ON THE NUMBER OF PRIMITIVE PYTHAGOREAN TRIANGLES WITH AREA LESS THAN n

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1. Introduction. In the preceding paper Lambek and Moser have shown that if $P_a(n)$ is the number of primitive Pythagorean triangles with area less than n then

$$(1) \qquad P_a(n) = c\,n^{1/2} + O(n^{1/3}),$$

where

$$c = \frac{\Gamma^2(1/4)}{2^{1/2}\pi^{5/2}}.$$

They conjecture that

$$(2) \qquad P_a(n) = c\,n^{1/2} - c'\,n^{1/3} + o(n^{1/3}),$$

and on the basis of a table due to Miksa they find

$$(3) \qquad c' \approx .295.$$

Our purpose here is to show that

$$(4) \qquad P_a(n) = c\,n^{1/2} - c'\,n^{1/3} + O(n^{1/4}\ln n),$$

where

$$(5) \qquad c' = -\frac{\zeta(1/3)\,(1+2^{-1/3})}{\zeta(4/3)\,(1+4^{-1/3})} \approx .297.$$

In the paper by Lambek and Moser, the problem of calculating $P_a(n)$ has been reduced to that of counting the number of lattice points $L(n)$ in the region R_1 defined by

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$$(6) \qquad xy(y^2 - x^2) < n, \ y > x > 0.$$

They obtain

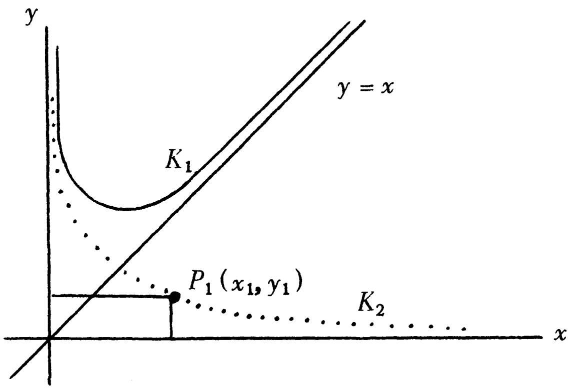
$$(7) \qquad L(n) = \frac{\Gamma^2(1/4)}{2^{5/2}\pi^{1/2}} n^{1/2} + O(n^{1/3}).$$

We shall obtain, in place of (7),

$$(8) \qquad L(n) = \frac{\Gamma^2(1/4)}{2^{5/2}\pi^{1/2}} n^{1/2} + (1 + 2^{-1/3})\,\zeta(1/3)\, n^{1/3} + O(n^{1/4}).$$

2. Proof of (8). Following is the graph of

$$(9) \qquad K_1 : xy(y^2 - x^2) = n.$$



From K_1 we obtain the curve K_2 by replacing y in K_1 by $y + x$ to get

$$(10) \qquad K_2 : xy(x + y)(2x + y) = n.$$

This transformation preserves the area and number of lattice points in R_1. So we count the lattice points in R_2 defined by

$$(11) \qquad xy(x + y)(2x + y) < n, \ x > 0, \ y > 0.$$

By Cardan's formulas, we obtain, from (10),

$$(12) \quad x = \left(\frac{n}{4y}\right)^{1/3} \left\{ \left[1 + \left(1 - \frac{y^8}{108n^2}\right)^{1/2} \right]^{1/3} + \left[1 - \left(1 - \frac{y^8}{108n^2}\right)^{1/2} \right]^{1/3} \right\} - \frac{y}{2}$$

and

$$(13)\qquad y=\left(\frac{n}{2x}\right)^{1/3}\left\{\left[1+\left(1-\frac{4x^8}{27n^2}\right)^{1/2}\right]^{1/3}+\left[1-\left(1-\frac{4x^8}{27n^2}\right)^{1/2}\right]^{1/3}\right\}-x.$$

In (13) take

$$x=x_1=\left(\frac{27}{4}\right)^{1/8}n^{1/4},$$

say, so that

$$y=y_1=\left(\left(\frac{64}{3}\right)^{1/8}-\left(\frac{27}{4}\right)^{1/8}\right)n^{1/4},$$

thus determining the point $p_1 : (x_1, y_1)$ on K_2.

Let square brackets denote the greatest integer function. Then from the figure we have

$$(14)\qquad L(n)=\sum_{x=1}^{[x_1]}[y]+\sum_{y=1}^{[y_1]}[x]-[x_1][y_1]-l(n),$$

where $l(n)$ is the number of lattice points on K_2. Now $l(n)$ is zero unless n is an integer N. For nonintegral n we can prove (8). For small positive ϵ we obtain (8) for, say, $N+\epsilon$ and $N-\epsilon$, so that trivially

$$(15)\qquad l(n)=O(n^{1/4}) \text{ for all real } n.$$

By definition,

$$(16)\qquad [x_1]=O(n^{1/4}),\ [y_1]=O(n^{1/4}),$$

so that we may drop the brackets in (14) with an error $O(n^{1/4})$. Then, by use of (15) and (16), (14) becomes

$$(17)\qquad L(n)=\sum_{x=1}^{[x_1]}y+\sum_{y=1}^{[y_1]}x-x_1y_1+O(n^{1/4}).$$

We shall estimate the above sums by the Euler-Maclaurin summation formula in the form:

$$(18) \quad \sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{1}{2} f(b) + \frac{1}{2} f(a) + \int_a^b \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx.$$

We obtain from (17):

$$(19) \quad L(n) = \int_1^{[x_1]} y\,dx + \int_1^{[y_1]} x\,dy - x_1 y_1 + O(n^{1/4})$$

$$+ \frac{1}{2} y([x_1]) + \frac{1}{2} y(1) + \frac{1}{2} x([y_1]) + \frac{1}{2} x(1)$$

$$+ \int_1^{[x_1]} \left(x - [x] - \frac{1}{2}\right) \frac{dy}{dx}\,dx + \int_1^{[y_1]} \left(y - [y] - \frac{1}{2}\right) \frac{dx}{dy}\,dy.$$

In the first two terms of (19), we may drop brackets with an error of $O(n^{1/4})$, so that, if A represents the entire area of R_2, we may replace the first three terms of (19) by

$$(20) \qquad A - \int_0^1 y\,dx - \int_0^1 x\,dy + O(n^{1/4}).$$

Now from (12) and (13) we have

$$(21) \qquad x = (n/4y)^{1/3} (2^{1/3} + O(y^{8/3}/n^{2/3})) + O(y)$$

$$= (n/2y)^{1/3} + O(y^{7/3}/n^{1/3}) + O(y),$$

and similarly

$$(22) \qquad y = (n/x)^{1/3} + O(x^{7/3}/n^{1/3}) + O(x).$$

Substituting in (20), we obtain

$$(23) \qquad A - 3n^{1/3}/2 + O(n^{-1/3}) + O(1)$$

$$- 3n^{1/3}/2^{4/3} + O(n^{-1/3}) + O(1) + O(n^{1/4})$$

$$= A - \frac{3}{2}(1 + 2^{-1/3}) n^{1/3} + O(n^{1/4}).$$

The fifth and seventh terms of (19) are $O(n^{1/4})$. Also

$$\frac{1}{2}y(1) = \frac{1}{2}n^{1/3} + O(1) \text{ and } \frac{1}{2}x(1) = n^{1/3}/2^{4/3} + O(1).$$

Differentiating the expansions of x and y in (21) and (22) we obtain

$$dx/dy = -n^{1/3}y^{-4/3}/3 \cdot 2^{1/3} + O(y^{4/3}/n^{1/3}) + O(1), \tag{24}$$

$$dy/dx = -n^{1/3}x^{-4/3}/3 + O(x^{4/3}/n^{1/3}) + O(1). \tag{25}$$

We now rewrite the last two terms of (19) as

$$\int_1^{n^{1/8}}\left(x-[x]-\frac{1}{2}\right)\frac{dy}{dx}\,dx + \int_{n^{1/8}}^{[x_1]}\left(x-[x]-\frac{1}{2}\right)\frac{dy}{dx}\,dx \tag{26}$$

$$+\int_1^{n^{1/8}}\left(y-[y]-\frac{1}{2}\right)\frac{dx}{dy}\,dy + \int_{n^{1/8}}^{[y_1]}\left(y-[y]-\frac{1}{2}\right)\frac{dx}{dy}\,dy.$$

Since $|dy/dx|$ is monotonic decreasing, we have, by the second mean value theorem for integrals, and (25),

$$\int_{n^{1/8}}^{[x_1]}\left(x-[x]-\frac{1}{2}\right)\frac{dy}{dx}\,dx = \left.\frac{dy}{dx}\right|_{x=n^{1/8}}\int_{n^{1/8}}^{h}\left(x-[x]-\frac{1}{2}\right)dx \tag{27}$$

$$= O(n^{1/6})\,O(1) = O(n^{1/6}).$$

Similarly

$$\int_{n^{1/8}}^{[y_1]}\left(y-[y]-\frac{1}{2}\right)\frac{dx}{dy}\,dy = O(n^{1/6}). \tag{28}$$

Substituting (24) and (25) in the remaining terms of (26) yields

$$\int_1^{n^{1/8}}\left(x-[x]-\frac{1}{2}\right)\frac{dy}{dx}\,dx = -\frac{n^{1/3}}{3}\int_1^{n^{1/8}}\left(x-[x]-\frac{1}{2}\right)x^{-4/3}\,dx + O(1) \tag{29}$$

$$= -\frac{n^{1/3}}{3}\int_1^{\infty}\left(x-[x]-\frac{1}{2}\right)x^{-4/3}\,dx + \frac{n^{1/3}}{3}\int_{n^{1/8}}^{\infty}\left(x-[x]-\frac{1}{2}\right)x^{-4/3}\,dx + O(1)$$

$$= -\frac{n^{1/3}}{3}\int_1^{\infty}\left(x-[x]-\frac{1}{2}\right)x^{-4/3}\,dx + O(n^{1/6}),$$

and similarly

$$(30)\qquad \int_1^{n^{1/8}} \left(y - [y] - \frac{1}{2}\right)\frac{dx}{dy}\, dy = -\frac{n^{1/3}}{3 \cdot 2^{1/3}} \int_1^{\infty} \left(y - [y] - \frac{1}{2}\right) y^{-4/3}\, dy$$

$$+ O(n^{1/6}).$$

Collecting the preceding results, we have

$$(31)\qquad L(n) = A - \frac{3}{2}(1 + 2^{-1/3})n^{1/3} + O(n^{1/4}) + O(n^{1/4}) + n^{1/3}/2 + O(1)$$

$$+ n^{1/3}/2^{4/3} + O(1) + (1 + 2^{-1/3})c_1 n^{1/3} + O(n^{1/6})$$

$$= A - (1 + 2^{-1/3})(1 - c_1)n^{1/3} + O(n^{1/4}),$$

where

$$(32)\qquad c_1 = \int_1^{\infty} \left(x - [x] - \frac{1}{2}\right) dx^{-1/3} = \zeta(1/3) + 1.$$

Now A is the area of R_2 and therefore the area of R_1. Its value as calculated by Lambek and Moser is

$$(33)\qquad A = c_2\, n^{1/2}, \quad c_2 = \frac{\Gamma^2(1/4)}{2^{5/2}\pi^{1/2}} .$$

Substitution from (32) and (33) in (31) yields (8).

3. Derivation of (4). Let $c_3 = -(1 + 2^{-1/3})\,\zeta(1/3)$, so that (31) becomes

$$(34)\qquad L(n) = c_2 n^{1/2} - c_3 n^{1/3} + O(n^{1/4}).$$

Following the notation of Lambek and Moser, we can write (34) as

$$(35)\qquad L(Rt) = c_2 t^2 - c_3 t^{4/3} + O(t).$$

From their equation (14) we have

$$(36)\qquad L'(Rt) = \sum_{i \geq 1} \mu(i) \left\{ c_2 \frac{t^2}{i^2} - c_3 \frac{t^{4/3}}{i^{4/3}} + O(t/i) \right\}$$

$$= \{6c_2/\pi^2 + O(1/t)\}t^2 - \{c_3/\zeta(4/3) + O(1/t^{1/3})\}t^{4/3} + O(t \ln t)$$

$$= \frac{6c_2}{\pi^2} t^2 - \frac{c_3}{\zeta(4/3)} t^{4/3} + O(t \ln t).$$

Then from their equations (6), (15), and our (36), we have

$$(37) \qquad P_a(n) = \sum_{i \geq 0} (-1)^i \left\{ \frac{6c_2 n^{1/2}}{\pi^2 2^i} - \frac{c_3 n^{1/3}}{\zeta(4/3) 4^{i/3}} + O\left(\frac{n^{1/4}}{4^{i/4}} \ln \frac{n}{4^i} \right) \right\}$$

$$= 4c_2 n^{1/2}/\pi^2 - \frac{c_3 n^{1/3}}{\zeta(4/3)(1 + 4^{-1/3})} + O(n^{1/4} \ln n)$$

$$= c\, n^{1/2} - c' n^{1/3} + O(n^{1/4} \ln n).$$

This is (4).

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