NOTE ON THE SCHWARZ TRIANGLE FUNCTIONS

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1. Introduction. In his classical investigation of the hypergeometric series, Schwarz discussed the function which maps the upper half w-plane onto a curvilinear triangle in the z-plane with angles $\delta\pi$, $\epsilon\pi$, $\eta\pi(\delta+\epsilon+\eta<1)$. The inverse, $w=\phi(z)$, of this function is automorphic with respect to the group got by reflecting the triangle in its sides, reflecting the new figure in *its* free sides, and so on. In order that this process shall lead to a properly discontinuous group, it is necessary and sufficient that $1/\delta$, $1/\epsilon$, $1/\eta$ be positive integers or ∞. We take in particular $\delta=1/q$, $\epsilon=1/2$, $\eta=0(q=3,4,5,\cdots)$, and place the triangle in the upper half plane with vertices at $-\exp \pi i/q$, i, and $i\infty$. The group $\Gamma(\lambda)$ of transformations is then generated by

$$S: z \longrightarrow z+\lambda \text{ and } T: z \longrightarrow -\frac{1}{z},$$

where $\lambda = 2\cos\pi/q\,(q=3,4,5,\cdots)$. (We restrict λ to this countable set from now on.) The automorphic function $\phi_\lambda(z)=\phi(z)$ having a simple pole at $z=i\infty$ thus has the period λ, and we normalize its Fourier expansion as follows:

$$(1.1)\qquad \phi_\lambda(z)=\phi(z)=x^{-1}+\sum_{n=0}^{\infty} c_n(\lambda)x^n,\ x=\exp\frac{2\pi i z}{\lambda}.$$

This makes $\phi(z)$ unique except for an additive constant; ϕ is called a triangle function.

Concerning the Fourier coefficients $c_n(\lambda)$, we wish to make the following observations:

I. *All the Fourier coefficients of any triangle function ϕ_λ are rational numbers.*

II. *The Fourier coefficients $c_n(\lambda)$ have the asymptotic value*

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$$(1.2)\qquad c_n(\lambda) \sim \sqrt{\frac{1}{2\lambda}}\ \frac{e^{4\pi\sqrt{n}/\lambda}}{n^{3/4}}, \qquad n \longrightarrow \infty.$$

Both results can be extended to a wider class of Fuchsian groups; this will be done in future publications.[1]

2. Proof of I. Let $z = \psi(w)$ be the function inverse to ϕ; that is, ψ maps the upper half w-plane onto the triangle in the z-plane. It is well known [**1**, p. 304 f] that ψ is the quotient of two independent solutions of the hypergeometric equation

$$(2.1)\qquad w(w-1)\frac{d^2z}{dw^2} + [(\alpha+\beta+1)w-\gamma]\frac{dz}{dw} + \alpha\beta z = 0,$$

where

$$\alpha = \beta = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),\quad \gamma = 1-\frac{1}{q}.$$

In this case ($\alpha = \beta$), Fricke [**2**, p. 115, (18)] has given an explicit representation of a system of independent solutions valid at $w = \infty$:

$$(2.2)\qquad \begin{aligned} Z_1 &= w^{-\alpha}F(\alpha,\ \alpha-\gamma+1,\ 1;\ 1/w),\\ Z_2 &= w^{-\alpha}[F_1(\alpha,\ \alpha-\gamma+1;\ 1/w) - \log w \cdot F(\alpha,\ \alpha-\gamma+1,\ 1;\ 1/w)], \end{aligned}$$

where F is the ordinary hypergeometric series, and F_1 is a series with coefficients rational in α, β [**2**, p. 114, (15)],

$$F_1(\alpha, \beta; u) = \frac{\alpha\cdot\beta}{1.1}\left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{2}{1}\right)u + \cdots.$$

Both series converge for $|w| > 1$.

For our purposes we take for $z = \psi(w)$ the combination

$$-\frac{2\pi i z}{\lambda} = -\frac{Z_2}{Z_1} = \log w - \frac{F_1}{F} = \log w + \frac{A_1}{w} + \frac{A_2}{w^2} + \cdots,$$

[1]When $\phi(z)$ is Klein's absolute modular invariant $J(z)$, (1.2) is an immediate consequence of the Petersson-Rademacher [**3**, p. 202; **4**] series for $J(z)$.

where, as we see from (2.2), $A_1, A_2, \cdots$, are rational numbers. Hence with $x = \exp 2\pi i z/\lambda$, we have

$$x^{-1} = w \cdot \exp\left(\frac{A_1}{w} + \frac{A_2}{w^2} + \cdots\right) = w\left(1 + \frac{B_1}{w} + \frac{B_2}{w^2} + \cdots\right),$$

where again the B_n are rational.

We now invert this equation, setting

$$w = \phi(z) = x^{-1}(1 + c_0 x + c_1 x^2 + \cdots), \tag{2.3}$$

and have:

$$w^{-1} = x(1 + d_0 x + d_1 x^2 + \cdots),$$

$$x^{-1} = x^{-1}(1 + c_0 x + c_1 x^2 + \cdots)(1 + B_1 x(1 + d_0 x + d_1 x^2 + \cdots)$$
$$+ B_2 x^2(1 + d_0' x + d_1' x^2 + \cdots) + \cdots).$$

The last equation determines the c_n uniquely in a step-by-step manner. They clearly are rational numbers. Furthermore, (2.3) agrees with (1.1). This proves I.

3. Proof of II. From (1.1) we have

$$c_n(\lambda) = \frac{1}{\lambda}\int_C \phi(z) e^{-2\pi i n z/\lambda}\, dz \qquad (n > 0),$$

where C is a path connecting any two points in the upper half plane at the same height and at a distance λ apart. We take C to be the horizontal line

$$z = x + \frac{i}{N}, \quad |x| \le \frac{\lambda}{2};$$

$N > 0$ will eventually be taken of the order of $\sqrt{n}$.

The line C cuts a finite number of fundamental regions of $\Gamma(\lambda) = R_1, R_2, \cdots, R_s$; the corresponding segments are $l_1, l_2, \cdots, l_s$. Thus

$$\lambda c_n(\lambda) = \sum_{j=1}^{s} \int_{l_j} \phi(z) e^{-2\pi i n z/\lambda}\, dz.$$

There is a unique substitution

$$z' = \frac{a_j z + b_j}{c_j z + d_j} \tag{3.1}$$

of $\Gamma(\lambda)$ which carries R_j into R_0, the standard fundamental region with cusp at $i\infty$; the coefficients a_j, b_j, $\cdots$ are real, and $c_j \neq 0$. Thus because of the invariance on ϕ on Γ, we get

$$\lambda c_n(\lambda) = \sum_{j=1}^{s} \int_{l_j} \phi(z') e^{-2\pi i n z/\lambda}\, dz,$$

where z' lies in R_0.

Now, by (1.1), write

$$\phi(z) = e^{-2\pi i z/\lambda} + \psi(z), \ \psi(z) = \sum_{0}^{\infty} c_m e^{2\pi i m z/\lambda};$$

then

$$\lambda c_n(\lambda) = \sum_{j=1}^{s} \int_{l_j} e^{-2\pi i (z' + nz)/\lambda}\, dz + \sum_{j=1}^{s} \int_{l_j} \psi(z') e^{-2\pi i n z/\lambda}\, dz \tag{3.2}$$

$$= \sum_{j=1}^{s} H_j + S_2 = S_1 + S_2.$$

In the following estimates, A will denote a constant, not the same one at each appearance, independent of N and n but possibly depending on λ; θ is an absolute constant of modulus less than unity.

We know that $\psi(z')$ is bounded in R_0 because ϕ is regular in the upper half plane except for a simple pole at $i\infty$; put $|\psi(z')| \leq A$. Hence

$$|S_2| \leq A\, e^{2\pi n/N\lambda} \int_C |dz| \leqq A\, e^{2\pi n/N\lambda}. \tag{3.3}$$

The principal contribution to S_1 will come from the segment lying in the fundamental region, R_1 say, which is the map of R_0 by T: $z' = -1/z$. R_1 is bounded by an arc of the unit circle and by two arcs passing through the origin, the right-hand one having the equation

$$\left(\frac{x-1}{\lambda}\right)^2 + y^2 = \frac{1}{\lambda^2} \qquad (z = x + iy).$$

Hence the endpoints of l_1 are $\pm z_1$, where

$$z_1 = \frac{\theta A}{N^2} + \frac{i}{N}.$$

Let K be the circle, described counter clockwise, with center at the origin and passing through z_1, z_2, and L the larger of the arcs connecting z_1, z_2. We have

$$H_1 = \int_{l_1} = -\int_K \int_L = J_1 + J_2,$$

the integrands being the same as in the first term of the right member of (3.2).

The first integral on the right is calculated by the residue theorem. We have $z' = -1/z$, so

$$J_1 = -\int_K e^{2\pi i(1/z - nz)/\lambda}\, dz = -2\pi i \operatorname*{Res}_{z=0} \sum_{\mu=0}^{\infty} \frac{1}{\mu!}\left(\frac{2\pi i}{\lambda z}\right)^{\mu} \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\left(\frac{-2\pi inz}{\lambda}\right)^{\nu}$$

$$= \frac{2\pi}{\sqrt{n}} \sum_{\nu=0}^{\infty} \frac{(2\pi\sqrt{n}/\lambda)^{2\nu+1}}{\nu!(\nu+1)!} = \frac{2\pi}{\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right),$$

where I_1 is the Bessel function of the first kind with purely imaginary argument. To estimate J_2, we note that on L we have

$$|z|^2 = x^2 + y^2 = |z_1|^2.$$

Thus

$$|J_2| \le \int_L |e^{2\pi i(1/z - nz)/\lambda}\, dz| < 2\pi |z_1| \max_L \exp \frac{2\pi}{\lambda}\left(\frac{y}{\alpha^2 + y^2} + ny\right)$$

$$= 2\pi |z_1| \max_L \exp \frac{2\pi}{\lambda}\left(\frac{y}{|z_1|^2} + ny\right) = 2\pi |z_1| \exp \frac{2\pi}{\lambda N}\left(\frac{1}{|z_1|^2} + n\right),$$

so that

$$J_2 = \theta A N^{-2} \exp 2\pi(n+1)/N\lambda.$$

Putting these results together, we find that

$$(3.4) \qquad H_1 = \frac{2\pi}{\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right) + \theta A \exp 2\pi n/N\lambda .$$

We now estimate the summands of S_1 for which $j \neq 1$. Here the decisive point is that, in (3.1), $|c_j| > 1$ if $j \neq 1$. This is because $1/|c_j|$ is the radius of an isometric circle. The largest isometric circle in the strip $|\Re z| \leq \lambda/2$ is the one corresponding to the transformation $T : z \longrightarrow -1/z$, for which $c = 1$; all the others are smaller. From (3.1) we get, with $z' = x' + iy'$,

$$y' = \frac{y}{(c_j x + d_j)^2 + c_j^2 y^2} \leq \frac{1}{c_j^2 y} \leq \frac{1}{\gamma^2 y},$$

where $\gamma > 1$ is the minimum of $|c_2|, |c_3|, \cdots, |c_s|$. Hence

$$|H_j| \leq \int_{l_j} |e^{-2\pi i(z'+nz)/\lambda}||dz| \leq \int_{l_j} e^{-2\pi(1/\gamma^2 y + ny)/\lambda} dx$$

$$= |l_j| \cdot \exp \frac{2\pi}{\lambda}\left(\frac{N}{\gamma^2} + \frac{n}{N}\right) \qquad (j \neq 1),$$

where $|l_j|$ denotes the length of the segment l_j. Therefore,

$$(3.5) \qquad \sum_{j=2}^{s} |H_j| < \lambda \exp \frac{2\pi}{\lambda}\left(\frac{N}{\gamma^2} + \frac{n}{N}\right).$$

From (3.2), (3.3), (3.4), and (3.5), we now obtain

$$c_n(\lambda) = \frac{2\pi}{\sqrt{n}\lambda} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right) + \theta A \exp 2\pi(n+1)/N\lambda$$

$$+ \theta A \exp 2\pi\left(\frac{N}{\gamma^2} + \frac{n}{N}\right)\Big/\lambda .$$

The first term in the right member is asymptotic to

$$\frac{1}{\sqrt{2\lambda}} \cdot \frac{\exp 4\pi\sqrt{n}/\lambda}{n^{3/4}},$$

by a well-known formula for the Bessel function [5, p. 373]. The last term is made as small as possible by the choice $N = \gamma\sqrt{n}$, in which ase the exponent becomes $4\pi\sqrt{n}/\gamma\lambda$. Since $\gamma > 1$, this term, as well as the second one, is of lower order than the first term, and (1.2) follows.

References

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