}\left\{\sum_{n=1}^\infty b_n^q/(\alpha_n - \alpha_{n-1})^{q-1}\right\}^{1/q}.$$

with $0 < K(\alpha) \le p + q$.

Proof. Let $\alpha(x)$, $0 \le x < \infty$, be the polygonal function with vertices (n, α_n), $n = 0, 1, \cdots$. Set $f(x) = A_n \ge 0$, $g(x) = B_n \ge 0$ for $n - 1 \le x < n$, $n = 1, 2, \cdots$. By Theorem 2,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_m B_n \int_{m-1}^{m} \int_{n-1}^{n} \frac{d\alpha(x)\, d\alpha(y)}{\alpha(x+y)}$$

$$\le K(\alpha) \left\{ \sum_{m=1}^{\infty} A_m^p (\alpha_m - \alpha_{m-1}) \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} B_n^q (\alpha_n - \alpha_{n-1}) \right\}^{1/q},$$

with $0 < K(\alpha) \le p + q$. Since $\alpha(x)$ is nondecreasing, the double sum dominates

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m B_n}{\alpha_{m+n}} (\alpha_m - \alpha_{m-1})(\alpha_n - \alpha_{n-1}).$$

Setting $A_m(\alpha_m - \alpha_{m-1}) = a_m$, $B_m(\alpha_m - \alpha_{m-1}) = b_m$ gives (7).

As a special case of Theorem 3, we take $\alpha_0 = 0$, $\alpha_n - \alpha_{n-1} = 1/n$ for $n = 1, 2, \cdots$. Since, for $n = 2, 3, \cdots$,

$$\alpha_n = \sum_{k=1}^{n} d^{-1} < 1 + \log n \le \left(1 + \frac{1}{\log 2}\right) \log n,$$

we find that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \log(m+n) \le M \left(\sum_{m=1}^{\infty} m^{p-1} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} n^{q-1} b_n^q \right)^{1/q},$$

with $0 < M < (p + q)(1 + 1/\log 2)$. For $p = 2$, this is the inequality (1) with a slightly smaller bound for the constant.

4. A generalization to several variables. The alternative proof offered for Theorem 2 suggests that the inequalities of this paper can be stated for N variables, $N \ge 2$.

For example, we have:

THEOREM 2′. *Let* $\alpha(x)$ *be continuous, nonnegative, and nondecreasing for* $0 \le x < \infty$. *Let* $p_1 > 1, \cdots, p_N > 1$ *and* $p_1^{-1} + \cdots + p_N^{-1} = 1$. *If* $f_1(x) \ge 0, \cdots, f_N(x) \ge 0$ *for* $0 \le x < \infty$ *and*

$$\int_0^\infty [f_i(x)]^{p_i}\, d\alpha(x) < \infty$$

for $i = 1, \cdots, N$, *then*

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^N f_i(x_i)\, d\alpha(x_i)}{[\alpha(\sum_{i=1}^N x_i)]^{N-1}} \le K_N(\alpha) \prod_{i=1}^N \left\{ \int_0^\infty [f_i(x)]^{p_i} d\alpha(x) \right\}^{1/p_i}$$

with

$$0 < K_N(\alpha) \le \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{N-1} x_i^{-1/p_i} dx_i}{\{\max[x_1, \cdots, x_{N-1}, 1]\}^{N-1}} .$$

If $\alpha(x+y) \ge \alpha(x) + \alpha(y)$, *then*

$$K_N(\alpha) \le \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{N-1} x_i^{-1/p_i} dx_i}{[1 + \sum_{i=1}^{N-1} x_i]^{N-1}} = M_N .$$

If $\alpha(0) = 0$, $\alpha(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$ *and* $\alpha(x+y) \le \alpha(x) + \alpha(y)$, *then* $K_N(\alpha) \ge M_N$.

The proof is patterned on that of Theorem 322 of [**1**, p. 231].

RFERENCE

1. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge, England, 1952.

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