# A CLASS OF GENERALIZED WALSH FUNCTIONS

H. E. CHRESTENSON

**1. Introduction.** Let $\alpha$ denote a fixed integer, $\alpha \geq 2$, and put $\omega = \exp(2\pi i/\alpha)$.

DEFINITION 1. The *Rademacher functions of order* $\alpha$ are defined by

$$\phi_0(x) = \omega^k \text{ if } k/\alpha \leq x < (k+1)/\alpha,\ k = 0, \cdots, \alpha - 1;$$

and for $n \geq 0$

$$\phi_n(x+1) = \phi_n(x) = \phi_0(\alpha^n x).$$

DEFINITION 2. The *Walsh functions of order* $\alpha$ are defined by

$$\psi_0(x) = 1,$$

and if $n = a_1 \alpha^{n_1} + \cdots + a_m \alpha^{n_m}$ where $0 < a_j < \alpha$ and $n_1 > n_2 > \cdots > n_m$, then

$$\psi_n(x) = \phi_{n_1}^{a_1}(x) \cdots \phi_{n_m}^{a_m}(x).$$

For convenience we let $\Psi_\alpha$ denote the set of Walsh functions of order $\alpha$. We may observe that $\Psi_2$ is the orthonormal system of functions defined by Walsh [4]. R.E.A.C. Paley's proof that $\Psi_2$ is orthonormal and complete in $L(0,1)$ may be modified by the reader to establish the same properties for $\Psi_\alpha$, $\alpha = 3, 4, \cdots$. [3; pp. 242-244].

It is the purpose of this paper to study Fourier expansions in the sets $\Psi_\alpha$. The results obtained here will include known results for ordinary Walsh Fourier series, most of which are contained in a paper of N. J. Fine [1]. In fact, most

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of the properties of Fourier expansions in $\Psi_2$ are shared by expansions in $\Psi_\alpha$.

The system $\Psi_\alpha$ is in fact the character group of $G_\alpha$, the countable product of cyclic groups of order $\alpha$, transferred to the unit interval. The operation +, introduced in § 2, is precisely the image of the group operation. Some of our results and many of our methods readily admit interpretations in $G_\alpha$, although little mention of these will be made in the text. For example, in Lemma 1 we prove that the Haar integral in the group corresponds to the Lebesgue integral on $(0,1)$.

Using an obvious abbreviation, we summarize our most important results: (i) The $W_\alpha FS$ of $f(x)$ converges to $f(x)$ a.e. if $f(x)$ is of bounded variation, and the convergence tests of Dini and Dini-Lipschitz are valid. (ii) If $f(x)$ has variation $V$ and if $c_k$ is the coefficient of $\psi_k(x)$ in the $W_\alpha FS$ of $f(x)$, then $|c_k| \leq Vk^{-1} \csc \pi/\alpha$. (iii) The continuity of $f(x)$ is a sufficient condition for the uniform $(C, 1)$ summability of the $W_\alpha FS$.

**2. Notation and preliminary results.** Define

$$I_{n,k} = I_{n,k}(\alpha) = \{x : k\,\alpha^{-n} \leq x < (k+1)\,\alpha^{-n}\},$$

$k = 0, \cdots, \alpha^n - 1$, $n = 1, 2, \cdots$. Then if $\phi_n(x)$ is the $n$th Rademacher function of order $\alpha$, $\phi_n(x) = \omega^k$ if $x \in I_{n+1,k}$.

The term, *$\alpha$-adic rational*, will denote any number of the form $k\alpha^{-n}$ where $k$ and $n$ are integers. Thus if $x$ has the base $\alpha$ expansion

$$\sum_{j=1}^{\infty} x_j\,\alpha^{-j}, \quad 0 \leq x_j < \alpha,$$

where the terminating expansion is taken in case $x$ is an $\alpha$-adic rational, we see that $\phi_n(x) = \omega^{x_{n+1}}$.

We introduce a binary operation, denoted by $\dot{+}$, and defined as follows: If $0 \leq a < 1$ and $0 \leq x < 1$, and if $a$ and $x$ have base $\alpha$ expansions

$$\sum_{1}^{\infty} a_j\alpha^{-j} \text{ and } \sum_{1}^{\infty} x_j\alpha^{-j}$$

respectively, then $a \dot{+} x$ will denote the number

$$\sum_{1}^{\infty} y_j \alpha^{-j}$$

where $y_j \equiv a_j + x_j \pmod{\alpha}$, $0 \leq y_j < \alpha$. If we agree to take the terminating expansions for $\alpha$-adic rationals whenever possible, it follows that for any fixed $a$ and all $n \geq 0$ $\phi_n(a \dotplus x) = \phi_n(a)\,\phi_n(x)$, a.e. The exceptional values occur when $a \dotplus x$ is the infinite expansion of an $\alpha$-adic rational. It is also true that $\psi_n(a \dotplus x) = \psi_n(a)\,\psi_n(x)$, a.e.

LEMMA 1. *If $f(x) \in L(0,1)$ then $f(a \dotplus x) \in L(0,1)$ and*

$$\int_0^1 f(x)\,dx = \int_0^1 f(a \dotplus x)\,dx.$$

The reader will have no difficulty in modeling a proof after the proof in the case $\alpha = 2$ [**1**, p. 379].

If $f(x) \in L(0,1)$ and if

$$c_n = \int_0^1 f(t)\,\overline{\psi}_n(t)\,dt$$

we say that $\sum_0^\infty c_n\,\psi_n(x)$ is the $W_\alpha FS$ of $f(x)$. Let $s_k(x)$ denote the $k$th partial sum of this series, so that

$$s_k(x) = \int_0^1 f(t) \sum_{0}^{k-1} \psi_j(x)\,\overline{\psi}_j(t)\,dt = \int_0^1 f(t)\,D_k(x,t)\,dt$$

where the kernel $D_k(x,t)$ is defined accordingly. We will write $D_k(t) = D_k(0,t)$. Observe that for all $k \leq \alpha^n$, $D_k(x,t) = D_k(x',t')$ provided only that $x$ and $x'$ are in the same $I_{n,r}$ and that $t$ and $t'$ are in the same $I_{n,r'}$.

Let $z = z(x,n)$ be that number satisfying

$$x \dotplus z = 0 \tag{2.1}$$

except when this relation determines $z$ as the nonterminating expansion of an $\alpha$-adic rational. In these cases let the first $n$ digits in the expansion of $z$ be determined by (2.1), and let the remaining digits be zeros. For all $k \leq \alpha^n$ we have for almost all $t$

$$(2.2)\quad D_k(x,t)=\sum_{0}^{k-1}\psi_j(x)\,\overline{\psi}_j(t)=\sum\overline{\psi}_j(z)\,\psi_j(t)=\sum\overline{\psi}_j(z\dot{+}t)=D_k(z\dot{+}t).$$

If we use Lemma 1 we have the following useful result.

$$(2.3)\qquad s_k(x)=\int_0^1 D_k(z\dot{+}t)\,f(t)\,dt$$

$$=\int_0^1 D_k(x\dot{+}z\dot{+}t)f(x\dot{+}t)\,dt=\int_0^1 D_k(t)f(x\dot{+}t)\,dt.$$

Unless otherwise stated all functions will be assumed to be periodic and integrable on $(0,1)$.

**3. Convergence.**

LEMMA 2.

$$D_{\alpha^n}(t)=\begin{cases}\alpha^n \text{ if } t\in I_{n,0},\\ 0 \quad \text{otherwise.}\end{cases}$$

*Proof.* We have from the definitions

$$(3.1)\qquad D_{\alpha^n}(t)=\sum_{r=0}^{\alpha^n-1}\overline{\psi}_r(t)=\prod_{r=0}^{n-1}[1+\overline{\phi}_r(t)+\cdots+\overline{\phi}_r^{\alpha-1}(t)].$$

If $t\in I_{n,0}$ each $\phi_r(t)=1$, while if $t\notin I_{n,0}$ at least one factor in the product vanishes. (The $p$th factor is zero if $\phi_p(t)\neq 1$.)

By translating under $\dot{+}$ we see that Lemma 2 has the following equivalent form: *If $\rho=\rho(x,n)$ is such that $x\in I_{n,\rho}$ then*

$$D_{\alpha^n}(x,t)=\begin{cases}\alpha^n \quad \text{if } t\in I_{n,\rho},\\ 0 \quad \text{otherwise.}\end{cases}$$

As an immediate consequence we have

THEOREM 1. *If $f(x)\in L(0,1)$ then $\lim_{n\to\infty}s_{\alpha^n}(x)=f(x)$* a.e. *In particular, $s_{\alpha^n}(x)\longrightarrow f(x)$ at a point of continuity of $f(x)$ and the convergence is uniform in a closed interval of continuity. If $x$ is an $\alpha$-adic rational then $s_{\alpha^n}(x)\longrightarrow f(x)$ provided $x$ is a point of right hand continuity of $f(x)$.*

Additional usefulness of Lemma 2 is seen from the identity

$$(3.2)\quad D_n(x,t)=\sum_{j=1}^{m}\left\{\phi_{n_1}^{a_1}(x)\bar{\phi}_{n_1}^{a_1}(t)\cdots\phi_{n_{j-1}}^{a_{j-1}}(x)\bar{\phi}_{n_{j-1}}^{a_{j-1}}(t)\right.$$

$$\left.D_{\alpha^{n_j}}(x,t)\left[1+\phi_{n_j}(x)\bar{\phi}_{n_j}(t)+\cdots+\phi_{n_j}^{a_j-1}(x)\bar{\phi}_{n_j}^{a_j-1}(t)\right]\right\},$$

where the base $\alpha$ expansion of $n$ is given in Definition 2. To prove (3.2) notice that

$$(3.3)\quad D_n(x,t)=D_{\alpha^{n_1}}(x,t)+\sum_{r=0}^{n-\alpha^{n_1}-1}\psi_{\alpha^{n_1}+r}(x)\,\bar{\psi}_{\alpha^{n_1}+r}(t)$$

$$=D_{\alpha^{n_1}}(x,t)+\phi_{n_1}(x)\,\bar{\phi}_{n_1}(t)\,D_{n-\alpha^{n_1}}(x,t).$$

By using (3.3) recursively we obtain (3.2).

The usual method of establishing convergence of the full sequence of partial sums of the $W_\alpha FS$ will be to reduce the convergence of $s_n(x)$ to that of $s_{\alpha^{n_1}}(x)$ by showing that $s_{\alpha^{n_1}}(x)-s_n(x)\longrightarrow 0$ as $n\longrightarrow\infty$, where $\alpha^{n_1}\le n<\alpha^{n_1+1}$. In the following lemma we use the notation of Definition 2, with the additional convention of writing $N$ for $n_1$.

LEMMA 3. *Let $\nu$ be a fixed positive integer and let $x\in I_{\nu,\rho}$. Then if $\sigma\neq\rho$*

$$(3.4)\qquad \lim_{n\to\infty}\int_{I_{\nu,\sigma}}[D_n(x,t)-D_{\alpha^N}(x,t)]f(t)\,dt=0.$$

*If also $y\in I_{\nu,\rho}$ and $N\ge\nu$, then*

$$(3.5)\qquad \left|\int_{y}^{(\rho+1)\alpha^{-\nu}}[D_n(x,t)-D_{\alpha^N}(x,t)]\,dt\right|<\alpha,$$

*and in case $y=\rho\alpha^{-\nu}$, the integral (3.5) vanishes.*

*Proof.* In proving (3.4) we may suppose $N\ge\nu$. Let $r$ be chosen so that $n_r\ge\nu>n_{r+1}$; in case $n_m\ge\nu$ take $r=m$. By Lemma 2 all $D_{\alpha^k}(x,t)=0$ for $t\in I_{\nu,\sigma}$ and $k\ge\nu$. Thus $D_n(x,t)=D_n(x,t)-D_{\alpha^N}(x,t)$ and by (3.2) this is a sum of $m-r$ terms, each of which is, for $t\in I_{\nu,\sigma}$, a constant multiple of

$$\overline{\phi}_{n_1}^{a_1}(t)\cdots\overline{\phi}_{n_r}^{a_r}(t)=\overline{\psi}_{M(n)}(t),$$

say. A careful inspection of (3.2) shows that the sum of the moduli of the coefficients of $\overline{\psi}_{M(n)}(t)$ is bounded independent of $n$. Also, $M(n)\longrightarrow\infty$ as $n\longrightarrow\infty$. We have now reduced (3.4) to a theorem of Mercer [2, p. 17].

The inequality (3.5) is proved by writing $I_{\nu,\rho}$ as a sum of $I_{N,s}$. On each $I_{N,s}$ the integrand is a linear combination of $\overline{\phi}_N^b(t)$, $0<b<\alpha$. On each complete $I_{N,s}$ contained in $(y,(\rho+1)\alpha^{-\nu})$ the integral vanishes. The remainder of the interval of integration has length less than $\alpha^{-N}$, and from (3.3) we see that the integrand is numerically less than $\alpha^{N+1}$.

THEOREM 2. *If $f(x)$ is of bounded variation and continuous from the right on $[0,1)$, then as $n\longrightarrow\infty$, $s_n(x)\longrightarrow f(x)$ at every point of continuity and at every $\alpha$-adic rational. If $x$ is an $\alpha$-adic irrational which is a point of discontinuity, $s_n(x)$ does not converge.*

*Proof.* To prove convergence it is sufficient to show that for $f(t)$ monotonic

$$s_n(x)-s_{\alpha N}(x)=\int_0^1[D_n(x,t)-D_{\alpha N}(x,t)]f(t)\,dt\longrightarrow 0.$$

Write this integral as

$$\int_{I_{\nu,\rho}}+\int_{CI_{\nu,\rho}}[D_n(x,t)-D_{\alpha N}(x,t)]f(t)\,dt=J_1+J_2,$$

where $C$ denotes the complement taken with respect to $(0,1)$. By the second theorem of the mean, there is $y\in I_{\nu,\rho}$ such that

$$J_1=f(\rho\alpha^{-\nu}+0)\int_{\rho\alpha^{-\nu}}^{y}[D_n-D_{\alpha N}]\,dt$$

$$+f((\rho+1)\alpha^{-\nu}-0)\int_y^{(\rho+1)\alpha^{-\nu}}[D_n-D_{\alpha N}]\,dt.$$

By (3.5)

$$(3.6)\qquad |J_1|\le\alpha|f((\rho+1)\alpha^{-\nu}-0)-f(x)|+\alpha|f(x)-f(\rho\alpha^{-\nu}+0)|<\epsilon/2$$

for $\nu$ sufficiently large and for $n\ge\alpha^{\nu}$, since $f(x+0)=f(x)=f(x-0)$. If

$x$ is an $\alpha$-adic rational, first choose $\nu$ large enough so that $\rho\alpha^{-\nu} = x$, so that only right hand continuity is involved in (3.6). With $\nu$ fixed, $J_2 \longrightarrow 0$ as $n \longrightarrow \infty$ by (3.4).

Notice that for convergence at $x$, the hypothesis of bounded variation is needed only in a neighborhood of $x$.

The proof of the second part of Theorem 2 will be omitted, except to note that it is sufficient to consider the $W_\alpha FS$ of $f(x)$, $f(x) = 0$ if $0 \leq x < a$, $f(x) = 1$ if $a < x \leq 1$, where $a$ is an $\alpha$-adic irrational. The partial sums of the $W_\alpha FS$ of $f(x)$ may be explicitly written in terms of the digits in the base $\alpha$ expansion of $a$, and the assertion follows directly.

Lemmas 2 and 3 provide a direct proof of the theorem of localization for $W_\alpha FS$.

THEOREM 3. *If $f(x) = g(x)$ a.e. for $a - \epsilon < x < a + \epsilon$, then the $W_\alpha FS$ of $f(x)$ and $g(x)$ are equiconvergent at $a$. If $a$ is an $\alpha$-adic rational it is sufficient that $f(x) = g(x)$ a.e. for $a < x < a + \epsilon$.*

LEMMA 4. *The kernel $D_k(x, t)$ satisfies*

$$\int_0^1 D_k(x, t)\, dt = 1, \tag{3.7}$$

*and for* $0 < t < 1$

$$|D_k(t)| < \alpha/t. \tag{3.8}$$

*Proof.* The first assertion is obvious.

For a proof of (3.8) the reader is referred to Fine's paper [**1**; pp. 391, 392].

THEOREM 4. *If for a fixed $x$,*

$$\frac{f(t) - c}{t - x} \in L(x - \delta,\ x + \delta) \text{ for some } \delta > 0,$$

*then $s_n(x) \longrightarrow c$.*

*Proof.* Suppose the base $\alpha$ expansion of $x$ does not end in an infinite sequence of ones. Let $z$ be determined by (2.1). Then we have, using (2.2) and (3.7)

$$s_n(x) - c = \int_{|t-x|<h<\delta} [f(t) - c] D_n(z \dotplus t)\, dt$$

$$+ \int_{|t-x|>h} [f(t) - c] D_n(x, t)\, dt = J_1 + J_2 .$$

One may verify that

$$(3.9) \qquad |x - t| \leq \alpha(z \dotplus t).$$

Thus, with (3.8), we have

$$|J_1| \leq \alpha^2 \int_{|t-x|<h} \frac{|f(t) - c|}{|t - x|}\, dt < \epsilon$$

for $h$ sufficiently small. With $h$ fixed, $J_2 \longrightarrow 0$ by Theorem 3 and the remark below equation (3.6).

In case $x$ is of the form excluded in the argument above, the proof must be modified. We put $z = z(x, n)$ where $z(x, n)$ is defined in § 2. Inequality (3.9) may not be satisfied on a set $F_n \subset (x - \delta, x + \delta)$. One may show that $F_n$ is a subset of an interval of length $\alpha^{-n}$, so

$$|J_1| \leq \alpha^2 \int_{|t-x|<h} \frac{|f(t) - c|}{|t - x|}\, dt + n \int_{F_n} |f(t) - c|\, dt = J_1' + J_1''.$$

$J_1' < \epsilon$ as before, and with $h$ fixed,

$$J_1'' \leq n\alpha^{-n} \int_{F_n} \frac{|f(t) - c|}{|t - x|}\, dt \longrightarrow 0$$

and $J_2 \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 1 and equation (2.2) provide a proof that

$$\int_0^1 |D_k(x, t)|\, dt = \int_0^1 |D_k(t)|\, dt \quad \text{for all } x \in (0, 1).$$

We put $L_k = \int_0^1 |D_k(t)|\, dt$, the $k$th Lebesgue constant of the system $\Psi_\alpha$.

LEMMA 5. *The Lebesgue constants satisfy* $L_k = O(\log k)$, *where the* $O$ *depends upon* $\alpha$.

*Proof.* By Lemma 4, $|D_k(t)| \leq \min(\alpha/t, k)$. Thus

$$L_k \leq \int_0^{\alpha/k} k\,dt + \int_{\alpha/k}^1 \alpha/t\,dt = O(\log k).$$

In the statement of the next theorem, $W(\delta; f)$ is the modulus of continuity of $f(x)$;

$$W(\delta; f) = \sup_{|h| \leq \delta,\ 0 \leq x < 1} |f(x+h) - f(x)|.$$

THEOREM 5. *If $f(x)$ satisfies $W(\delta; f) = o((\log \delta^{-1})^{-1})$ as $\delta \longrightarrow 0$, then $s_n(x) \longrightarrow f(x)$ uniformly.*

*Proof.* For this proof, write $n = a\alpha^k + k'$ where $0 < a < \alpha$, $0 \leq k' < \alpha^k$. Since

$$s_n - s_{\alpha^k} = (s_n - s_{a\alpha^k}) + (s_{a\alpha^k} - s_{\alpha^k}) = S_1 + S_2,$$

it is sufficient to show that $S_1 \longrightarrow 0$ and $S_2 \longrightarrow 0$ uniformly. By using Lemma 2 and (3.3) we obtain

$$S_2 = \int_{I_{k,\rho}} [\phi_k(x)\,\overline{\phi}_k(t) + \cdots + \phi_k^{a-1}(x)\,\overline{\phi}_k^{a-1}(t)]\alpha^k f(t)\,dt,$$

where $\rho$ is chosen so that $x \in I_{k,\rho}$. Since $f(x)$ is uniformly continuous, $S_2 \longrightarrow 0$ as $k \longrightarrow \infty$. Again using (3.3),

$$S_1 = \int_0^1 \phi_k^a(x)\,\overline{\phi}_k^a(t)\,D_{k'}(x,t)\,f(t)\,dt.$$

Replacing $t$ by $t \dot{+} b\alpha^{-k-1}$, we have

$$S_1 = \omega^{-ab} \int_0^1 \phi_k^a(x)\,\overline{\phi}_b^a(t) D_{k'}(x,t) f(t \dot{+} b\alpha^{-k-1})\,dt,$$

so by subtraction

$$S_1(1 - \omega^{ab}) = \phi_k^a(x) \int_0^1 D_{k'}(x,t)\,\overline{\phi}_k^a(t)[f(t) - f(t \dot{+} b\alpha^{-k-1})]\,dt.$$

If $b$ is chosen so that $|1 - \omega^{ab}| \geq 3^{1/2}$, this becomes

$$|S_1|\,3^{1/2} \leq W(\alpha^{-k};\, f)\, L_k' = o(1),$$

where we have used Lemma 5.

## 4. Fourier coefficients.

THEOREM 6. *If*

$$f(x) \sim \sum_0^\infty c_n \psi_n(x),$$

*then*

$$f(a \dot{+} x) \sim \sum_0^\infty d_n \psi_n(x)$$

*where* $d_n = c_n \psi_n(a)$.

*Proof.* This is a consequence of Lemma 1 and the relation $\psi_n(a \dot{+} x) = \psi_n(a)\, \psi_n(x)$, a.e.

By using Theorem 6 and the scheme from the proof of Theorem 5 we may establish the following.

THEOREM 7. *If*

$$f(x) \sim \sum_0^\infty c_j \psi_j(x),$$

*then*

$$|c_n| \leq 3^{1/2}\, W((\alpha - 1)/n;\, f).$$

There is a similar result with $W$ replaced by the integral modulus of continuity.

As a corollary to Theorem 7 there is the following.

THEOREM 8. *If* $f(x) \in \mathrm{Lip}(\eta)$, *then* $c_n = O(n^{-\eta})$ *where the* $O$ *depends upon* $\alpha$.

For the next lemma we define

$$J_n(x) = \int_0^x \psi_n(t)\,dt$$

and we write $n = a\alpha^k + k'$, where $0 < a < \alpha$, $0 \leq k' < \alpha^k$.

LEMMA 6. *For* $n \geq 0$ *and all* $x$,

$$|J_n(x)| < n^{-1} \csc \pi/\alpha.$$

*Proof. If* $x \in I_{k,\rho}$ we have, from elementary properties of $\psi_n(x)$,

$$(4.1) \qquad |J_n(x)| = \left|\int_{\rho\alpha^{-k}}^x \psi_n(t)\,dt\right| = \left|\psi_{k'}(\rho\alpha^{-k})\int_{\rho\alpha^{-k}}^x \phi_k^a(t)\,dt\right|.$$

If $\tau$ is defined by the relation $x \in I_{k+1,\tau}$, we have by a direct calculation

$$\left|\int_{\rho\alpha^{-k}}^x \phi_k^a(t)\,dt\right| \leq \max\left\{\left|\int_{\rho\alpha^{-k}}^{\tau\alpha^{-k-1}} \phi_k^a(t)\,dt\right|, \left|\int_{\rho\alpha^{-k}}^{(\tau+1)\alpha^{-k-1}} \phi_k^a(t)\,dt\right|\right\}$$

$$\leq \max\left\{\alpha^{-k-1}\left|\frac{1-\omega^{a\tau}}{1-\omega}\right|, \alpha^{-k-1}\left|\frac{1-\omega^{a(\tau+1)}}{1-\omega}\right|\right\}$$

$$\leq \alpha^{-k-1} \csc \pi/\alpha < n^{-1} \csc \pi/\alpha.$$

THEOREM 9. *If* $f(x)$ *has total variation* $V$ *then*

$$|c_n| \leq Vn^{-1} \csc \pi/\alpha.$$

*Proof.* Since $J_n(0) = J_n(1) = 0$,

$$(4.2) \qquad c_n = -\int_0^1 \bar{J}_n(x)\,df(x),$$

and the theorem is now seen to be a consequence of Lemma 6.

For $\alpha = 2$, Theorem 9 was proved by N. J. Fine [**1**, p. 383] and in this case $\csc \pi/\alpha = 1$. That this factor is necessary when $\alpha > 2$ is seen from the following example. For an arbitrary positive integer $k$ define $n = \alpha^{k+1} - 1$. Let $\beta$ denote the integral part of $\alpha/2$ and put $\zeta = \beta\alpha^{-k-1}$ and $\xi = \zeta + \beta/\alpha$. Let $f(x)$ represent the characteristic function of the interval $[\zeta, \xi)$. By using (4.1) and (4.2) we may calculate $c_k$. It turns out that

$$|c_k| = [B(\alpha)/2]^2 \ \alpha^{-n-1} \csc \pi/\alpha \ V,$$

where $B(\alpha) = \max_{0 < b < \alpha} |1 - \omega^b|$ so that $3^{1/2} \leq B(\alpha) \leq 2$.

**5. $(C, 1)$ summability.** Let $\sigma_k(x)$ represent the $k$th $(C, 1)$ mean of $\{s_n(x)\}$, and define the kernel,

$$F_k(x, t) = k^{-1} \sum_{1}^{k} D_r(x, t).$$

We will write $F_k(0, t) = F_k(t)$.

LEMMA 7. *For $k \geq 1$, $\int_0^1 F_k(x, t)\,dt = 1$, and for $0 < t < 1$, $|F_k(t)| < \alpha/t$.*

*Proof.* These properties follow directly from the corresponding properties of $D_k(x, t)$.

LEMMA 8. *There is a constant $M$ such that for all $k \geq 0$*

$$\int_0^1 |F_{\alpha^k}(x, t)|\,dt \leq M.$$

*Proof.* Write $n$ in the form $n = a\alpha^k + k'$ where $0 < a < \alpha$ and $0 \leq k' \leq \alpha^k$. By a somewhat tedious calculation involving repeated use of (3.2) we obtain

$$(5.1) \quad nF_n(t) = [1 + \cdots + \bar{\phi}_k^{a-1}(t)]\alpha^k F_{\alpha^k}(t) + \bar{\phi}_k^a(t) k' F_{k'}(t)$$
$$+\{1 + [1 + \bar{\phi}_k(t)] + \cdots + [1 + \cdots + \bar{\phi}_k^{a-2}(t)]\}\alpha^k D_{\alpha^k}(t)$$
$$+ [1 + \cdots + \bar{\phi}_k^{a-1}(t)] k' D_{\alpha^k}(t).$$

If we take $k' = \alpha^k$ and $a = \alpha - 1$ in (5.1) we obtain

$$(5.2) \qquad \alpha^{k+1} F_{\alpha^{k+1}}(t) = R_k(t)\,\alpha^k F_{\alpha^k}(t) + Q_k(t)\,\alpha^k D_{\alpha^k}(t)$$

where

$$(5.3) \qquad R_k(t) = \begin{cases} \alpha \text{ if } \phi_k(t) = 1, \\ 0 \text{ otherwise}, \end{cases}$$

and

$$(5.4) \qquad Q_k(t) = \begin{cases} \alpha(\alpha-1)/2 & \text{if } \phi_k(t) = 1, \\ \alpha/(1-\bar{\phi}_k(t)) & \text{otherwise}. \end{cases}$$

By applying a simple induction argument to (5.2) we obtain

$$(5.5) \qquad \alpha^{k+1} F_{\alpha^{k+1}}(t) = Q_k(t)\, \alpha^k D_{\alpha^k}(t) + \sum_{r=1}^{k} R_k(t) R_{k-1}(t) \cdots R_r(t) Q_{r-1}(t)\, \alpha^{r-1} D_{\alpha^{r-1}}(t) + \prod_{r=0}^{k} R_r(t).$$

Let

$$S = \sum_{r=1}^{\alpha-1} |1 - \omega^r|^{-1},$$

then equations (5.3)-(5.5) enable us to show that

$$\alpha^{k+1} \int_0^1 |F_{\alpha^{k+1}}(t)|\, dt \le \alpha^k[(\alpha-1)/2 + S] + 1 + [(\alpha-1)/2 - S] \sum_1^k \alpha^{r-1},$$

from which the lemma follows.

Observe that by setting $k = 0$ in (5.2) we see that for $\alpha > 2$ the kernels $F_{\alpha^k}(t)$ are not positive. Fine showed that in case $\alpha = 2$, $F_{\alpha^k}(t) \ge 0$ [**1**, p. 396].

LEMMA 9. *If $t$ is not of the form $t = d\alpha^{-m}$, $m \ge 1$, $0 \le d < \alpha$, then* $\lim_{k\to\infty} F_k(t) = 0$.

*Proof.* Let $t$ be given and choose $n$ so that $\alpha^{-n} < t < \alpha^{-n+1}$. Write $k = p\alpha^n + q$ where $0 \le q < \alpha^n$. Then

$$kF_k(t) = \sum_{r=0}^{p-1} \sum_{s=1}^{\alpha^n} D_{r\alpha^n+s}(t) + \sum_{s=1}^{q} D_{p\alpha^n+s}(t).$$

One can show that $D_{r\alpha^n+s}(t) = D_{\alpha^n}(t) D_r(\alpha^n t) + \psi_r(\alpha^n t) D_s(t)$. This gives

$$D_{r\alpha^n+s}(t) = \bar{\psi}_r(\alpha^n t) D_s(t),$$

so that

$$kF_k(t) = \alpha^n F_{\alpha^n}(t) D_p(\alpha^n t) + \overline{\psi}_p(\alpha^n t) qF_q(t).$$

Put $b$ equal to the integral part of $\alpha^n t$. Since $0 < \alpha^n t - b < 1$, we have by Lemma 4

$$|D_p(\alpha^n t)| \leq \alpha(\alpha^n t - b)^{-1}.$$

Using Lemma 7 we obtain

$$|kF_k(t)| \leq \alpha^{n-2} t^{-1}(\alpha^n t - b)^{-1} + q\alpha t^{-1},$$

from which the conclusion follows.

THEOREM 10. *If* $f(x)$ *is continuous then* $\sigma_{\alpha^k}(x) \longrightarrow f(x)$ *uniformly.*

*Proof.* It follows from (2.3) and Lemma 7 that

$$(5.6) \qquad \sigma_n(x) - f(x) = \int_0^1 F_n(t)[f(x \dotplus t) - f(x)]dt.$$

By applying Lemmas 7-9 together with a standard argument we can show that

$$\int_0^1 |F_{\alpha^k}(t)| \; |f(x \dotplus t) - f(x)| dt \longrightarrow 0 \text{ uniformly}.$$

THEOREM 11. *If* $f(x)$ *is continuous then* $\sigma_n(x) \longrightarrow f(x)$ *uniformly.*

*Proof.* Let the base $\alpha$ expansion of $n$ be given in Definition 2. From (5.1) we obtain the estimate

$$(5.7) \qquad |nF_n(t)| \leq \sum_{r=1}^{m} \{a_r \alpha^{n_r} |F_{\alpha^{n_r}}(t)| + \frac{1}{2} a_r(a_r+1)\alpha^{n_r} D_{\alpha^{n_r}}(t)\}.$$

Let $\epsilon_k = \epsilon_k(x)$ represent the larger of

$$\int_0^1 |F_{\alpha^k}(t)| \; |f(x \dotplus t) - f(x)| dt$$

and

$$\int_0^1 D_{\alpha k}(t)\, |f(x \dotplus t) - f(x)|\, dt,$$

so that by Theorems 1 and 10 $\epsilon_k \longrightarrow 0$ uniformly. Using (5.6) and (5.7)

$$|\sigma_n(x) - f(x)| \leq \alpha \sum_{r=1}^{m} a_r \alpha^{n_r} n^{-1} \epsilon_{n_r} = \delta_n, \text{ say.}$$

One may readily verify that the transformation which sends $\{\epsilon_k\}$ into $\{\delta_n\}$ is regular, so that $\delta_n \longrightarrow 0$ uniformly, and the theorem is proved.

It is interesting to note that by virtue of a well known consequence of the Banach-Steinhaus theorem [**5**, p. 99], Theorem 11 implies that $\int_0^1 |F_n(t)|\, dt \leq M$.

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UNIVERSITY OF OREGON AND WHITMAN COLLEGE