

REMARKS ON THE BOREL PROPERTY

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1. Introduction. In this note we study some implications of the results obtained in [4] and [5]. To make the remarks essentially self-contained, we summarize the background and results on which the sequel depends.

Let T denote an arbitrary method of summability corresponding to a real matrix (a_{nk}) , by means of which a sequence $\{s_k\}$ is said to be summable- T to s if each of the series in

$$t_n = \sum_{k=1}^{\infty} a_{nk} s_k \quad (n = 1, 2, 3, \dots)$$

is convergent, and if $t_n \rightarrow s$.

We are concerned with the class X of all sequences $x = \{\alpha_k\}$, where the α_k are 0 or 1 with infinitely many 1's. A biunique mapping of the class X into the real interval

$$Y = (0 < y \leq 1)$$

is obtained by defining y as the dyadic fraction $0 \cdot \alpha_1 \alpha_2 \alpha_3 \dots$ corresponding to

$$x = (\alpha_1, \alpha_2, \alpha_3, \dots),$$

and conversely. This enables us to employ the phrase, "almost all sequences of 0's and 1's," by which is meant a subset of X for which the corresponding subset of Y has Lebesgue measure one.

If almost all sequences of 0's and 1's are summable- T to the value $1/2$, we say that T has the Borel property, or that $T \in (BP)$.

The following theorems are proved in [5].

(1.1) THEOREM. *In order that $T \in (BP)$, the following conditions are necessary:*

Received March 30, 1953, and in revised form December 24, 1953.

Pacific J. Math. 4 (1954), 227-242

$$(1.2) \quad \sum_{k=1}^{\infty} a_{nk} \text{ converges for each } n \text{ and tends to } 1 \text{ as } n \rightarrow \infty;$$

$$(1.3) \quad A_n = \sum_{k=1}^{\infty} a_{nk}^2 < \infty \text{ for each } n;$$

$$(1.4) \quad \lim_{n \rightarrow \infty} a_{nk} = 0 \text{ for each } k;$$

$$(1.5) \quad \lim_{n \rightarrow \infty} A_n = 0.$$

We observe that conditions (1.2) and (1.4) are among the familiar Silverman-Toeplitz conditions for the regularity of T . The remaining condition, namely,

$$(1.6) \quad \sum_{k=1}^{\infty} |a_{nk}| = O(1),$$

is not necessary in order that T have the Borel property [5, p. 403].

(1.7) THEOREM. *In order that $T \in (BP)$, the conditions (1.2) and*

$$(1.8) \quad A_n = o(1/\log n),$$

are sufficient.

It is shown in [5] that no condition of the form (1.8), with $\log n$ replaced by $\psi(n)$, is necessary. However, (1.8) is the best possible condition of its kind, in the sense that there exists a (regular and triangular) matrix (c_{nk}) that does not have the Borel property, and which is such that

$$(\log n) \sum_{k=1}^n c_{nk}^2 \rightarrow 2$$

as $n \rightarrow \infty$ [5, p. 404].

On the basis of these results we proceed to derive a number of criteria sufficient, respectively, to ensure that the methods of Riesz, Nörlund, and Hausdorff shall have the Borel property.

In the Hausdorff case the conditions are necessary as well as sufficient. (The more specialized methods of Abel, Borel, Cesàro, and Euler are known

[4] to have the Borel property.) Finally, we consider an extension to general methods of the results of Buck and Pollard [1] concerning the $(C, 1)$ -summability of a sequence and its subsequences.

2. Simple Riesz means. Let (R, p_k) denote the method of simple Riesz means defined by the matrix

$$a_{nk} = p_k/P_n,$$

where $p_k \geq 0$, $p_1 > 0$, and

$$P_n = p_1 + p_2 + \cdots + p_n \quad (k = 1, 2, \dots, n; n = 1, 2, 3, \dots).$$

The conditions (1.2) and (1.6) are automatically satisfied, and (1.4) reduces to $P_n \rightarrow \infty$ as $n \rightarrow \infty$. Thus:

(2.1) *If $(R, p_k) \in (BP)$, then (R, p_k) is regular, and the latter is equivalent to $P_n \rightarrow \infty$.*

In case the sequence $\{p_k\}$ is nonincreasing ($p_k \downarrow$) we have the following result, the proof of which is independent of Theorem (1.7).

(2.2) *In order that $(R, p_k \downarrow) \in (BP)$, it is necessary and sufficient that it be regular.*

The necessity follows from (2.1), and the sufficiency follows from the facts that a regular $(R, p_k \downarrow)$ includes $(C, 1)$ [6, p. 111], and $(C, 1)$ has the Borel property.

The criterion that follows is given in [4], where it is shown that the convergence of (2.4) is not necessary.

(2.3) *In order that a regular $(R, p_k) \in (BP)$, it is sufficient that the series*

$$(2.4) \quad \sum_{k=1}^{\infty} (p_k/P_k)^2$$

be convergent.

The next two results throw light on the range of applicability of (2.3).

(2.5) *In order that $(R, p_k) \in (BP)$, where $p_k = O(P_k^{1-\epsilon})$ for some $\epsilon (0 < \epsilon < 1)$, it is necessary and sufficient that it be regular.*

The necessity is clear from (2.1). To prove the sufficiency we note that, since $P_k \rightarrow \infty$, the series $\sum p_k/P_k^\alpha$ is convergent for $\alpha > 1$ by the Abel-Dini theorem [6, p.299]. The conclusion then follows from (2.3) and the relation

$$\sum_{k=1}^n (p_k/P_k)^2 = \sum_{k=1}^n (p_k/P_k^{1+\epsilon}) (p_k/P_k^{1-\epsilon}) = O\left(\sum_{k=1}^n p_k/P_k^{1+\epsilon}\right).$$

(2.6) *If (R, p_k) is regular, and (2.4) converges, then there is a constant c such that $P_n \leq \exp(cn^{1/2})$ for all n sufficiently large.*

From the Schwarz inequality we obtain

$$\left[\sum_{k=1}^n (p_k/P_k) \cdot 1 \right]^2 \leq \sum_{k=1}^n (p_k/P_k)^2 \sum_{k=1}^n 1 = n \sum_{k=1}^n (p_k/P_k)^2.$$

Consequently, for all n such that $P_n > 1$,

$$\sum_{k=1}^n (p_k/P_k)^2 \geq \frac{1}{n} \left(\sum_{k=1}^n p_k/P_k \right)^2 = \frac{\log^2 P_n}{n} \frac{(\sum_{k=1}^n p_k/P_k)^2}{\log^2 P_n},$$

where the last fraction, by a theorem of Cesàro [6, p.301], tends to 1 as $n \rightarrow \infty$. Then we must have

$$\log P_n = O(n^{1/2}),$$

from which the conclusion follows at once.

As a corollary of (2.5) we obtain the following result, which includes (2.2) as a special case.

(2.7) *In order that $(R, p_k) \in (BP)$, where $\{p_k\}$ is bounded, it is necessary and sufficient that it be regular.*

For the case of an unbounded sequence $\{p_k\}$, (2.5) provides a lower estimate, and (2.6) an upper estimate, for the admissible rate of increase in so far as the criterion (2.3) is concerned. That some restriction is necessary, regardless of the criterion, is shown by the example of (R, e^k) . The latter is regular, but fails to have the Borel property since the necessary condition (1.5) is violated. We shall see, however, as a consequence of the following considerations, that the restriction imposed by (2.6) is not in general essential.

According to Theorem (1.1), the condition

$$A_n = \sum_{k=1}^n p_k^2 / P_n^2 = o(1)$$

is necessary in order that $(R, p_k) \in (BP)$. This condition can be characterized as follows.

(2.8) *In order that $A_n = o(1)$, the conditions (i) $P_n \rightarrow \infty$, and (ii) $p_n = o(P_n)$, are necessary and sufficient.*

The sufficiency follows from the inequality

$$A_n \leq \frac{1}{P_n} \sum_{k=1}^n p_k (p_k / P_k),$$

since the right side is the regular (R, p_k) -transform of the null sequence $\{p_k / P_k\}$. The necessity is clear from the relations

$$(p_1 / P_n)^2 \leq A_n \quad \text{and} \quad (p_n / P_n)^2 \leq A_n.$$

It is doubtful whether (i) and (ii) are sufficient in order that $(R, p_k) \in (BP)$, but the question remains open. However, from Theorem (1.7) we obtain the following sufficient condition, together with the corollaries (2.11) and (2.13).

(2.9) *The method (R, p_k) will have the Borel property if*

$$(2.10) \quad \sum_{k=1}^n p_k^2 = o(P_n^2 / \log n).$$

(2.11) *If $M_n = \max(p_1, p_2, \dots, p_n)$, and*

$$(2.12) \quad M_n = o(P_n / \log n),$$

then $(R, p_k) \in (BP)$.

For we have

$$A_n \leq M_n \sum_{k=1}^n p_k / P_n^2 = M_n / P_n,$$

so that (2.12) implies (2.10).

The next criterion may be compared with (2.5) if we write the condition given there in the form

$$p_n = O(P_n/P_n^\epsilon).$$

Then in case p_n is nondecreasing ($p_n \uparrow$) we can replace P_n^ϵ , which is $\geq (p_1 n)^\epsilon$, by $\log n$, provided we replace "O" by "o."

(2.13) *The method (R, p_k) will have the Borel property if*

$$(2.14) \quad p_n = o(P_n/\log n).$$

Now let

$$p_k = \exp k^{1/2}.$$

Then

$$P_n \cong 2n^{1/2} p_n,$$

the final condition in (2.6) is violated, and hence the criterion (2.3) fails. On the other hand, the conditions in (2.13) are satisfied, and hence the method $(R, \exp k^{1/2}) \in (BP)$. It is of interest to observe in passing that this method is definitely weaker than $(C, 1)$. This is a consequence of the known facts that $(R, p_k \uparrow)$ is always included in $(C, 1)$, and will be equivalent to $(C, 1)$ if and only if $p_n = O(P_n/n)$.

Since condition (2.14), and therefore (2.10), does not imply the convergence of (2.4), the question of the converse arises. Consider the following example, in which the notation $P(i)$ is used as alternative to P_i . Let

$$p_k = 1 \quad (k = 1, 2, \dots, 2^{16} - 1),$$

and

$$p_k = P(2^{m^4} - 1) / [m^2(\log 2)^{1/2} - 1] \quad \text{for } 2^{m^4} \leq k < 2^{(m+1)^4} \quad (m = 2, 3, 4, \dots).$$

Then it is not difficult to verify that $\{p_k\}$ is nondecreasing and unbounded, that the series (2.4) is convergent, and that

$$(\log n) \sum_{k=1}^n p_k^2 / P_n^2 \geq (\log n) (p_n / P_n)^2 = 1 \text{ for } n = 2^{m^4} \quad (m = 2, 3, 4, \dots).$$

Consequently, of the criteria (2.3) and (2.9) (or (2.13)), neither includes the other.

For the case of an unbounded sequence $\{p_k\}$ it would be of interest to have a sharper limitation on the permissible rate of increase than is provided by (2.3) and (2.9). At the present time, however, we know of no such result.

3. Nörlund means. Let (N, p_k) denote the method of Nörlund means corresponding to the matrix $a_{nk} = p_{n-k+1} / P_n$, where the sequence $\{p_k\}$ satisfies the defining conditions for (R, p_k) , as given above. The conditions (1.2) and (1.6) are again automatically satisfied, but (1.4) in this case reduces to $p_n = o(P_n)$. We therefore have:

(3.1) *If $(N, p_k) \in (BP)$, then (N, p_k) is regular, and the latter is equivalent to $p_n = o(P_n)$.*

A comparison of the matrices of (R, p_k) and (N, p_k) suggests that the behavior of one for increasing (or decreasing) $\{p_k\}$ will correspond in some sense to the behavior of the other for decreasing (or increasing) $\{p_k\}$. This observation is supported by the next result, and, to a certain extent, by those that follow.

(3.2) *In order that $(N, p_k \uparrow) \in (BP)$ it is necessary and sufficient that it be regular.*

The necessity is clear, and the sufficiency is implied by the fact that a regular $(N, p_k \uparrow)$ includes $(C, 1)$ [2, p. 67].

Since the expression for A_n is the same in each case, we note that (2.8), (2.9), and (2.11) apply to (N, p_k) . With this in mind we shall refer to them as (2.8)*, (2.9)*, and (2.11)*. It remains an open question here also whether (i) and (ii) of (2.8)* are sufficient in order that $(N, p_k) \in (BP)$. We remark that (2.13) is, of course, valid for (N, p_k) , but the fact is of no interest in view of (3.2).

In the general case of (N, p_k) , where $\{p_k\}$ may be unbounded, the criteria (2.9)* and (2.11)* are all we can state at the present time. If $\{p_k\}$ is subject to suitable restrictions the criteria that follow may be obtained. As the first of these we have the following analogue of (2.7).

(3.3) If $p_k = O(1)$ and $\liminf p_k > 0$, then $(N, p_k) \in (BP)$.

The conditions imply the existence of positive constants c_1, c_2, k_0 such that

$$p_k \leq c_1 \quad (\text{all } k) \quad \text{and} \quad p_k \geq c_2 \quad (k > k_0).$$

Then

$$P_n \geq P_{k_0} + (n - k_0)c_2 \quad \text{for } n > k_0,$$

so that

$$A_n \leq c_1 / [P_{k_0} + (n - k_0)c_2].$$

The conclusion follows from (2.9)*.

The condition, $\liminf p_k > 0$, in (3.3) can be removed if P_n increases faster than $\log n$.

(3.4) If $p_k = O(1)$ and $\log n = o(P_n)$, then $(N, p_k) \in (BP)$.

This follows from (2.9)* in view of the fact that $A_n = O(1/P_n)$.

If $p_k \rightarrow p \neq 0$, the conditions of (3.3) are satisfied. If $p = 0$, the following holds.

(3.5) If $p_k = o(1)$ and $\log n = O(P_n)$, then $(N, p_k) \in (BP)$.

It follows that $P_n \rightarrow \infty$, and hence that

$$P_n A_n = \sum_{k=1}^n p_k(p_k)/P_n = o(1),$$

since we have here the regular (R, p_k) -transform of the null sequence $\{p_k\}$. Consequently,

$$A_n \log n = [(\log n)/P_n](P_n A_n) = o(1),$$

and the conclusion follows from (2.9)*.

Finally, if we strengthen the first condition in (3.4) or (3.5), the second can be weakened, as follows.

(3.6) If $\sum_{k=1}^{\infty} p_k^2 < \infty$ and $\log n = o(P_n^2)$, then $(N, p_k) \in (BP)$.

This follows from (2.9)* and the inequality

$$A_n \leq \sum_{k=1}^{\infty} p_k^2 / P_n^2.$$

In the event that p_k decreases monotonically to zero, it would be of interest to know if criteria sharper than the preceding exist. It appears likely that such is the case.

We point out that (3.5) and (3.2) combine to give another proof that $(C, \alpha > 0) \in (BP)$ [4, p. 557 and p. 561]. For (C, α) is a Nörlund method, where p_k^α is the binomial coefficient $C_{k-1+\alpha-1, k-1}$, and $P_n^\alpha = C_{n-1+\alpha, n-1}$. The conclusion follows from (3.5), if $0 < \alpha < 1$, and from (3.2) if $\alpha \geq 1$.

4. Hausdorff means¹. We denote by (H, α) the convergence-preserving method of Hausdorff means defined by the matrix of elements

$$h_{nk} = \int_0^1 T_{nk}(u) d\alpha(u) \quad (k = 0, 1, 2, \dots, n; n = 0, 1, 2, \dots),$$

where

$$T_{nk}(u) = C_{n,k} u^k (1-u)^{n-k} \quad \text{on } U = (0 \leq u \leq 1),$$

and $\alpha(u)$ is a function of bounded variation on U , normed by the condition $\alpha(0) = 0$. The regularity of (H, α) , which we do not assume, is characterized by the well-known conditions $\alpha(0^+) = \alpha(0)$ and $\alpha(1) = 1$.

We make use of the following estimate [3, p. 111].

(4.1) *There is a constant K such that for $n = 1, 2, 3, \dots$, $k = 0, 1, 2, \dots, n$, and $0 < u < 1$, we have*

$$T_{nk}(u) < K[nu(1-u)]^{-1/2}.$$

¹Theorem (4.4) and its proof were communicated to the author by George Piranian in a letter of May 13, 1953. With his permission the original and partial results of this section have been replaced by this complete result. We understand that the same theorem has been obtained independently by G. G. Lorentz.

From (4.1) and Theorem (1.7) we deduce the following lemma.

(4.2) *If (H, α) is regular, and there exists a $\delta > 0$ such that $\alpha(u) = 0$ for $0 \leq u \leq \delta$ and such that $\alpha(u) = 1$ for $1 - \delta \leq u \leq 1$, then $(H, \alpha) \in (BP)$.*

Using (4.1), we obtain

$$|h_{nk}| < K \int_{\delta}^{1-\delta} [nu(1-u)]^{-1/2} |d\alpha(u)| \leq K_1 n^{-1/2} \quad (k = 0, 1, 2, \dots, n),$$

where K_1 is a constant. Consequently,

$$h_{nk}^2 \leq K_1 |h_{nk}| n^{-1/2},$$

and since $\sum_k |h_{nk}| = O(1)$ it follows that $A_n = O(n^{-1/2})$. The conclusion is apparent from Theorem (1.7).

The following proposition is an immediate consequence of (4.2).

(4.3) *If $\alpha(u)$ is of bounded variation on U , and there exists a $\delta > 0$ such that $\alpha(u) = 0$ for $0 \leq u \leq \delta$ and such that $\alpha(u) = \alpha(1)$ for $1 - \delta \leq u \leq 1$, then almost all sequences of 0's and 1's are summable- (H, α) to $\alpha(1)/2$.*

(4.4) *In order that $(H, \alpha) \in (BP)$ it is necessary and sufficient that (h_{nk}) be a regular matrix whose diagonal elements h_{nn} tend to zero.*

To prove the necessity we assume $(H, \alpha) \in (BP)$ and apply Theorem (1.1). Condition (1.2) reduces to $\alpha(1) = 1$, and condition (1.4) for $k = 0$ implies $\alpha(0^+) = \alpha(0)$. Therefore (H, α) is regular. The condition (1.5) evidently implies $h_{nn} \rightarrow 0$, and this completes the proof of necessity.

To show that the stated conditions are sufficient we assume for the moment that $\alpha(u)$ is nondecreasing, and we introduce two sequences of nonnegative and nondecreasing functions defined as follows for $m = 1, 2, 3, \dots$. On the intervals

$$(0 \leq u \leq 2^{m-1}), \quad (2^{m-1} \leq u \leq 2^m), \quad (2^m \leq u \leq 1),$$

the function $\alpha_{m1}(u)$ is defined as

$$0, \quad \alpha(u) - \alpha(2^{m-1}), \quad \alpha(2^m) - \alpha(2^{m-1}),$$

respectively. On the intervals

$$(0 \leq u \leq 1 - 2^{-m}), (1 - 2^{-m} \leq u \leq 1 - 2^{-m-1}), (1 - 2^{-m-1} \leq u \leq 1),$$

the function $\alpha_{m2}(u)$ is defined as

$$0, \alpha(u) - \alpha(1 - 2^{-m}), \alpha(1 - 2^{-m-1}) - \alpha(1 - 2^{-m}),$$

respectively. We need now the known [7, p.189] and easily established fact that $h_{nn} \rightarrow 0$ is equivalent to the continuity of $\alpha(u)$ at $u = 1$. Setting

$$\alpha_m(u) = \alpha_{m1}(u) \cap \alpha_{m2}(u),$$

and using the continuity of $\alpha(u)$ at $u = 0$ and $u = 1$, one readily sees that the series $\sum_1^\infty \alpha_m(u)$ converges uniformly to $\alpha(u)$ for $0 \leq u \leq 1$.

Since the method (H, α_m) satisfies the conditions of (4.3) it follows that almost all sequences of 0's and 1's are summable- (H, α_m) to the value $\alpha_m(1)/2$. Let X_m denote the latter set of sequences, and let X^* denote the intersection of all the sets X_m . Consider the transformations

$$y_n(x) = \sum_{k=0}^n h_{nk} x_k, \quad y_n^m(x) = \sum_{k=0}^n h_{nk}^m x_k \quad (\text{all } m \text{ and } n),$$

where (h_{nk}) and (h_{nk}^m) are, respectively, the matrices of (H, α) and (H, α_m) , and where $x = \{x_k\}$ is any sequence in X^* . For the double sequence

$$s_{np}(x) \equiv \sum_{m=1}^p y_n^m(x),$$

the inequality

$$0 \leq y_n(x) - s_{np}(x) \leq \sum_{p+1}^\infty \alpha_m(1)$$

is easily seen to hold, and this establishes the relation

$$\lim_p s_{np}(x) = y_n(x),$$

uniformly in n . On the other hand, we have

$$\lim_n s_{np}(x) = \sum_1^p \alpha_m(1)/2$$

for each p . These facts imply the existence and equality of the corresponding iterated limits, and we therefore obtain

$$\lim_n \gamma_n(x) = \sum_1^{\infty} \alpha_m(1)/2 = \alpha(1)/2 = 1/2$$

for each $x \in X^*$, that is, for almost all x . This proves the theorem when $\alpha(u)$ is nondecreasing.

Finally, if $\alpha(u)$ is any function of bounded variation defining a regular (H, α) , let $\alpha(u) = P(u) - N(u)$ where $P(u)$ and $N(u)$ are, respectively, the positive and negative variations of $\alpha(u)$ on $[0, u]$. Evidently $P(1) > 0$, and we may assume $N(1) > 0$. Then

$$\alpha(u) = P(1) \frac{P(u)}{P(1)} - N(1) \frac{N(u)}{N(1)} \equiv a_1 \alpha_1(u) - a_2 \alpha_2(u),$$

where α_1 and α_2 are regular and nondecreasing generating functions, each continuous at $u = 1$. Since

$$(H, \alpha) = a_1(H, \alpha_1) - a_2(H, \alpha_2) \text{ with } a_1 - a_2 = \alpha(1) = 1,$$

the proof is complete.

We point out the connection between Theorem (4.4) and a theorem of Lorentz [7, p. 189]. According to Lorentz a (bounded) sequence $\{s_k\}$ is *almost convergent* to s if

$$\lim_p (s_{n+1} + s_{n+2} + \cdots + s_{n+p})/p = s$$

holds uniformly in n . He calls a method T *strongly regular* if it evaluates all almost convergent sequences. His necessary and sufficient conditions for the strong regularity of a convergence preserving (H, α) are precisely those of Theorem (4.4). Hence, (H, α) has the Borel property if, and only if, it is strongly regular.

5. Summability of $\{s_k \alpha_k(y)\}$. If $\{s_k\}$ is a given sequence, a biunique mapping of its infinite subsequences $\{s_{k_i}\}$ onto the interval $Y = (0 < y \leq 1)$ may be obtained by defining $y = 0 \cdot \alpha_1 \alpha_2 \alpha_3 \cdots$ (radix 2) by means of the equations

$$\alpha_k = 1 \ (k = k_i) \text{ and } \alpha_k = 0 \ (k \neq k_i).$$

The inverse correspondence is evident if we agree to use only the infinite representation of y . The phrase "almost all subsequences of $\{s_k\}$ " will then mean that the corresponding subset of Y has measure one.

The following results are due to Buck and Pollard [1].

(5.1) THEOREM. *If $\{s_k\}$ is $(C, 1)$ -summable to s , and*

$$(5.2) \quad \sum_{k=1}^{\infty} s_k^2/k^2 < \infty,$$

then almost all of the subsequences are $(C, 1)$ -summable to s .

(5.3) THEOREM. *If almost all of the subsequences of $\{s_k\}$ are $(C, 1)$ -summable, then $\{s_k\}$ is itself $(C, 1)$ -summable to a value s , and almost all of the subsequences are in turn $(C, 1)$ -summable to s . Moreover, it follows that*

$$(5.4) \quad \sum_{k=1}^n s_k^2 = o(n^2).$$

(5.5) THEOREM. *A bounded sequence is $(C, 1)$ -summable if and only if almost all of its subsequences are $(C, 1)$ -summable.*

Tsuchikura [8] has recently shown that the condition (5.2) can be weakened to

$$(5.6) \quad \sum_{k=1}^n s_k^2 = o(n^2/\log \log n),$$

and that "o" in the latter cannot be replaced by "O."

In order to extend these theorems to a general method $T = (a_{nk})$ it appears to be necessary to replace the subsequences $\{s_{k_i}\}$ by the sequences $\{s_k \alpha_k(y)\}$, where $\alpha_k(y)$ is the k th digit in the infinite dyadic expansion of y . For $(C, 1)$, and apparently for $(C, 1)$ only, the summability of $\{s_k \alpha_k(y)\}$ can be given the elegant formulation in terms of subsequences.

When $\{s_{k_i}\}$ is replaced by $\{s_k \alpha_k(y)\}$ it should be noticed, in the event that $\{s_k \alpha_k(y)\}$ is summable to a constant almost everywhere, that we can expect this generalized limit to be only *half* that of $\{s_k\}$, since, in the asymptotic sense, half the terms of $\{s_k \alpha_k(y)\}$ are zeros in almost all cases.

(5.7) Let $T = (a_{nk})$ be any matrix method, and let $\{s_k\}$ be summable- T to $s \neq 0$. If

$$(5.8) \quad \sum_{k=1}^{\infty} a_{nk}^2 s_k^2 = o(1/\log n),$$

then almost all of the sequences $\{s_k \alpha_k(y)\}$ are summable- T to $s/2$. In (5.8), “ o ” cannot be replaced by “ O .”

Set $a_{nk}^* = a_{nk} s_k / s$ and let T^* be defined by the matrix (a_{nk}^*) . Then T^* satisfies the conditions of Theorem (1.7), so that almost all sequences of 0’s and 1’s are summable- T^* to $1/2$. This is equivalent to the theorem.

To see that the result is not true when “ o ” is replaced by “ O ,” choose for (a_{nk}) the matrix (c_{nk}) mentioned in § 1, and for $\{s_k\}$ the sequence all of whose terms are 1.

For the case in which $s = 0$, the following holds.

(5.9) Let $T = (a_{nk})$ be a method satisfying (1.2) and (1.8), and let $\{s_k\}$ be summable- T to 0. If (5.8) holds then almost all of the sequences $\{s_k \alpha_k(y)\}$ are summable- T to 0.

Set $b_{nk} = a_{nk}(s_k + 1)$ and let T' be defined by the matrix (b_{nk}) . It is clear that (b_{nk}) satisfies condition (1.2). Moreover,

$$\begin{aligned} \sum_{k=1}^{\infty} b_{nk}^2 &\leq \sum_{k=1}^{\infty} a_{nk}^2 (|s_k| + 1)^2 = \sum_{|s_k| \leq 1} + \sum_{|s_k| > 1} \\ &\leq 4 \left(\sum_{k=1}^{\infty} a_{nk}^2 + \sum_{k=1}^{\infty} a_{nk}^2 s_k^2 \right) = o(1/\log n), \end{aligned}$$

by (1.8) and (5.8). Then $T' \in (BP)$ by Theorem (1.7); and since $T \in (BP)$ by assumption, the conclusion follows from the relation

$$\sum_{k=1}^{\infty} a_{nk} s_k \alpha_k(y) = \sum_{k=1}^{\infty} b_{nk} \alpha_k(y) - \sum_{k=1}^{\infty} a_{nk} \alpha_k(y).$$

It is an open question here whether “ o ” can be replaced by “ O ” in (5.8).

(5.10) *Suppose that $T = (a_{nk})$ satisfies (1.4), and that almost all of the sequences $\{s_k \alpha_k(y)\}$ are summable- T , say to $s(y)$. Then $\{s_k\}$ is summable- T to a value s , and $s(y) = s/2$ almost everywhere. Moreover, it follows that*

$$(5.11) \quad \sum_{k=1}^{\infty} a_{nk}^2 s_k^2 = o(1).$$

The proof of this result is easily obtained by making appropriate modifications in the proof of Theorem (5.3) [1, p. 926], and combining the latter with the proof of (1.5) [5, pp. 401-402].

If we take T as $(C, 1)$, then (5.7) and (5.9) give the equivalent of Theorem (5.1), with the condition (5.2) replaced by the weaker condition

$$\sum_{k=1}^n s_k^2 = o(n^2/\log n).$$

The latter, however, is stronger than the "best" condition, (5.6), of Tsuchikura. On the other hand, (5.10) applied to $(C, 1)$ yields the equivalent of Theorem (5.3), and (5.11) reduces to (5.4).

As a corollary of (5.7), (5.9), and (5.10), we obtain the following extension of Theorem (5.5).

(5.12) *Let T be a method satisfying (1.2) and (1.8). Then a bounded sequence $\{s_k\}$ is summable- T if and only if almost all of the sequences $\{s_k \alpha_k(y)\}$ are summable- T .*

To formulate the next two results we introduce the Rademacher functions, $R_k(y)$, defined as $1 - 2\alpha_k(y)$, where $\alpha_k(y)$ is defined as above. It is shown in [5] that the condition (1.8) is sufficient to imply the convergence to 0, almost everywhere, of $\sum_k a_{nk} R_k(y)$. This remark gives rise to the following observations.

(5.13) *Let T and $\{s_k\}$ be such that (5.8) holds. Then the sequence $\{s_k R_k(y)\}$ is summable- T to 0 almost everywhere. Moreover, the "o" in (5.8) cannot be replaced by "O."*

(5.14) *Let T satisfy (1.8). Then for every bounded sequence $\{s_k\}$, the sequence $\{s_k R_k(y)\}$ is summable- T to 0 almost everywhere. The "o" in (1.8) cannot be replaced by "O."*

The final statement in each case here may be verified from the remark following the proof of (5.7), by taking into account the relation

$$\sum_{k=1}^n c_{nk} \cdot 1 \cdot R_k(y) = \sum_{k=1}^n c_{nk} - 2 \sum_{k=1}^n c_{nk} \cdot 1 \cdot \alpha_k(y).$$

Since the sequence $\{R_k(y)\}$ represents a random arrangement of +1's and -1's, the observation (5.14) is of interest from the following point of view. If T is a given matrix method, and a bounded sequence $\{s_k\}$ is given arbitrarily, then it is extremely unlikely that $\{s_k\}$ will be summable- T , even if T is regular. On the other hand, if T satisfies (1.8), and the signs of the given s_k are varied at random, then the probability is 1 that the resulting sequence $\{\pm s_k\}$ will be summable- T to 0.

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