THE RATE OF INCREASE OF REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENTIAL-DIFFERENCE EQUATIONS OF THE FIRST ORDER

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1. Introduction. It is the purpose of this paper to prove several theorems describing the rate of increase, as $t \longrightarrow +\infty$, of real solutions of algebraic differential-difference equations of the form

$$(1) \qquad P(t, u(t), u'(t), u(t+1), u'(t+1)) = 0.$$

In this equation, and throughout this paper, $P(t, u, v, \cdots)$ denotes a polynomial in the variables $t, u, v, \cdots$, with real coefficients, and a prime denotes differentiation with respect to t. In order to explain the significance and limitations of these theorems, it is first necessary to summarize the work, by other investigators, which suggested the present discussion.

In 1899, E. Borel, [1], published a memoir in which he studied the magnitude of solutions of algebraic differential equations. His result, as later improved by E. Lindelöf, [4], is quoted here for reference:

Let $u(t)$ be a real function which is defined and which has a continuous first derivative for all t larger than t_0, and which satisfies the first order algebraic differential equation

$$(2) \qquad P(t, u(t), u'(t)) = 0$$

for $t > t_0$. Then there is a positive number k, which depends only on P, such that

$$|u(t)| < \exp(t^k/k)$$

for $t \geq t_0$.

It is noteworthy that it is impossible to prove a result of the above type for higher order equations. For a discussion of this point, refer to Vijayaraghavan, [7].

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Extensions of the Borel-Lindelöf method to difference equations have already been effected by Lancaster, [3], and Shah, [5] and [6]. Shah demonstrated that no theorem comparable to that of Borel and Lindelöf can be obtained for the class of algebraic difference equations of the form

$$P(t, u(t), u(t+1)) = 0. \tag{3}$$

For, let $g(t)$ be an arbitrary increasing function which becomes indefinitely large as $t \longrightarrow +\infty$. Shah proved that it is possible to construct an equation of the type (3) with a real solution $u(t)$ which exists and is continuous for $t \geq t_0$ and which exceeds $g(t)$ at each point of a sequence $\{t_n\}$ such that $t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$. The situation with respect to higher order equations is similar. Shah did, however, obtain the following weaker results concerning the possible rate of growth of solutions of (3):

There exists a positive number A, which depends only on the polynomial P, with the following property: if $u(t)$ is, for all $t \geq t_0$, a real continuous solution of (3), then there is no number T such that[1]

$$|u(t)| > e_2(At) \qquad \textit{for all } t \geq T.$$

That is, for each such solution $u(t)$ there is a sequence $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that

$$|u(t)| \leq e_2(At) \tag{4}$$

for $t = t_1, t_2, \cdots$. If $u(t)$ is a real, continuous, monotonic solution of (3) for $t \geq t_0$, then there exists a number $\tau \geq t_0$ such that (4) holds for all $t \geq \tau$.

We shall now turn to a discussion of the class of differential-difference equations of the form (1). We first make the following definition.

DEFINITION. A real function $u(t)$ will be said to be a *proper solution* of a differential-difference equation (1) if there exists a number t_0 such that $u(t)$ exists and is a solution of (1) for all $t \geq t_0$, and such that $u(t)$ has a continuous first derivative for $t \geq t_0$.

In view of Shah's results on difference equations, it is not to be expected that a theorem analogous to the Borel-Lindelöf theorem should hold for first

[1] We here employ the notation $e_2(x) = \exp(e^x)$ which was adopted by G. H. Hardy, [2].

order differential-difference equations. However, it might be expected that a result like that of Shah could be obtained for equations of the class (1). This is not the case, as is shown by the following theorem.

THEOREM 1. *Let $g(t)$ be an arbitrary increasing function which becomes indefinitely large as $t \longrightarrow +\infty$. It is possible to construct an algebraic differential-difference equation of the form*

$$(1) \qquad P(t, u(t), u'(t), u(t+1), u'(t+1)) = 0$$

which has a proper solution $u(t)$ which exceeds $g(t)$ for all t. This statement remains valid if equation (1) is replaced by the equation

$$(5) \qquad P(t, u(t), u'(t), u'(t+1)) = 0.$$

Proof. We shall prove this theorem at once by constructing a suitable example. Define a function $u(t)$ as follows. Let $u(t) = g(n+2) + 1$ in the interval $[n, n+1]$, for $n = 0, 2, 4, \cdots$. In the intervals $[n, n+1]$, where $n = 1, 3, 5, \cdots$, let $u(t)$ be any continuous, non-decreasing function which has a continuous first derivative, and for which

$$u(n) = g(n+1) + 1, \; u(n+1) = g(n+3) + 1, \; u'(n) = u'(n+1) = 0.$$

It is clear that the function so defined satisfies the equation

$$(6) \qquad u'(t)u'(t+1) = 0$$

for all $t > 0$. Furthermore, $u(t)$ is non-decreasing for all t and $u(t) > g(t)$ for $t \geq 0$. Since equation (6) is in the class of equations of the form (1), and in the class of equations of the form (5), the proof of Theorem 1 is complete.

This theorem is in sharp contrast to those for algebraic differential or difference equations. It shows that no bound at all can be placed on the rate of growth of solutions of differential-difference equations of the form (1). The same difficulty intrudes even if we speak only of monotone solutions.

It is, however, possible to obtain useful bounds on the rate of growth of solutions of less general classes of differential-difference equations. We observe first of all that, according to Theorem 1, no results like those of Borel or Shah can be obtained for the class of equations of the form (5). We shall, however, prove analogous results for equations of the following types:

$$(7) \qquad P(t, u(t), u'(t+1)) = 0$$

$$(8) \qquad P(t, u(t), u'(t), u(t+1)) = 0$$

$$(9) \qquad P(t, u'(t), u(t+1)) = 0.$$

Even for such equations it is not possible to establish a theorem like the Borel-Lindelöf theorem. This may be seen from the following simple counterexample. Let $g(t)$ be an arbitrary real, continuous, increasing function which becomes indefinitely large as $t \longrightarrow +\infty$. Let m be any non-negative integer. Let $u(t) = t^m$ for t in the intervals $[2n, 2n+1]$, $n = 0, 1, 2, \cdots$. For t in the intervals $(2n+1, 2n+2)$, $n = 0, 1, 2, \cdots$, let $u(t)$ be defined in any convenient fashion for which $u'(t)$ is continuous and $u(2n+3/2) > g(2n+3/2)$. This function $u(t)$ exceeds $g(t)$ for arbitrarily large values of t, and satisfies each of the following equations for all $t > 0$:

$$(10) \qquad \{u'(t+1) - m(t+1)^{m-1}\}\{u(t) - t^m\} = 0$$

$$(11) \qquad [u(t+1) - (t+1)^m][u'(t) - mt^{m-1}] = 0.$$

Note that (10) is an equation in the class (7) and equation (11) is in the class (8) and (9). Furthermore, all the above remarks are correct for $m = 0$, in which case (10) and (11) are equations with constant coefficients. The following theorem has therefore been proved.

THEOREM 2. *Let $g(t)$ be an arbitrary increasing function which becomes indefinitely large as $t \longrightarrow +\infty$. It is possible to construct a first order algebraic differential-difference equation of the form*

$$(7) \qquad P(t, u(t), u'(t+1)) = 0$$

with a proper solution $u(t)$ which exceeds $g(t)$ at each point of a sequence $\{t_n\}$ for which $t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$. The statement remains true if (7) is replaced by equation (8) or equation (9), or by one of the equations

$$(12) \qquad P(u(t), u'(t+1)) = 0$$

$$(13) \qquad P(u'(t), u(t+1)) = 0.$$

Although we cannot establish theorems of the Borel-Lindelöf type for the classes of equations mentioned above, we have proved several results analogous to those of Shah. These results are stated in Theorems 3, 5, and 6 of § 3 below. Moreover, in Theorem 4, stated below, we have proved a theorem of the Borel-Lindelöf type for a certain subclass of equations of the type (7). No theorems are given in this paper for equations with higher order derivatives or differences, since results like those mentioned above can be obtained only for rather special classes of such equations.

2. Lemmas. In this section, we shall prove several lemmas which will be required in proving the theorems of § 3.

LEMMA 1. *Suppose that $u(t)$ is, for all $t \geq t_0$, a positive function with a continuous first derivative. Let A and B be two positive numbers for which $B < e^A$. Let C be an arbitrary non-negative number. Suppose that there is a sequence $\{\tau_n\}$ for which $\tau_n \longrightarrow +\infty$ as $n \longrightarrow \infty$ and for which $u(\tau_n) \geq e_2(A\tau_n)$. Then there exists a sequence $\{t_n\}$ for which $t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$ and for which*

$$u'(t_n + 1) > t_n^C u(t_n)^B.$$

Proof. Assume that $u(t)$ is a positive function with a continuous first derivative, and that

$$u'(t+1) \leq t^C u(t)^B \tag{14}$$

for all $t \geq T$. We shall prove that as a consequence there is a number T_2 such that

$$u(t) < e_2(At) \tag{15}$$

for $t \geq T_2$. This will prove Lemma 1. We divide the proof of (15) into two cases.

Case 1. We assume that $B > 1$. We may, of course, suppose that T is as large as is convenient; choose T so large that

$$B^{j-1} \log T > j - 1 \qquad (j = 1, 2, 3, \cdots). \tag{16}$$

This is certainly true for j sufficiently large if $\log T > 0$, and by choosing T large enough we can ensure that it is true for all j. Then for $j = 1, 2, 3, \cdots$,

$$(2T)^{B^{j-1}} \geq 2T^{B^{j-1}} > T + e^{j-1} \geq T + j. \tag{17}$$

Having chosen T, define

$$M' = \max_{T \leq t \leq T+1} u(t), \qquad M = \max(M', 1).$$

We shall now prove by induction that

$$u(t) \leq M^{B^n} \prod_{j=0}^{n} (T+j)^{(C+1)B^{n-j}} \tag{18}$$

for $T+n \leq t \leq T+n+1$ $(n = 0, 1, 2, \cdots)$. This is evident for $n = 0$. Suppose that (18) has been proved for $n = k-1$ $(k \geq 1)$. Then by (18) and (14)

$$u'(t+1) \leq (T+k)^C M^{B^k} \prod_{j=0}^{k-1} (T+j)^{(C+1)B^{k-j}}$$

for $T+k-1 \leq t \leq T+k$. Upon observing that the right hand side of inequality (18) is an increasing function of n, and employing (14) again, we get

$$u(T+k) \leq u(T+1) + \int_T^{T+k-1} u'(t+1)\,dt$$

$$\leq M + (T+k)^C (k-1) M^{B^k} \prod_{j=0}^{k-1} (T+j)^{(C+1)B^{k-j}}$$

$$\leq (T+k)^C (T+k-1) M^{B^k} \prod_{j=0}^{k-1} (T+j)^{(C+1)B^{k-j}}.$$

On integrating the first inequality under (18) from $T+k-1$ to t, where $t \leq T+k$, and combining with the inequality just derived, we obtain

$$u(t+1) \leq t(T+k)^C M^{B^k} \prod_{j=0}^{k-1} (T+j)^{(C+1)B^{k-j}} \qquad (T+k-1 \leq t \leq T+k).$$

Replacing t by $T+k$ in the right member of the above inequality, we see that (18) is valid for $n = k$. This completes the inductive proof of (18).

We now employ (17). (18) takes the form

$$u(t) \leq \left[M(2T)^{(n+1)(C+1)}\right]^{B^n} = e_2\,[n \log B + \log\{(1+n)(C+1)\log(2T) + \log M\}]$$

for $T+n \leq t \leq T+n+1$. Let $R = \max(2T, M)$. Then

$$u(t) \leq e_2\,[n \log B + \log(nC+n+C+2) + \log\log R]$$

for $T+n \leq t \leq T+n+1$. Since $\log B < A$ by hypothesis, (15) follows.

Case 2. We now assume that $B \leq 1$. Using the same method as in Case 1, we can easily prove by induction that

$$(19) \qquad u(t) \leq M \prod_{j=0}^{n} (T+j)^{C+1}$$

for $T+n \leq t \leq T+n+1$ $(n = 0, 1, 2, \cdots)$. Hence

$$u(t) \leq M(T+n)^{(n+1)(C+1)} \leq Mt^{(C+1)(t-T+1)}$$

for $T+n \leq t \leq T+n+1$. (15) follows at once. This completes the proof of Lemma 1.

LEMMA 2. *Suppose that* $u(t)$ *is, for all* $t \geq t_0$, *a positive non-decreasing function with a continuous first derivative, and that* $u(t) \geq e_2(At)$ *for* $t \geq t_0$. *Let* B *and* C *be any non-negative numbers for which* $B + C < e^A$, *and let* D *be any non-negative number. Then, given any positive number* ϵ, *there exists a sequence* $t_1, t_2, \cdots$ $(t_n \longrightarrow +\infty$ *as* $n \longrightarrow \infty)$ *such that*

$$(20) \qquad u(t_n+1) \geq u(t_n)^B u'(t_n)^C$$

$$(n = 1, 2, \cdots).$$

$$(21) \qquad t_n^D u(t_n) \leq u'(t_n) \leq u(t_n)^{1+\epsilon}$$

Proof. We divide the proof into two cases.

Case 1. Suppose that $u'(t) \geq t^D u(t)$ for all sufficiently large t, say for $t \geq t_0$. It will be sufficient to prove the lemma for values of ϵ so small that $(B+C)(1+\epsilon) < e^A$. Let ϵ be any such number, and let $\alpha = (B+C)(1+\epsilon)$.

Borel, [1], proved that if a function $u(t)$ is, for all $t \geq t_0$, a positive, non-decreasing function with a continuous first derivative, then, given any positive number ϵ, $u'(t) \geq u(t)^{1+\epsilon}$ at most on a set of intervals the sum of whose lengths is a finite number (which depends on ϵ). This result will hereafter be referred to as Borel's Lemma.

If $u(t)$ satisfies the hypotheses of Lemma 2, then by Borel's Lemma there is a number $T \geq t_0$ such that $u'(t) \leq u(t)^{1+\epsilon}$ for all $t \geq T$, except for t belonging to a set E of intervals of total length less than 1/2. We can now choose a number $\tau > T$ such that no point of the sequence $\tau, \tau+1, \tau+2, \cdots$, belongs to E. It follows that (21) holds for each point $t_n = \tau + n$. We shall now show that (20) holds at the points of an infinite subsequence of the sequence $\{\tau + n\}$. If this is not true, there is an integer N such that

$$u(\tau + n + 1) < u(\tau + n)^B u'(\tau + n)^C \qquad \text{for all } n \geq N.$$

This implies that

$$u(\tau + n + 1) < u(\tau + n)^\alpha \qquad \text{for } n \geq N.$$

It follows that

$$u(\tau + N + m) < e_2[m \log \alpha + \log \log u(\tau + N)]$$

for $m = 1, 2, 3, \cdots$. Since $\log \alpha < A$, this contradicts the hypothesis that $u(t) \geq e_2(At)$ for $t \geq t_0$. It follows that there is an infinite subsequence of the sequence $\{\tau + n\}$ at which (20) is valid. This completes the proof in Case 1.

Case 2. The alternative to the supposition of Case 1 is that $u'(t) < t^D u(t)$ for arbitrarily large values of t. We define α as in Case 1, and again suppose ϵ so small that $\log \alpha < A$. From the fact that $u(t) \geq e_2(At)$ it follows that $u'(t) > t^D u(t)$ for arbitrarily large t. By continuity of $u(t)$ and $u'(t)$, $u'(t) < t^D u(t)$ in open intervals, and $u'(t) \geq t^D u(t)$ in closed intervals. Let the open intervals be

$$(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n), \cdots$$

$(a_1 \geq t_0)$ and let the closed intervals be

$$[b_1, a_2], \ [b_2, a_3], \cdots, [b_n, a_{n+1}], \cdots.$$

Note that $a_n \longrightarrow +\infty$ and $b_n \longrightarrow +\infty$, and that

$$u'(a_n) = a_n^D u(a_n) \text{ and } u'(b_n) = b_n^D u(b_n). \tag{22}$$

By Borel's Lemma, $u'(t) \leq u(t)^{1+\epsilon}$ for all t except for t in a set E of open intervals of finite total length. Let E_n be the subset of E contained in $[b_n, a_{n+1}]$ and let L_n be the sum of the lengths of the intervals of E_n. Then $\lim L_n = 0$ as $n \longrightarrow \infty$. We shall prove that there are arbitrarily large values of n for which there is at least one point t_n in the interval $[b_n, a_{n+1}]$ such that

$$u(t_n + 1) \geq u(t_n)^B u'(t_n)^C$$

and such that t_n is not in E_n. The proof will be by contradiction. Assume the contrary. Then there is a positive integer N such that, for every $n \geq N$,

$$u(t+1) < u(t)^B u'(t)^C \tag{23}$$

for all t which are in $[b_n, a_{n+1}]$ but not in E_n.

First we suppose that $0 < \alpha \leq 1$. Since $u(t) \geq e_2(At)$, we may select an integer $p \geq N$ such that

$$u(b_n)^\epsilon > b_n^D \qquad \text{for all } n \geq p.$$

Equations (22) therefore imply that

$$u(b_n)^{1+\epsilon} > u'(b_n) \qquad \text{for all } n \geq p.$$

Hence b_n is not in E_n if $n \geq p$. Consequently (23) implies that

$$u(b_p+1) < u(b_p)^B u'(b_p)^C < u(b_p)^\alpha \leq u(b_p).$$

But $u(t)$ is non-decreasing. Thus we have reached a contradiction, and (23) cannot be true if $0 < \alpha \leq 1$.

Suppose, then, that $\alpha > 1$. Just as before, we may select an integer $p \geq N$ such that b_n is not in E_n for $n \geq p$. We also choose p so large that $L_n < 1$ for $n \geq p$ and so large that

$$\max_{\zeta \geq b_p} \frac{(D+1)\zeta^D}{\alpha^\zeta \log \alpha} < \alpha^{-b_p} \log u(b_p). \tag{24}$$

This is possible because the right-hand member becomes indefinitely large as $p \longrightarrow +\infty$, since $u(t) \geq e_2(At)$ and $A > \log \alpha$, and because the maximum in the left member is finite. Define $c_p = b_p$. We shall now employ an inductive method to establish the existence of a sequence $c_p, c_{p+1}, c_{p+2}, \cdots$, for which

$$\log u(c_{p+i}) \leq \alpha^{c_{p+i} - c_p + \sum \delta_j} \log u(c_p) \tag{25}$$

$(i = 0, 1, 2, \cdots)$, where the summation is over all $j \geq p$ for which $b_j \leq c_{p+i-1}$, and where the δ_j are defined below. In the first place, it is clear that (25) is true for $i = 0$. Suppose that we have established the existence of points c_{p+1}, $c_{p+2}, \cdots, c_{p+k-1}$ $(k \geq 1)$ for which (25) holds. There are now two possibilities:

(a) One possibility is that the point c_{p+k-1} lies in an interval $[b_q, a_{q+1}]$ for some value of q. If this is so, c_{p+k-1} may lie in E_q, or it may not. Let $\epsilon_{q,1}$ be the smallest non-negative number such that $c_{p+k-1} - \epsilon_{q,1}$ is in $[b_q, a_{q+1}]$ but not in E_q. Such a number exists, since b_q is not in E_q. Then, by (23),

$$u(c_{p+k-1} + 1 - \epsilon_{q,1}) < u(c_{p+k-1} - \epsilon_{q,1})^B u'(c_{p+k-1} - \epsilon_{q,1})^C.$$

By Borel's Lemma and the fact that $u(t)$ is non-decreasing, this gives rise to

$$u(c_{p+k-1} + 1 - \epsilon_{q,1}) < u(c_{p+k-1})^{\alpha}.$$

Since $\epsilon_{q,1} < L_q < 1$, the points c_{p+k-1} and $c_{p+k-1} + 1 - \epsilon_{q,1}$ cannot lie in the same interval of E_q. If $c_{p+k-1} + 1 - \epsilon_{q,1} > a_{q+1}$, we define $c_{p+k} = c_{p+k-1} + 1 - \epsilon_{q,1}$. If not, we proceed as follows. Let $\epsilon_{q,2}$ be the smallest non-negative number such that $c_{p+k-1} + 1 - \epsilon_{q,1} - \epsilon_{q,2}$ is in $[b_q, a_{q+1}]$ but not in E_q. Using (23) again, we find that

$$u(c_{p+k-1} + 2 - \epsilon_{q,1} - \epsilon_{q,2}) < u(c_{p+k-1} + 1 - \epsilon_{q,1} - \epsilon_{q,2})^{\alpha}$$

and therefore that

$$\log u(c_{p+k-1} + 2 - \epsilon_{q,1} - \epsilon_{q,2}) < \alpha^2 \log u(c_{p+k-1}).$$

We continue in this manner, obtaining a sequence of points

$$c_{p+k-1},\ c_{p+k-1} + 1 - \epsilon_{q,1},\ c_{p+k-1} + 2 - \epsilon_{q,1} - \epsilon_{q,2},\ \cdots,$$

no two of which can lie in the same interval of E_q. Let

$$c_{p+k-1} + r - \epsilon_{q,1} - \cdots - \epsilon_{q,r}$$

be the first point in this sequence which is larger than a_{q+1}; such a point must exist since

$$\delta_q = \epsilon_{q,1} + \epsilon_{q,2} + \cdots + \epsilon_{q,r} \leq L_q < 1.$$

Define

$$c_{p+k} = c_{p+k-1} + r - \epsilon_{q,1} - \cdots - \epsilon_{q,r}.$$

Then

$$\log u(c_{p+k}) < \alpha^r \log u(c_{p+k-1}) = \alpha^{c_{p+k}-c_{p+k-1}+\delta_q} \log u(c_{p+k-1}).$$

Combining this result with (25) for $i = k - 1$, we find that (25) holds for $i = k$, with the choice of c_{p+k} made above.

(b) The alternative to (a) is that c_{p+k-1} lies in an interval (a_q, b_q) for some value of q. (In this case $k \geq 2$, since $c_p = b_p$.) Now $u'(t) < t^D u(t)$ for all t in this interval. Hence, by integration,

$$u(b_q) < u(c_{p+k-1}) \exp(b_q^{D+1} - c_{p+k-1}^{D+1}). \tag{26}$$

In this case, we define $c_{p+k} = b_q$. We shall now show that, with this choice of c_{p+k}, (25) is satisfied for $i = k$. From the extended theorem of the mean and the inequality (24) we can deduce

$$c_{p+k}^{D+1} - c_{p+k-1}^{D+1} < (\alpha^{c_{p+k}} - \alpha^{c_{p+k-1}})\alpha^{-c_p} \log u(c_p).$$

This inequality will still be true if, in the right member, we place the additional factor

$$\alpha^{\sum \delta_j},$$

where the summation is over all $j \geq p$ for which $b_j \leq c_{p+k-2}$. Using this result, (26), and (25) for $i = k - 1$, we obtain

$$\log u(c_{p+k}) < c_{p+k}^{D+1} - c_{p+k-1}^{D+1} + \log u(c_{p+k-1}) < \alpha^{c_{p+k}-c_p+\sum \delta_j} \log u(c_p).$$

The inequality (25) for $i = k$ is an immediate consequence.

This completes the proof that there is a sequence of points c_{p+i} for which (25) is valid. It is clear that $c_{p+i} \longrightarrow +\infty$ as $i \longrightarrow \infty$. Since the sum of the δ_j $(j = 1, 2, \cdots)$ is no greater than the sum L of the lengths of all the intervals of E,

$$\log u(t) \leq \alpha^{t-c_p+L} \log u(c_p) = \exp[(t - c_p + L)\log \alpha + \log\log u(c_p)]$$

for $t = c_{p+i}$ $(i = 0, 1, 2, \cdots)$. Since $\log \alpha < A$, it is a consequence of the above inequality that there is a positive integer I such that $\log u(t) < \exp(At)$ for $t = c_{p+i}$ $(i \geq I)$. This contradicts the hypothesis of Lemma 2. Therefore (23) cannot be true if $\alpha > 1$. Hence, no matter what the value of α, there are arbitrarily large values of n for which there is at least one point t_n in the interval

$[b_n, a_{n+1}]$ such that

$$u(t_n + 1) \geq u(t_n)^B u'(t_n)^C,$$

and such that t_n is not in E_n. Since t_n is not in E_n, $u'(t_n) \leq u(t_n)^{1+\epsilon}$ for each such t_n. Since t_n lies in $[b_n, a_{n+1}]$, we have $u'(t_n) \geq t_n^D u(t_n)$. This completes the proof of the lemma.

LEMMA 3. *Suppose that $u(t)$ is, for all $t \geq t_0$, a positive function with a continuous first derivative, and that $u(t) \geq e_2(At)$ for all $t \geq t_0$. Let C be any non-negative number less than e^A. Then there is a sequence $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that*

$$u(t_n + 1) \geq u'(t_n)^C, \quad u'(t_n) \geq e^{t_n}.$$

Proof. First we suppose that there is a number $T \geq t_0$ such that $u'(t) \geq e^t$ for $t \geq T$. Then $u(t)$ is non-decreasing for $t \geq T$, and the result follows at once from Lemma 2.

On the other hand, suppose that $u'(t) < e^t$ for arbitrarily large values of t. Since $u(t) \geq e_2(At)$, $u'(t) > e^t$ for arbitrarily large values of t. Therefore there is a sequence of numbers $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that $u'(t_n) = \exp(t_n)$. There exists a positive integer N such that

$$u(t_n + 1) \geq e_2\{A(t_n + 1)\} > (e^{t_n})^C = u'(t_n)^C$$

for $n \geq N$. This completes the proof of Lemma 3.

3. Theorems. We can now state and prove the theorems alluded to in the last paragraph of the introduction. The first of these is the following.

THEOREM 3. *Consider any equation of the form*

$$P(t, u(t), u'(t+1)) = 0. \tag{7}$$

There exists a positive number A, which depends only on the polynomial P, with the following property: to each proper solution $u(t)$ of (7) there corresponds a sequence $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that

$$|u(t)| < e_2(At) \tag{4}$$

for $t = t_n$ ($n = 1, 2, 3, \cdots$). That is, if $u(t)$ is a proper solution of (7) then there *is no number* $T > 0$ *for which* $|u(t)| \geq e_2(At)$ *for all* $t \geq T$.

Proof. Equation (7) may be written in the form

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} T_{ijk} = 0 \tag{27}$$

where

$$T_{ijk} = a_{ijk} t^i u(t)^j u'(t+1)^k.$$

The a_{ijk} are real numbers independent of t. Among the terms T_{ijk} there is one term T_{pqr} selected in the following way:

(1) Choose $r = K$.

(2) Choose q to be the greatest of the values of j among all the terms T_{ijr}.

(3) Choose p to be the greatest of the values of i among all the terms T_{iqr}.

The term T_{pqr} so defined will be called the *principal term*.

Except for constant factors, the ratios T_{ijk}/T_{pqr} are of the following three possible types (excluding the ratio T_{pqr}/T_{pqr}):

(a)
$$\left\{\frac{t^{r_0} u(t)^{r_1}}{u'(t+1)}\right\}^{r-k}$$

where r_0 and r_1 are rational numbers and $r > k$.

(b)
$$\left\{\frac{t^{r_2}}{u(t)}\right\}^{q-j}$$

where r_2 is a rational number and $q > j$.

(c)
$$t^{i-p}$$

where $p > i$. Let R be the least non-negative number which is greater than or equal to the maximum value of r_1 for all ratios of type (a). Let A be any positive number such that $e^A > R$.

Now suppose that $u(t)$ is a proper solution of (7) and that $u(t) \geq e_2(At)$ for $t \geq T$. Choose B so that $R < B < e^A$. It follows from Lemma 1 that there exists a sequence $\{t_n\}$ for which $t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$ and for which $u'(t_n+1) > u(t_n)^B$. For each value $t = t_n$, the function $u(t)$ satisfies not only equation

(7), but also the equation

$$(28) \qquad \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \frac{T_{ijk}}{T_{pqr}} = 0.$$

Since $u(t) \geq e_2(At)$, all ratios of types (b) or (c) approach zero as $t_n \longrightarrow +\infty$. Each ratio of type (a) is bounded by

$$\left\{\frac{t^{r_0} u(t)^R}{u(t)^B}\right\}^{r-k}$$

when $t = t_n$, for appropriate values of r_0 and k. Since $B > R$ and $r > k$, each such ratio approaches zero on the sequence $\{t_n\}$. It now follows that we may find a positive integer N such that the sum of all ratios T_{ijk}/T_{pqr} is less than one in absolute value when $t = t_N$, whereas $T_{pqr}/T_{pqr} = 1$. Thus (28) cannot be satisfied at the point t_N. This contradiction shows that a proper solution $u(t)$ of (7) cannot satisfy $u(t) \geq e_2(At)$ for all $t \geq T$.

Moreover, a proper solution $u(t)$ of (7) cannot satisfy $u(t) \leq -e_2(At)$ for all $t \geq T$. For if it could, the function $U(t) = -u(t)$ would satisfy $U(t) \geq e_2(At)$ for $t \geq T$ and would be a proper solution of an equation of the type (7). We have just shown that this is impossible. Since a proper solution is continuous, this completes the proof of Theorem 3.

The following theorem gives a much stronger result than does Theorem 3, but for a smaller class of equations.

THEOREM 4. *Let $u(t)$ be a non-decreasing or non-increasing proper solution of an equation of the form*

$$(29) \qquad \sum_{i=0}^{I} a_{iLK} t^i u(t)^L u'(t+1)^K + \sum_{i,j,k} a_{ijk} t^i u(t)^j u'(t+1)^k = 0,$$

wherein the a_{ijk} are constants and the latter summation is a triple summation over the ranges $i = 0, 1, \cdots, I$; $j = 0, 1, \cdots, J$; $k = 0, 1, \cdots, K-1$. (L may be greater than J, equal to J, or less than J.) Then there exists a number $A > 0$, which depends only on the form of (29), and there exists a number $T > 0$, which depends on (29) and on $u(t)$, such that

$$(4) \qquad |u(t)| < e_2(At)$$

for all $t \geq T$.

Proof. The method used in the proof of Theorem 3 for selecting the principal term T_{pqr} leads to the choice $p = I$, $q = L$, $r = K$ for the equation (29). Except for constant factors, the ratios T_{ijk}/T_{pqr} are of the following two possible types (excluding the ratio T_{pqr}/T_{pqr}):

(a) $$\left\{\frac{t^{r_0}u(t)^{r_1}}{u'(t+1)}\right\}^{K-k}$$

where r_0 and r_1 are rational and $K > k$.

(b) $$t^{i-I}$$

where $I > i$. Define R, A, and B as in the proof of Theorem 3. Let C be any positive number for which $C/2$ is larger than the maximum value of r_0 for all ratios of type (a).

Now suppose that $u(t)$ is a proper, non-decreasing solution of (29) for which $u(t) \geq e_2(At)$ for a sequence $\{\tau_n\}$ of values of t for which $\tau_n \longrightarrow +\infty$ as $n \longrightarrow \infty$. It follows from Lemma 1 that there exists a sequence $\{t_n\}$ for which $t_n \longrightarrow +\infty$ and for which $u'(t_n + 1) > t_n^C u(t_n)^B$. For each value $t = t_n$, the function $u(t)$ satisfies not only equation (29), but also the equation (28) obtained by dividing by the principal term. But for $t = t_n$ all ratios of type (b) approach zero as $n \longrightarrow \infty$. Each ratio of type (a) is bounded by

$$\left\{\frac{t^{C/2}u(t)^R}{t^C u(t)^B}\right\}^{K-k}.$$

Since $B > R$ and $K > k$, and since $u(t_n) \longrightarrow +\infty$ as $t_n \longrightarrow +\infty$, each such ratio approaches zero. We thus obtain the same contradiction as in the proof of Theorem 3. No such solution $u(t)$ can exist. Therefore to each proper non-decreasing solution $u(t)$ there corresponds a $T > 0$ such that $|u(t)| < e_2(At)$ for all $t \geq T$.

If a proper, non-increasing solution $u(t)$ exists for which $u(t) \leq -e_2(At)$ for $t = \tau_n$ $(n = 1, 2, \cdots)$, where $\tau_n \longrightarrow +\infty$ as $n \longrightarrow \infty$, we define $U(t) = -u(t)$, and obtain the same contradiction. Therefore to each proper, non-increasing solution $u(t)$ there corresponds a $T > 0$ such that $|u(t)| < e_2(At)$ for all $t \geq T$. This completes the proof of Theorem 4.

Our next theorem is as follows.

THEOREM 5. *Consider any equation of the form*

$$P(t, u(t), u'(t), u(t+1)) = 0. \tag{8}$$

There exists a positive number A, which depends only on the polynomial P, with the following property: to each proper non-decreasing or non-increasing solution $u(t)$ of (8) there corresponds a sequence $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that

$$|u(t)| < e_2(At) \tag{4}$$

for $t = t_n$ ($n = 1, 2, \cdots$). That is, if $u(t)$ is any proper non-decreasing or non-increasing solution of (8), there is no number $T > 0$ for which $|u(t)| \geq e_2(At)$ for all $t \geq T$.

Proof. Equation (8) may be written in the form

$$\sum_{h=0}^{H} \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} T_{hijk} = 0$$

where

$$T_{hijk} = a_{hijk} t^h u(t)^i u'(t)^j u(t+1)^k.$$

The a_{hijk} are real numbers independent of t. We select a principal term T_{pqrs} in the following way. Let S be the set of all terms T_{hijk}. Let S_1 be the subset of S consisting of those terms for which $k = K$. Let M_1 be the maximum value of $i + j$ for all terms in S_1. Let S_2 be the set consisting of those terms of S_1 for which $i + j = M_1$. Let M_2 be the maximum value of j for all terms in S_2. Let S_3 be the set containing those terms of S_2 for which $j = M_2$. Let M_3 be the maximum value of h for all terms in S_3. There is a unique term in S_3 for which $h = M_3$. This term will be called the principal term. We shall use the symbol T_{pqrs} for it.

Except for constant factors, the ratios T_{hijk}/T_{pqrs} are of the following possible types (excluding the ratio T_{pqrs}/T_{pqrs}):

$$\left\{\frac{t^{r_0} u(t)^{r_1} u'(t)^{r_2}}{u(t+1)}\right\}^{s-k} \tag{a}$$

where r_0, r_1, and r_2 are rational numbers and $s > k$.

(b) $$t^{h-p}u(t)^{i-q}u'(t)^{j-r}$$

where $q+r > i+j$. Since i, j, q, and r are integers, terms of type (b) fall into one of the following two sub-classes.

(1) $$\frac{t^{h-p}u(t)^m}{u'(t)^{m+n}}$$

where m is an integer, $h-p$ is an integer, n is a positive integer, and $m+n$ is a non-negative integer.

(2) $$\frac{t^{h-p}u'(t)^m}{u(t)^{m+n}} = \left\{\frac{t^{r_3}u'(t)}{u(t)^{1+r_4}}\right\}^m$$

where m and n are positive integers, $h-p$ is an integer, r_3 is a rational number, and r_4 is a positive rational number.

(c) $$\left\{\frac{t^{r_5}u(t)}{u'(t)}\right\}^{r-j}$$

where r_5 is a rational number and $r > j$.

(d) $$t^{h-p}$$

where $p > h$.

Let R_0 be the maximum value of r_0 for all ratios of type (a). Let R_1' be the maximum value of r_1 for all ratios of type (a), and let $R_1 = \max(0, R_1')$. Let R_2' be the maximum value of r_2 for all ratios of type (a), and let $R_2 = \max(0, R_2')$. Let A be any number such that $e^A > R_1 + R_2$. Select any numbers B and C for which $B > R_1$, $C > R_2$, and $B + C < e^A$. Let R_3 be the maximum value of r_3 and let M be the maximum value of m for all ratios of type (b2). Let R_4 be the minimum value of r_4 for all ratios of type (b2). Let ϵ be any positive number less than $R_4/2$. Let R_5' be the maximum value of r_5 for all ratios of type (c), and let $R_5 = \max(0, R_5')$. Select any number D for which $D > R_5$.

Now assume that there exists a proper, non-decreasing solution $n(t)$ of (8) for which $u(t) \geq e_2(At)$ for all $t \geq t_0$. By Lemma 2 there exists a sequence $\{t_n\}$ such that (20) and (21) are satisfied. For each value $t = t_n$, $u(t)$ satisfies not only equation (8), but also the equation

$$\sum_{h=0}^{H}\sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K}\frac{T_{hijk}}{T_{pqrs}}=0.$$

Now since $u(t) \geq e_2(At)$ and $u'(t) \geq t^D u(t)$ when $t = t_n$, all ratios of types (bl), (c), and (d) approach zero as $t_n \longrightarrow +\infty$. Also, each ratio of type (a) is, according to (20), bounded by

$$\left\{\frac{t^{R_0}u(t)^{R_1}u'(t)^{R_2}}{u(t)^B u'(t)^C}\right\}^{s-k}$$

when $t = t_n$; and each ratio of type (b2) is, according to (21), bounded by

$$\left\{\frac{t^{R_3}u'(t)}{u(t)^{1+2\epsilon}}\right\}^M < \left\{\frac{t^{R_3}u'(t)}{u'(t)u(t)^{\epsilon}}\right\}^M$$

when $t=t_n$. Since $B > R_1$ and $C > R_2$, all these ratios tend to zero as $t_n \longrightarrow +\infty$. This conclusion yields a contradiction, just as in the proofs of the earlier theorems. Therefore no such solution $u(t)$ can exist.

The assumption that a proper, non-increasing solution $u(t)$ satisfies $u(t) \leq -e_2(At)$ for all $t \geq t_0$ may be shown to lead to a contradiction by defining $U(t) = -u(t)$.

The conclusion stated in Theorem 5 follows.

Our final theorem is the following.

THEOREM 6. *Consider any equation of the form*

$$P(t, u'(t), u(t+1)) = 0. \tag{9}$$

There exists a positive number A, which depends only on the polynomial P, with the following property: to each proper solution $u(t)$ of (9) there corresponds a sequence $t_1, t_2, \cdots$ ($t_n \longrightarrow +\infty$ as $n \longrightarrow \infty$) such that

$$|u(t)| < e_2(At) \tag{4}$$

for $t = t_n$ ($n = 1, 2, \cdots$). That is, if $u(t)$ is any proper solution of (9), there is no positive number T for which $|u(t)| \geq e_2(At)$ for all $t \geq T$.

Proof. Equation (9) may be written in the form (27), where

$$T_{ijk} = a_{ijk}t^i u'(t)^j u(t+1)^k.$$

The principal term T_{pqr} is selected as follows:

(1) $r = K$;

(2) q is the greatest of the values of j among all terms T_{ijr};

(3) p is the greatest of the values of i among all terms T_{iqr}.

By using Lemma 3, the proof of Theorem 6 may now be completed in much the same way as before. We omit the details.

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