AN EXAMPLE CONCERNING UNIFORM BOUNDEDNESS OF SPECTRAL MEASURES

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1. Introduction. Let $\mathfrak{X} = \{x\}$ be a Banach space with a norm $||x||$. A bounded linear operator E which maps $\mathfrak{X}$ into itself is called a *projection* if $E^2 = E$. We do not require that $||E|| \leq 1$, where

$$||E|| = \sup_{||x|| \leq 1} ||Ex||.$$

Let $\mathfrak{B} = \{\sigma\}$ be a Boolean algebra with a unit element 1. We denote the zero element of $\mathfrak{B}$ by 0, and two fundamental operations in $\mathfrak{B}$ by $\sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2$. A family $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ of projections $E(\sigma)$ of $\mathfrak{X}$ into itself is called an $\mathfrak{X}$-*spectral measure* on $\mathfrak{B}$ if the following conditions are satisfied: (i) $E(0) = 0$ (= zero operator), (ii) $E(1) = 1$ (= unit operator), (iii) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$ for any $\sigma_1, \sigma_2 \in \mathfrak{B}$, (iv) $\sigma_1 \cap \sigma_2 = 0$ implies $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)$. An $\mathfrak{X}$-spectral measure $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ is said to be *uniformly bounded* if there exists a constant $K < \infty$ such that $||E(\sigma)|| \leq K$ for all $\sigma \in \mathfrak{B}$.

Let $\mathfrak{B} = \{\sigma\}$, $\mathfrak{B}' = \{\sigma'\}$ be two Boolean algebras with a unit element, and let $\mathfrak{B}^* = \mathfrak{B} \otimes \mathfrak{B}'$ be the Kronecker product of $\mathfrak{B}$ and $\mathfrak{B}'$. Now $\mathfrak{B}^*$ may be considered as the Boolean algebra of all open-closed subsets σ^* of S^*, where $S^* = S \times S'$ is the topological Cartesian product of two Stone representation spaces S, S' of $\mathfrak{B}$, $\mathfrak{B}'$, respectively. Every element $\sigma^* \in \mathfrak{B}^*$ is expressible in the form:

$$\sigma^* = \bigcup_{i=1}^{n} \sigma_i \times \sigma_i', \tag{1.1}$$

where $\sigma_i \in \mathfrak{B}$, $\sigma_i' \in \mathfrak{B}'$ $(i = 1, \cdots, n)$.

Let $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ and $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$ be two $\mathfrak{X}$-spectral measures on $\mathfrak{B}$, $\mathfrak{B}'$, respectively, which are commutative with each other; that is,

$$E(\sigma)E'(\sigma') = E'(\sigma')E(\sigma)$$

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for any $\sigma \in \mathfrak{B}$, $\sigma' \in \mathfrak{B}'$. Let us put

$$(1.2) \qquad F(\sigma^*) = \sum_{i=1}^{n} E(\sigma_i) E'(\sigma_i')$$

if $\sigma^* \in \mathfrak{B}^*$ is of the form (1.1) and if $\sigma_i \times \sigma_i'$ $(i = 1, \cdots, n)$ are disjoint. Then it is easy to see that $F(\sigma^*)$ is uniquely determined (although the expression (1.1) with disjoint $\sigma_i \times \sigma_i'$ is not necessarily unique), and $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is an $\mathfrak{X}$-spectral measure on $\mathfrak{B}^*$; $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is called the *direct product $\mathfrak{X}$-spectral measure of* $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ *and* $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$.

It was asked by N. Dunford [2] whether the uniform boundedness of $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ and $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$ implies that of $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$. This question was answered in the affirmative by J. Wermer [5] in case $\mathfrak{X}$ is a Hilbert space. The main purpose of this note is to show that the answer is negative if $\mathfrak{X}$ is a general Banach space; that is, we want to prove the following proposition:

PROPOSITION. *There exists a Banach space $\mathfrak{X}$ and a commutative pair of uniformly bounded $\mathfrak{X}$-spectral measures for which the direct product $\mathfrak{X}$-spectral measure is not uniformly bounded.*

Such an example will be given in § 3. In our example, the Banach space $\mathfrak{X}$ is given as a cross product space $C(S) \circledast C(S')$ of two Banach spaces of continuous functions which will be defined in § 2. This Banach space is not reflexive and hence it remains open to decide whether the answer to the question is positive or negative in case $\mathfrak{X}$ is a reflexive Banach space.

2. The Banach space $C(S) \circledast C(S')$. Let $S = \{s\}$, $S' = \{s'\}$ be two compact Hausdorff spaces. Let $C(S)$, $C(S')$ be the Banach spaces of all complex-valued continuous functions $y(s)$, $z(s')$ defined on S, S' with the norms

$$\|y\|_\infty = \max_{s \in S} |y(s)|, \qquad \|z\|_\infty = \max_{s' \in S'} |z(s')|.$$

Let

$$S^* = S \times S' = \{s^* = (s, s') \mid s \in S, s' \in S'\}$$

be the topological Cartesian product of S and S', and let $C(S^*)$ be the Banach space of all complex-valued continuous functions

$$x(s^*) = x(s, s')$$

defined on S^* with the norm

$$||x||_\infty = \max_{s^* \in S^*} |x(s^*)|.$$

Now $C(S)$, $C(S')$ may be considered as closed linear subspaces of $C(S^*)$ by identifying $y(s) \in C(S)$, $z(s') \in C(S')$ with $x(s, s') \in C(S^*)$ defined by

$$x(s, s') = y(s), \quad x(s, s') = z(s'),$$

respectively.

Consider $C(S^*)$ as a normed ring with the norm $||x||_\infty$. Then $C(S)$ and $C(S')$ are closed subrings of $C(S^*)$. Let $C(S) \otimes C(S')$ be the subring of $C(S^*)$ algebraically generated by $C(S)$ and $C(S')$; that is, the set of all functions $x(s, s') \in C(S^*)$ of the form:

$$x(s, s') = \sum_{i=1}^{n} y_i(s) z_i(s'), \tag{2.1}$$

where $y_i(s) \in C(S)$, $z_i(s') \in C(S')$ $(i = 1, \cdots, n)$. From the Stone-Weierstrass theorem it follows that $C(S) \otimes C(S')$ is dense in $C(S^*)$.

Let us now introduce a new norm on $C(S) \otimes C(S')$ defined by

$$|||x||| = \inf \sum_{i=1}^{n} ||y_i||_\infty \cdot ||z_i||_\infty, \tag{2.2}$$

where inf is taken for all possible representations of $x(s, s') \in C(S) \otimes C(S')$ in the form (2.1).

It is easy to see that $|||x|||$ is a norm on $C(S) \otimes C(S')$ and satisfies

$$||x||_\infty \leq |||x|||$$

for all $x(s, s') \in C(S) \otimes C(S')$. Let $C(S) \circledast C(S')$ be the completion of $C(S) \otimes C(S')$ with respect to the norm $|||x|||$. The completion $C(S) \circledast C(S')$ is obtained from $C(S) \otimes C(S')$ by means of Cauchy sequences in $C(S) \otimes C(S')$ with respect to the norm $|||x|||$. Since a Cauchy sequence with respect to $|||x|||$ is a Cauchy sequence with respect to $||x||_\infty$, we may consider $C(S) \circledast C(S')$ as a subset of $C(S^*)$:

LEMMA 1. *Let $C(S) \circledast C(S')$ be the set of all functions $x_0(s^*) \in C(S^*)$ for which there exists a sequence $\{x_n(s^*) \mid n = 1, 2, \cdots\}$ of functions from*

$C(S) \otimes C(S')$ *with the following properties*:

(i) $\lim_{n \to \infty} \|x_n - x_0\|_\infty = 0$, *that is* $\lim_{n \to \infty} x_n(s^*) = x_0^*(s)$ *uniformly on* S^*;

(ii) $\lim_{m,n \to \infty} |||x_m - x_n||| = 0$, *that is*, $\{x_n \mid n = 1, 2, \cdots\}$

is a Cauchy sequence with respect to the norm $|||x|||$.

If we put

$$|||x_0||| = \lim_{n \to \infty} |||x_n|||,$$

then $C(S) \circledast C(S')$ *is a Banach space with respect to the norm* $|||x|||$ *and contains* $C(S) \otimes C(S')$ *as a dense subset.*

The proof is easy and so it is omitted. It is interesting to observe that $C(S) \circledast C(S')$ is a normed ring with respect to the norm $|||x|||$.

$C(S) \circledast C(S')$ is called the *minimal cross product Banach space* of $C(S)$ and $C(S')$. It is easy to see that the minimal cross product Banach space $\mathfrak{Y} \circledast \mathfrak{Z}$ of any two Banach spaces $\mathfrak{Y}$ and $\mathfrak{Z}$ can be defined in a similar way. $\mathfrak{Y} \circledast \mathfrak{Z}$ is one of the cross product Banach spaces defined and discussed by R. Schatten and J. von Neumann [**3**; **4**].

3. Construction of an example. Let us now consider the case when both S and S' are Cantor sets. Let $S = S'$ be the set of all real numbers s of the form

$$s = 2\left\{\frac{\epsilon_1(s)}{3} + \frac{\epsilon_2(s)}{3^2} + \cdots + \frac{\epsilon_n(s)}{3^n} + \cdots\right\}, \tag{3.1}$$

where $\epsilon_n(s) = 0$ or 1 $(n = 1, 2, \cdots)$. Let $\mathfrak{B} = \{\sigma\}$ be the Boolean algebra of all open-closed subsets σ of S.

Let $S^* = S \times S$ be the Cartesian product of S with itself, and let $\mathfrak{B}^* = \{\sigma^*\}$ be the Boolean algebra of all open-closed subsets σ^* of S^*. It is clear that $\mathfrak{B}^* = \mathfrak{B} \otimes \mathfrak{B}$; that is, $\mathfrak{B}^*$ consists of all subsets σ^* of S^* which are expressible in the form (1.1), where $\sigma_i, \sigma_i' \in \mathfrak{B}$ $(i = 1, \cdots, n)$.

For each $\sigma \in \mathfrak{B}$, let $\phi_\sigma(s)$ be the characteristic function of σ, and put

$$E(\sigma)x(s, s') = \phi_\sigma(s)x(s, s'), \quad E'(\sigma)x(s, s') = \phi_\sigma(s')x(s, s').$$

It is clear that $E(\sigma)$, $E'(\sigma)$ are projections of $\mathfrak{X} = C(S) \circledast C(S')$ into itself, and that $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$, $\{E'(\sigma) \mid \sigma \in \mathfrak{B}\}$ are $\mathfrak{X}$-spectral measures on $\mathfrak{B}$. Both of these spectral measures are uniformly bounded since $E(\sigma)$, $E'(\sigma)$ have norm 1 for any $\sigma \in \mathfrak{B}$ with $\sigma \neq 0$. Since

$$E(\sigma)E'(\sigma') = E'(\sigma')E(\sigma)$$

for any $\sigma, \sigma' \in \mathfrak{B}$, we can consider the direct product $\mathfrak{X}$-spectral measure $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$, defined on $\mathfrak{B}^* = \mathfrak{B} \otimes \mathfrak{B}$. We shall show that $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is not uniformly bounded.

Let us define a sequence of functions $\{\rho_n(s^*) \mid n = 0, 1, 2, \cdots\}$ defined on $S^* = S \times S$ as follows: $\rho_0(s^*) \equiv 1$ on S^*, and

$$\rho_n(s^*) \equiv \rho_n(s, s') = (-1)^{\sum_{k=1}^{n} \epsilon_k(s)\epsilon_k(s')}, \tag{3.2}$$

where $\epsilon_k(s)$ is the kth coefficient in the expansion (3.1) of s. It is easy to see that $\rho_n(s^*)$ takes only the values ± 1 and belongs to $C(S) \otimes C(S')$ for $n = 0, 1, 2, \cdots$. Let us put

$$\sigma_n^* = \{s^* \mid \rho_n(s^*) = 1\} \qquad (n = 0, 1, 2, \cdots).$$

Then $\sigma_n^* \in \mathfrak{B}^*$ for $n = 0, 1, 2, \cdots$, and it is easy to see that

$$\rho_n = (2F(\sigma_n^*) - I)\rho_0 \qquad (n = 0, 1, 2, \cdots).$$

Thus, in order to prove the proposition of § 1, it suffices to prove the following lemma:

LEMMA 2. *Let S be the Cantor set. Let $\{\rho_n(s^*) \mid n = 1, 2, \cdots\}$ be a sequence of functions defined on $S^* = S \times S$ by (3.2). Then*

$$\lim_{n \to \infty} |||\rho_n||| = \infty,$$

where the norm $|||\rho_n|||$ of ρ_n is defined by (2.2).

In order to prove this lemma, let us put

$$\tau(s) = \frac{\epsilon_1(s)}{2} + \frac{\epsilon_2(s)}{2^2} + \cdots + \frac{\epsilon_n(s)}{2^n} + \cdots. \tag{3.3}$$

Then $t = \tau(s)$ is a mapping of S onto the closed unit interval

$$I = \{t \mid 0 \leq t \leq 1\}$$

which is one-to-one except for a countable set. Let

$$\mu(\sigma) = m(\tau(\sigma))$$

be a measure defined on $B = \{\sigma\}$ which corresponds to the Lebesgue measure m on I. Let us consider the L^2-space $L^2(S;\mu)$ on S with respect to the measure μ, where the norm is given by

$$(3.4) \qquad \|y\|_2 = \left\{\int_S |y(s)|^2 \mu(ds)\right\}^{1/2}.$$

Let $\sigma_i^{(n)}$ be the open-closed subset of S consisting of all $s \in S$ such that

$$(3.5) \qquad \frac{\epsilon_1(s)}{2} + \cdots + \frac{\epsilon_n(s)}{2^n} = \frac{i-1}{2^n} \qquad (i = 1, \cdots, 2^n).$$

We observe that

$$\mu(\sigma_i^{(n)}) = 2^{-n} \qquad (i = 1, \cdots, 2^n)$$

and that $\rho_n(s, s')$ is constant ($= \epsilon_{ij}^{(n)} = \pm 1$) on each $\sigma_i^{(n)} \times \sigma_j^{(n)}$ $(i, j = 1, \cdots, 2^n)$. Further, if we put

$$(3.6) \qquad \rho_j^{(n)}(s) = \rho_n(s, s')$$

for $s \in S$ and $s' \in \sigma_j^{(n)}$ $(j = 1, \cdots, 2^n)$, that is, $\rho_j^{(n)}(s) = \epsilon_{ij}^{(n)}$ if $s \in \sigma_i^{(n)}$, then the functions $\rho_j^{(n)}(s)$ $(j = 1, \cdots, 2^n)$ form an ortho-normal set in $L^2(S;\mu)$. Consequently, by Bessel's inequality,

$$(3.7) \qquad \int_S \left| \int_S \rho_n(s, s')y(s)\mu(ds)\right|^2 \mu(ds')$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} \left| \int_S \rho_j^{(n)}(s)y(s)\mu(ds)\right|^2$$

$$\leq \frac{1}{2^n} \|y\|_2^2$$

for any $y(s) \in L^2(S;\mu)$. From this it follows that

$$(3.8)\quad \left| \int_S \int_S \rho_n(s,s')y(s)z(s')\mu(ds)\mu(ds') \right|^2$$

$$\leqq \left\{ \int_S \left| \int_S \rho_n(s,s')y(s)\mu(ds) \right| \cdot |z(s')|\,\mu(ds') \right\}^2$$

$$\leqq \int_S \left| \int_S \rho_n(s,s')y(s)\mu(ds) \right|^2 \mu(ds') \cdot \int_S |z(s')|^2\,\mu(ds')$$

$$\leqq \frac{1}{2^n} \cdot \|y\|_2^2 \cdot \|z\|_2^2$$

$$\leqq \frac{1}{2^n} \|y\|_\infty^2 \cdot \|z\|_\infty^2$$

for any $y(s)$, $z(s) \in C(S)$. From (3.8) it follows further that

$$(3.9)\quad \left| \int_S \int_S \rho_n(s,s')x(s,s')\mu(ds)\mu(ds') \right| \leqq \sqrt{\frac{1}{2^n}} \cdot |||x|||$$

for any $x(s,s') \in C(S) \otimes C(S')$. Since

$$\rho_n(s,s') \in C(S) \otimes C(S') \text{ and } (\rho_n(s,s'))^2 = 1$$

on $S \times S'$, we obtain, by setting $x(s,s') = \rho_n(s,s')$ in (3.9), that

$$(3.10)\quad |||\rho_n||| \geq \sqrt{2^n} \qquad (n = 1, 2, \cdots),$$

and hence $\lim_{n\to\infty} |||\rho_n||| = \infty$.

4. Remarks. Let us consider the bounded linear operators T, T' defined on $C(S) \circledast C(S')$ by

$$(4.1)\quad Tx(s,s') = f(s)x(s,s'),$$

$$(4.2)\quad T'x(s,s') = f(s')x(s,s'),$$

where $f(s)$ is a continuous function defined on S by

$$(4.3)\quad f(s) = 3\left\{ \frac{\epsilon_1(s)}{4} + \frac{\epsilon_2(s)}{4^2} + \cdots + \frac{\epsilon_n(s)}{4^n} + \cdots \right\}.$$

It is easy to see that T, T' are spectral operators of scalar type and are given by

$$(4.4) \qquad T = \int_S f(s)E(ds),$$

$$(4.5) \qquad T' = \int_S f(s')E'(ds'),$$

where $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ and $\{E'(\sigma) \mid \sigma \in \mathfrak{B}\}$ are a commutative pair of uniformly bounded spectral measures defined in § 3.

It is possible to show that $T + T'$ is not a spectral operator of scalar type. In order to show this we first observe that the range S^{**} of $f(s) + f(s')$ on $S^* = S \times S'$ is a totally disconnected set. Let $\mathfrak{B}_0^*$ be the Boolean algebra of all open-closed subsets σ^* of S^* os the form:

$$\sigma^* = \{s^* = (s, s') \mid f(s) + f(s') \in \sigma^{**}\},$$

where σ^{**} is an open-closed subset of S^{**}. It suffices to show that the family of projections $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}_0^*\}$ is not uniformly bounded.

For each n, let $\{\eta_i^{(n)} \mid i = 1, 2, \cdots\}$ be a sequence of period 2^n; thus

$$\eta_{i+2^n}^{(n)} = \eta_i^{(n)} \qquad (i = 1, 2, \cdots).$$

Further, let the sequence consist only of $+1$ and -1 such that $(\eta_i^{(n)}, \cdots, \eta_{i+n-1}^{(n)})$ runs through all 2^n different sequences of length n consisting of $+1$ and -1 as i runs through $1, \cdots, 2^n$. The existence of such a sequence was proved by N. G. de Bruijn [**1**]. Let us put

$$(4.6) \qquad \pi_n(s^*) = \pi_n(s, s') = \eta_{i+j-1}^{(n)},$$

if $s \in \sigma_i^{(n)}$, $s' \in \sigma_j^{(n)}$ $(i, j = 1, \cdots, 2^n)$. Then $\{\pi_n(s^*) \mid n = 1, 2, \cdots\}$ is a sequence of functions from $C(S) \otimes C(S')$ taking only the values $+1$ and -1 such that the set

$$\sigma_n^* = \{s^* \mid \pi_n(s^*) = +1\} \in \mathfrak{B}_0^* \qquad \text{for } n = 1, 2, \cdots.$$

Thus, by the same reason as in § 3, it suffices to show that

$$\lim_{n \to \infty} |||\pi_n||| = \infty.$$

Let us put

$$\pi_j^{(n)}(s) = \pi_n(s, s')$$

if $s' \in \sigma_j^{(n)}$. Then $\{\pi_j^{(n)}(s) \mid j = 1, \cdots, 2^n\}$ is a set of functions from $L^2(S; \mu)$ such that

$$\{\pi_i^{(n)}(s), \cdots, \pi_{i+n-1}^{(n)}(s)\}$$

is an orthonormal system for $i = 1, \cdots, 2^n - n + 1$. This follows from the fact that

$$j + 1 \leqq k \leqq j + n - 1$$

implies

$$\text{(4.7)} \qquad \int_S \pi_j^{(n)}(s)\, \pi_k^{(n)}(s)\, \mu(ds)$$

$$= \frac{1}{2^n} \sum_{i=1}^{2^n} \eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = 0.$$

(The last equality holds because

$$\eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = +1$$

happens 2^{n-1} times and

$$\eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = -1$$

happens 2^{n-1} times as i runs through $1, \cdots, 2^n$.)

Thus, for any $y \in L^2(S; \mu)$, Bessel's inequality

$$\text{(4.8)} \qquad \sum_{j=i}^{i+n-1} \left| \int_S \pi_j^{(n)}(s)\, y(s)\, \mu(ds) \right|^2 \leqq \|y\|_2^2$$

holds for $i = 1, \cdots, 2^n - n + 1$, and hence

$$(4.9)\qquad \int_S \left| \int_S \pi_n(s, s')y(s)\mu(ds)\right|^2 \mu(ds')$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} \left| \int_S \pi_j^{(n)}(s)y(s)\mu(ds)\right|^2$$

$$\leqq \frac{1}{2^n}\left(\left[\frac{2^n}{n}\right] + 1\right) \|y\|_2^2$$

$$\leqq \left(\frac{1}{n} + \frac{1}{2^n}\right) \|y\|_2^2 \leqq \frac{2}{n} \|y\|_2^2.$$

From this follows, exactly as in § 3, that

$$(4.10)\qquad \left| \int_S \int_S \pi_n(s, s')x(s,s')\mu(ds)\mu(ds')\right| \leqq \sqrt{\frac{2}{n}}\ |||x|||$$

for any $x(s, s') \in C(S) \otimes C(S')$, and hence

$$(4.11)\qquad |||\pi_n||| \geq \sqrt{\frac{n}{2}}$$

for $n = 1, 2, \cdots$.

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