ON GROUPS OF ORTHONORMAL FUNCTIONS (I)

N. J. FINE

1. Introduction. Recently Civin [3, 4] and Chrestenson [2] have considered three specific systems of orthornormal functions on the unit interval which form multiplicative groups. They have shown that (subject to further restrictions) these systems are essentially characterized by their group structure. In this paper we propose to remove the topological restrictions on the base space and the group-theoretic restrictions on the system of functions.

Let $(\Omega, \mathfrak{Z}, m)$ be an abstract measure space[1], with m a countably-additive measure defined on the σ-ring $\mathfrak{Z}$, and $m(\Omega) = 1$. We may, and shall, assume that m is complete. Let

$$F = \{f_\alpha\} \qquad (\alpha = 0, 1, 2, \cdots)$$

be a family of complex-valued measurable functions on Ω, satisfying

$$(1) \qquad \int_\Omega f_\alpha \overline{f_\beta}\, dm = \delta_{\alpha\beta} \qquad (\alpha, \beta \geq 0),$$

$$(2) \qquad f_\alpha \overline{f_\beta} \in F \qquad (\alpha, \beta \geq 0).$$

We shall prove the following theorem:

THEOREM 1. *If $(\Omega, \mathfrak{Z}, m)$ and F are as above, then there exists a (unique) compact Abelian group H, satisfying the second axiom of countability, and a transformation T defined almost everywhere on Ω into H, such that*

(3) *the outer ν-measure of $Z = T(\Omega)$ is 1, and Z is dense in H, where ν denotes the Haar measure on H with $\nu(H) = 1$;*

(4) *for every ν-measurable set $M \subset H$, $T^{-1}(M) \in \mathfrak{Z}$ and $m(T^{-1}(M)) = \nu(M)$;*

---

[1] For the general measure- and group-theoretic concepts considered here, see [6] and [7].

Received August 29, 1953. This paper was written while the author was a National Science Foundation Post-doctoral Fellow, 1953-54.

Pacific J. Math. 5 (1955), 51-59

(5) *the functions* $f_\alpha T^{-1}$ *are single-valued on* Z, *and may be extended to* H *to form the character group of* H.

The transformation T *is onto if and only if*

(6) *for every sequence* $\{\omega_n\} \in \Omega$ *such that*

$$f_\alpha(\omega_n) \longrightarrow v_\alpha \qquad (\alpha \geq 0),$$

there exists $\omega \in \Omega$ *such that*

$$f_\alpha(\omega) = v_\alpha \qquad (\alpha \geq 0).$$

The transformation T *is one-to-one almost everywhere if and only if*

(7) *for almost all* $\omega \in \Omega$, $f_\alpha(\omega') = f_\alpha(\omega)$, $(\alpha \geq 0)$, *implies* $\omega' = \omega$.

In the examples considered by Civin and Chrestenson, $\Omega = I$, the unit interval, and m is Lebesgue measure. In [3], the conditions on F imply easily that it is isomorphic with the group of Walsh functions[2]. H is then the dyadic group. We have shown [5, § 2] that there is a mapping λ of H onto I which is one-to-one almost everywhere, measure-preserving, and carries the characters of H into the Walsh functions. The combined mapping λT of I onto I therefore takes F into the Walsh functions, is one-to-one almost everywhere, and $(\lambda T)^{-1}$ preserves measure, provided that (6) and (7) hold. This is exactly Civin's Theorem 3 of [3]. In [2], F is isomorphic to Ψ_α, the group of generalized Walsh functions of order α defined in [1]. H is then the α-adic group, the countable direct product of cyclic groups of order α. A mapping λ similar to that mentioned above obviously exists, and Chrestenson's result in [2] follows. In [4], F is infinite cyclic, so H is the group of reals mod 1, which we can map onto I in an obvious way. The character group of H is generated by exp $(2\pi i x)$, and if $f_1(x)$ is the generator of F, our results show that

$$f_1(x) = \exp(2\pi i c(x)), \quad 0 \leq c(x) < 1,$$

almost everywhere, and that $c(x)$ is equimeasurable with x.

This last result of Civin's shows that the distribution of $f_1(x)$ is uniform on the unit circle in the complex plane. We may also consider the joint distribution of the f_α in the general framework of Theorem 1. We shall prove the following result:

---

[2] For a treatment of the Walsh functions and for further references, see [5].

THEOREM 2. *Under the conditions of Theorem* 1, *if* $f_1, \cdots, f_n \in F$ *satisfy no relations of the form*

$$f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} = 1 \qquad (\textit{almost everywhere}) \tag{8}$$

other than the obvious ones imposed by their orders, then they are statistically independent functions. The marginal distribution of f_α *is uniform if* f_α *has infinite order, and assigns measure* $1/r$ *to the rth roots of unity if* f_α *is of finite order r.*

The general situation is only slightly more complicated. We have:

THEOREM 3. *Under the conditions of Theorem* 1, *for any set of functions* $f_1, \cdots, f_n \in F$, *there exists a statistically independent set* $f_{\alpha_1}, \cdots, f_{\alpha_n} \in F$ *such that almost everywhere*

$$f_\alpha = \prod_{j=1}^{n} f_{\alpha_j}^{c_{\alpha j}} \qquad (\alpha = 1, \cdots, n). \tag{9}$$

The matrix $(c_{\alpha j})$ *has integer elements and determinant* 1. *It can be constructed as soon as all the relations of the form* (8) *are given.*

2. Proof of Theorem 1. First we show that $|f_\alpha(\omega)| = 1$ for almost all $\omega \in \Omega$. By (2), $|f_\alpha|^2 \in F$, so $|f_\alpha|^{2n} \in F$ $(n = 1, 2, \cdots)$. Hence, by (1),

$$\int_\Omega |f_\alpha|^{4n} dm = 1 \qquad (n = 1, 2, \cdots).$$

Therefore

$$m\{\omega : |f_\alpha(\omega)| > 1\} = 0.$$

If

$$A = \{\omega : |f_\alpha(\omega)| = 1\},$$

then

$$1 = \int_A dm + \int_{\Omega - A} |f_\alpha|^{4n} dm \longrightarrow m(A),$$

so $m(A) = 1$. We redefine the f_α so that $|f_\alpha(\omega)| = 1$ everywhere. Now the function (say) $f_0 \equiv 1 \in F$, so for every β,

$$f_\beta^{-1} = \overline{f}_\beta = f_0 \overline{f}_\beta \in F.$$

Hence F is a multiplicative group.

Now define an equivalence relation on Ω by

$$(10) \qquad \omega_1 \sim \omega_2 \Longleftrightarrow f_\alpha(\omega_1) = f_\alpha(\omega_2) \qquad (all\ \alpha).$$

Let X denote the set of equivalence classes x, and let ρ be the natural mapping of Ω on X. Define

$$\mathfrak{S} = \{A : A \subset X,\ \rho^{-1}(A) \in \mathfrak{Z}\},$$

and for $A \in \mathfrak{S}$, set $\mu(A) = m\rho^{-1}(A)$. Then $(X, \mathfrak{S}, \mu)$ is also a complete measure-space, with $\mu(X) = 1$. Every function f on Ω which is constant on each equivalence class yields a function g on X, defined by

$$g(x) = f(\rho^{-1}(x)),$$

and conversely. If one is measurable so is the other, and

$$(11) \qquad \int_\Omega f(\omega)\,dm = \int_X g(x)\,d\mu.$$

In particular, the system

$$G = \{g_\alpha\} = \{f_\alpha \rho^{-1}\}$$

satisfies (1) and (2) with respect to $(X, \mathfrak{S}, \mu)$, and G and F are isomorphic. In addition, G separates X; that is,

$$(12) \qquad x_1 \neq x_2 \Longrightarrow g_\alpha(x_1) \neq g_\alpha(x_2) \text{ for some } \alpha.$$

This follows directly from (10). We assign to G the discrete topology.

Now let H be the character group of G. Since G is discrete and countable, H is compact and satisfies the second axiom of countability. To each $x \in X$ there corresponds in H an element $h = \phi(x)$, defined by

$$(13) \qquad h(g_\alpha) = g_\alpha(x) \qquad (\alpha \geq 0).$$

The mapping ϕ is one-to-one, in view of (12). If we assign to X the topology defined by neighborhoods

$$(U, J) = \{x : x \in X,\ g_\alpha(x) \in U \ \text{for}\ \alpha \in J\}, \tag{14}$$

where U is an open set on the unit circle in the complex plane, and J is a finite set, then ϕ becomes a homeomorphism of X into H. We denote by Z the image $\phi(X)$. If w is a continuous function on H, then $\tilde{w}$, defined by

$$\tilde{w}(x) = w(\phi(x)),$$

is continuous on X. We shall now show that $\tilde{w}$ is measurable and that

$$\int_X \tilde{w}\, d\mu = \int_H w\, d\nu, \tag{15}$$

where ν is the normalized Haar measure on H.

By the duality theorem, G is isomorphic with the character group of H, the correspondence $g_\alpha \leftrightarrow \chi_\alpha$ being given by

$$\chi_\alpha(h) = h(g_\alpha), \qquad h \in H.$$

We observe that $\tilde{\chi}_\alpha = g_\alpha$. Now the continuous function w may be approximated uniformly by linear combinations of characters:

$$P_n(h) = \sum_{\alpha=0}^{r_n} C_\alpha^{(n)} \chi_\alpha(h) \longrightarrow w(h).$$

Hence, by the orthonormality of the χ_α,

$$C_0^{(n)} = \int_H P_n(h)\, d\nu \longrightarrow \int_H w(h)\, d\nu. \tag{16}$$

But

$$\tilde{P}_n(x) = \sum_{\alpha=0}^{r_n} C_\alpha^{(n)} g_\alpha(x) \longrightarrow \tilde{w}(x),$$

also uniformly. Therefore $\tilde{w}$ is measurable on X, and

$$\int_X \tilde{P}_n(x)\, d\mu \longrightarrow \int_X \tilde{w}(x)\, d\mu. \tag{17}$$

Since the g_α are orthonormal on X, the left side of (17) is $C_0^{(n)}$. Our assertion (15) then follows from (16) and (17).

We shall now prove that $\phi^{-1}(M) \in \mathcal{S}$ for every ν-measurable set $M \subset H$, and that

$$\nu(M) = \mu(\phi^{-1}(M)).$$

Suppose first that M is closed. There exists a decreasing sequence $\{V_n\}$ of neighborhoods of the identity e, with intersection $\{e\}$. The open sets MV_n have M as their intersection, and

$$\nu(MV_n) \longrightarrow \nu(M).$$

The set $C_n = H - MV_n$ is closed and disjoint from M. By Urysohn's lemma, there exists a continuous function w_n satisfying

$$\begin{aligned} w_n(h) &= 1, & h &\in M, \\ &= 0, & h &\in C_n, \\ 0 \le w_n(h) &\le 1, & h &\in H. \end{aligned}$$

The corresponding function $\widetilde{w}_n$ satisfies

$$\begin{aligned} w_n(x) &= 1, & x &\in \phi^{-1}(M), \\ &= 0, & x &\in \phi^{-1}(C_n), \\ 0 \le \widetilde{w}_n(x) &\le 1, & x &\in X. \end{aligned}$$

The set $\phi^{-1}(M)$ is measurable in X, since its characteristic function is the limit of $\widetilde{w}_n$, and similarly for $\phi^{-1}(C_n)$. Also,

$$\nu(M) = \lim \int_H w_n d\nu,$$

$$\mu(\phi^{-1}(M)) = \lim \int_X \widetilde{w}_n d\mu.$$

The equality of these measures follows from (15). Thus our assertion is true for closed sets, hence for all Borel sets. If M is now any measurable set in H, there exist Borel sets A and B, such that

$$A \subset M \subset B \text{ and } \nu(A) = \nu(B),$$

by the regularity of Haar measure. Hence

$$\phi^{-1}(A) \subset \phi^{-1}(M) \subset \phi^{-1}(B) \quad \text{and} \quad \mu(\phi^{-1}(A)) = \mu(\phi^{-1}(B)).$$

The measurability of $\phi^{-1}(M)$ follows from the completeness of μ, and the equality of the measures is then obvious.

If we take for M any measurable set containing Z, then we have

$$\nu(M) = \mu(\phi^{-1}(M)) = \mu(X) = 1.$$

Hence the outer measure of Z is 1. Since a nonempty open set in H has positive measure, it follows that Z is dense in H. The condition that $Z = H$ is equivalent, therefore, to the compactness of X. Recalling (14), we see that this condition may be expressed in terms of the f_α by (6). Condition (7) is equivalent to the mapping ρ being one-to-one almost everywhere. Hence, if we put $T = \phi\rho$, and recall that ϕ is one-to-one, we see that Theorem 1 is proved. (The uniqueness of H follows from (5) and the duality theorem.)

If (6) and (7) are satisfied, we can say somewhat more about T. Since ϕ is now a homeomorphism onto H, the image of a Borel set is also a Borel set, and therefore belongs to $\mathfrak{M}$, the class of ν-measurable sets. If $\mathfrak{Z}_0$ is the σ-ring of Borel sets in X, and $\overline{\mu}_0$ is the completion of the restriction of μ to $\mathfrak{Z}_0$, then ϕ is a measure-preserving transformation from $(X, \overline{\mathfrak{Z}}_0, \overline{\mu}_0)$ to $(H, \mathfrak{M}, \nu)$. If $\mathfrak{Z}_0$ is the least σ-ring for which all the f_α are measurable, and $\overline{m}_0$ is the completion of the restriction of m to $\mathfrak{Z}_0$, it is easily verified that ρ is a measure-preserving transformation from $(\Omega, \overline{\mathfrak{Z}}_0, \overline{m}_0)$ to $(X, \overline{\mathfrak{Z}}_0, \overline{\mu}_0)$. Finally, T is a measure-preserving transformation from $(\Omega, \mathfrak{Z}_0, m_0)$ to $(H, \mathfrak{M}, \nu)$.

2. Proofs of Theorems 2 and 3. Let $f_1, \cdots, f_n$ satisfy the conditions of Theorem 2. By Theorem 1, it is sufficient to consider the distribution of the corresponding characters $\chi_1, \cdots, \chi_n$. Writing

$$\chi_\alpha(h) = \exp(2\pi i A_\alpha(h)),$$

where the $A_\alpha(h)$ are reals mod 1, we see that the combined mapping

$$A(h) = (A_1(h), \cdots, A_n(h))$$

is a homomorphism of H into the n-dimensional torus T^n, realized as n-tuples $(X_1, \cdots, X_n)$, the X_α being reals mod 1. The image $H' = A(H)$ is a closed subgroup of T^n, and is therefore definable by a system of relations

$$(18) \qquad \sum_{\alpha=1}^{n} b_{K\alpha} X_\alpha \equiv 0 \pmod 1,$$

where the $b_{K\alpha}$ are integers. The corresponding relations, with X_α replaced by $A_\alpha(h)$, must hold on H. By assumption, the only such relations are of the form

$$(19) \qquad \sum_{\alpha=1}^{n} d_\alpha A_\alpha(h) \equiv 0 \pmod 1,$$

where the d_α are multiples of the orders p_α of χ_α, if finite, and 0 otherwise. Thus H' decomposes into a direct product of copies of T^1 and cyclic groups of order $p_\alpha > 0$, given by

$$(20) \qquad p_\alpha X_\alpha \equiv 0 \pmod 1.$$

The normalized Haar measure on H' is the product measure ν'. It is easily verified that νA^{-1} is also a normalized Haar measure on H'. By the uniqueness theorem, we have $\nu A^{-1} = \nu'$, and Theorem 2 is proved.

The proof of Theorem 3 is exactly the same up to (18). But now nontrivial relations may exist. Equations (18) may be brought to canonical form (see [7, §6]) by an integral unimodular substitution carrying the coordinates $\{X_\alpha\}$ into $\{Y_j\}$, say:

$$(21) \qquad d_j Y_j \equiv 0 \pmod 1,$$

where

$$(22) \qquad Y_j = \sum_{\alpha=1}^{n} e_{j\alpha} X_\alpha \qquad (j = 1, \cdots, n).$$

The corresponding functions

$$(23) \qquad f_{a_j} = \prod_{\alpha=1}^{n} f_\alpha^{e_{j\alpha}} \qquad (j = 1, \cdots, n)$$

satisfy the conditions of Theorem 2 and are therefore statistically independent. If $(c_{\alpha j})$ is the inverse of the matrix $(e_{j\alpha})$, equations (9) hold, and Theorem 3 is proved.

REFERENCES

1. H. E. Chrestenson, *A class of generalized Walsh functions*, abstract, Bull. Amer. Math. Soc. **59** (1953), 391-392.

2. H. E. Chrestenson, *Some groups of orthonormal functions*, abstract, Bull. Amer. Math. Soc. **59** (1953), 392.

3. Paul Civin, *Multiplicative closure and the Walsh functions*, Pacific J. Math. **2** (1952), 291-295.

4. ———, *Orthonormal cyclic groups*, Pacific J. Math. **4** (1954), 481-482.

5. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372-414.

6. P. R. Halmos, *Measure theory*, New York, 1950.

7. L. Pontrjagin, *Topological groups*, Princeton, 1939.

UNIVERSITY OF PENNSYLVANIA AND
THE INSTITUTE FOR ADVANCED STUDY