

ON THE RENEWAL EQUATION

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Introduction. Recently Chung and Pollard [3] considered the following problem: Let $X_i (i=1, 2, \dots)$ denote independent identically distributed random variables having the distribution function $F(x)$ with mean

$$m = \int x dF(x) \quad (0 < m)$$

and let

$$S_n = \sum_{k=1}^n X_k.$$

Define

$$u(\zeta) = \sum_{n=1}^{\infty} \Pr \{ \zeta < S_n \leq \zeta + h \},$$

if X is not a lattice random variable then they show that $\lim_{\zeta \rightarrow \infty} u(\zeta) = h/m$. The above authors imposed the restriction that the distribution F possess an absolutely continuous part. T. E. Harris by written communication and independently D. Blackwell [2] show that this restriction was unnecessary. Of course, as can be verified directly, $u(\zeta)$ satisfies a renewal type equation

$$(*) \quad u(\zeta) - \int_{-\infty}^{\infty} u(\zeta - t) dF(t) = \int_{\zeta}^{\zeta+h} dF(t) = g(\zeta).$$

The existence of solutions and the limiting behavior for bounded solutions of such renewal type equations which involve positive and negative values of t has not been treated.

Feller [10] and later Täcklind [12] have developed many Tauberian results for the cases where all the functions $u(\zeta)$, $F(\zeta)$ and $g(\zeta)$ considered are

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zero for ζ negative. This reduces (*) to the classical renewal equation where Laplace transform methods can be exploited. Doob [6] and Blackwell [1] discussed the same type of renewal equation from the point of view of probability theory and appealed to the ergodic theory of Markoff chains.

In this work we shall show that most problems of the general renewal equation can be reduced to an application of the general Wiener theorem and the properties of slowly oscillating functions. Our methods are thoroughly analytic and apply to situations which do not necessarily correspond to probability models. Moreover, a complete analysis of (*) shall be given concerning existence and asymptotic behavior of solutions with results describing rates of convergence under suitable assumptions. Erdős, Pollard and Feller [7] and later Feller [9] in the study of recurrent events did apply the Wiener theorem to some discrete analogues of (*) and these examples have served to suggest to this writer this general unified approach. Most of the results of Täcklind who dealt with the classical renewal equation use deep methods of Fourier analysis. These results are illuminated and in many instances subsumed by our methods. Finally, in the course of revising this paper it has come to our attention that W. L. Smith very recently [11] independently has discussed the classical one-sided renewal equation from the point of view of Wiener's general Tauberian theorem. His treatment and this investigation supplement each other in many respects. We employ the basic properties of slowly oscillating functions while Smith uses Pitt's extension of the Wiener theorem.

Some fundamental differences appear between the general renewal equation (*) and the type of renewal equation studied in [8], [13] and [11]. For example, solutions to (*) need not exist and when they do exist there are, in general, infinitely many bounded and unbounded solutions. This complicates the analysis of the asymptotic behavior of solutions of (*). In fact, solutions $u(\zeta)$ can be found for certain examples which oscillate infinitely as $|\zeta| \rightarrow \infty$. Even when we restrict ourselves to bounded solutions to (*), the abundance of such solutions necessitates a careful analysis which does not occur in the handling of the one-sided renewal equation. (See the beginning of § 3.)

In § 2 we present a complete treatment of the discrete renewal equation

$$(**) \quad u_n - \sum_{k=\infty}^{\infty} a_{n-k} u_k = b_n.$$

In this case necessary and sufficient conditions are given to insure the existence of bounded solutions to (**). Asymptotic limit theorems for bounded

solutions to (**) are obtained and appropriate conditions are indicated which yield results about the rates of convergence of such solutions as $n \rightarrow \infty$.

The general equation (*) is treated in § 3 where the existence and limit theorems for bounded solutions of (*) are given. The Plancherel and Hausdorff-Young theorems are used to establish the existence of bounded solutions to (*). Limit theorems are analyzed and rates of convergence are obtained. Some applications are made to the classical renewal equation.

The relationship of Wiener's Tauberian theorem to ideal theory motivated the content of § 4. This last section indicates a new avenue of approach to the meaning of the renewal equation.

Finally, I wish to express my gratitude to James L. McGregor for his helpful discussions in the preparation of this manuscript.

2. Discrete renewal equation. This section is devoted to a complete analysis of the renewal equation

$$(1) \quad u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k = b_n.$$

The convolution of two sequences $\{x_n\}$ and $\{y_n\}$ is denoted by

$$x * y = \left\{ \sum_{k=-\infty}^{\infty} x_{n-k} y_k \right\}$$

This product operation is well defined whenever, for example, at least one of the sequences is an absolutely convergent series while the other sequence is uniformly bounded. Equation (1) can thus be written as

$$(2) \quad u - a * u = b,$$

We suppose hereafter, that the sequences $\{a_n\}$ and $\{b_n\}$ have the property that $a_n \geq 0$, $\sum a_n = 1$ and $\sum |b_n| < \infty$ and that u_n represents a solution of (2). In general, there exist many solutions of (2) which complicates the study of the asymptotic behavior of solutions $\{u_n\}$ of the renewal equations. We first investigate the general problem of the existence of solutions of (1). To this end, we introduce the linear operation T which can be applied to any sequence $\{c_n\}$ which forms an absolutely convergent series. Precisely, let

$$T\{c_n\} = \{(Tc)_n\}$$

where

$$(Tc)_n = \begin{cases} \sum_{i=n+1}^{\infty} c_i & n \geq 0 \\ -\sum_{-\infty}^n c_i & n < 0. \end{cases}$$

Let

$$\sigma_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and define the linear functional $\phi_0(c) = \sum_{n=-\infty}^{\infty} c_n$. We note the following relation for future use

$$(3) \quad \phi_0(c) \sigma - \sigma * c = Tc.$$

The operation T can be repeated provided that the resulting sequence $\{Tc\}$ is an absolutely convergent series. If, for example, $\sum_{n=-\infty}^{\infty} |n^k c_n| < \infty$, then $T^k c$ is well defined. Moreover, we observe for later reference that if

$$\sum |n^k c_n| < \infty,$$

then

$$\lim_{|n| \rightarrow \infty} |n^k (Tc)_n| = 0.$$

We now impose two very fundamental assumptions.

ASSUMPTION A. *The greatest common divisor of the indices n where $a_n > 0$ is 1.*

ASSUMPTION B. *The series $\sum |na_n| < \infty$ and $\sum_{n=-\infty}^{\infty} na_n = m \neq 0$. (For definiteness we take $m > 0$.)*

Many of the following results can be extended to the case where the g.c.d. of the indices n where $a_n > 0$ is $d > 1$. We leave this task to the interested reader. However, Assumption B is indispensable for the validity of many of the subsequent results. Some results can be extended by suitable modifications to $m = \infty$.

An important tool to be used frequently is the following lemma.

LEMMA 1. *If Assumptions A and B are satisfied, then there exists a sequence $\{r_n\}$ with*

$$\sum |r_n| < \infty \text{ and } r * Ta = \delta$$

where $\delta = \{\delta_n^0\}$. (The sequence δ is the identity element with respect to the $*$ multiplication.)

Proof. For the sequence $\{a_n\}$ let $a(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$. The relation (3) implies for $0 < \theta < 2\pi$

$$\frac{1 - a(\theta)}{1 - e^{i\theta}} = \sum_{-\infty}^{\infty} (Ta)_n e^{in\theta} = Ta(\theta).$$

Assumption A implies that $Ta(\theta) \neq 0$ for $\theta \neq 0$ and $|\theta| < 2\pi$. Assumption B yields that $Ta(0) \neq 0$ and the fact that $\sum_{-\infty}^{\infty} |(Ta)_n| < \infty$. By virtue of Wiener's Tauberian theorem

$$\frac{1}{(Ta)(\theta)} = \sum_{-\infty}^{\infty} r_n e^{in\theta}$$

defines an absolutely convergent Fourier series. The conclusion of Lemma 1 is now evident from this last relation.

We now proceed to discuss the existence of solutions to (1) or (2).

THEOREM 1. *If Assumptions A and B are satisfied, then there exists a bounded solution of (1). Any two bounded solutions of (1) differ by a fixed constant.*

Proof. We seek a bounded solution of

$$(2) \quad u - a * u = b.$$

Multiplying formally (2) by σ and using (3) we obtain $u * (Ta) = \sigma * b$ and hence by Lemma 1

$$u = r * \sigma * b.$$

The sequence $r * \sigma * b$ is a bounded sequence and it is easily verified

provides a solution for relation (2). To establish the second half of the theorem it is sufficient to show that

$$(4) \quad u - a * u = 0$$

possess only constant bounded solutions. Let $(\Delta u) = (u_n - u_{n+1})$. It follows readily that (4) implies $(Ta) * \Delta u = 0$. Multiplication by r yields that $(\Delta u) = 0$ and hence the result sought for.

We now show that in general, nonbounded solutions of (4) and therefore of (1), can be found. This is illustrated by the following example. Although the example is special, the technique is general and the reader can easily construct many other such examples.

Let $a_1 = 1/2$, $a_2 = 1/2$ and $a_i = 0$ for $i \neq 1, 2$. Equation (4) becomes

$$u_n = \frac{1}{2} u_{n-1} + \frac{1}{2} u_{n-2} \quad \text{all } n.$$

We can prescribe u_0 and u_1 arbitrarily and therefore we obtain a two-dimensional set of solutions. However, by virtue of Theorem 1 only a one-dimensional set of bounded solutions exists. Hence, unbounded solutions also exist. The unbounded solution oscillates infinitely as $n \rightarrow -\infty$.

It is worth showing that a converse to Theorem 1 can be obtained.

THEOREM 2. *If*

$$\sum_{k=-\infty}^{\infty} b_k > 0, \quad \sum |na_n| < \infty \quad \text{but} \quad \sum_{-\infty}^{\infty} na_n = 0,$$

then there exists no bounded solutions to (2) provided that

$$\sum_{n=-\infty}^{\infty} b_n > 0, \quad a_1 > 0 \quad \text{and} \quad a_{-1} > 0.$$

Proof. Suppose to the contrary that $\{u_n\}$ is a bounded solution to (2). Let $\lambda = \overline{\lim}_{n \rightarrow \infty} u_n$ then there exists a subsequence $u_{n_i} \rightarrow \lambda$. By virtue of a standard probability argument (see [10, p. 260]), it follows that $\lim_{n_i \rightarrow \infty} u_{n_i-k} = \lambda$ for each integer k . A similar subsequence m_i can be found such that $\lim_{m_i \rightarrow -\infty} u_{m_i-k} = u$, where $u = \overline{\lim}_{m \rightarrow -\infty} u_m$. As in Theorem 1 we obtain that $(Ta) * \Delta u = b$. Summing from m_i to n_i gives

$$\sum_{k=-\infty}^{\infty} (u_{n_i-k} - u_{m_i-k}) (Ta)_k = \sum_{k=m_i}^{n_i} b_k.$$

Allowing $n_i \rightarrow \infty$ and $m_i \rightarrow -\infty$, it follows readily since $\sum |(Ta)_k| < \infty$ that

$$0 < \sum_{-\infty}^{\infty} b_k = (\lambda - u) \sum_k (Ta)_k = (\lambda - u) \sum_{n=-\infty}^{\infty} na_n = 0$$

a contradiction.

REMARK. Theorem 2 can be established using the weaker Assumption A in place of the hypotheses that $a_1 > 0$ and $a_{-1} > 0$. We omit the details

Having discussed the question of existence we now turn to investigate the asymptotic properties of bounded solutions to (2). Throughout the remainder of this section we assume that Assumptions A and B are satisfied. A useful result which we state here for later purposes is the following well known Abelian theorem.

LEMMA 2. *If $\{r_n\}$ is such that $\sum_{n=-\infty}^{\infty} |r_n| < \infty$, $\{\omega_n\}$ is bounded and $\lim_{n \rightarrow \infty} \omega_n = 0$, then*

$$\lim_{n \rightarrow \infty} \sum_{-\infty}^{\infty} \omega_{n-k} r_k = 0.$$

The following theorem is a simple Tauberian result for solutions of (2).

THEOREM 3. *If u_n is a bounded solution to (1), then $\lim_{n \rightarrow \infty} u_n$ and $\lim_{u_n \rightarrow -\infty} u_n$ exist.*

Proof. By Theorem 1, it is sufficient to prove the result for the special solution

$$u = r * b * \sigma.$$

For this special solution, we have

$$u_n = \sum_{k=-\infty}^n (r * b)_k.$$

Hence the limit exists, in fact,

$$\lim_{n \rightarrow -\infty} u_n = 0, \quad \lim_{n \rightarrow \infty} u_n = \sum_{-\infty}^{\infty} (r * b)_k = \frac{\phi_0(b)}{m}$$

Q.E.D. To obtain more precise results let

$$v = u - \frac{\phi_0(b)}{m}$$

where $u = r * b * \sigma$ is the unique bounded solution for which $u_n \rightarrow 0$ as $n \rightarrow \infty$. From the proof of Theorem 3 it is clear that $v_n \rightarrow 0$ as $|n| \rightarrow \infty$. It is easy to show that

$$(5) \quad Ta * v = -Tb + \frac{\phi_0(b)}{m} T^2a$$

or

$$v = -r * \left[Tb - \frac{\phi_0(b)}{m} T^2a \right].$$

Hence if we assume in addition to A and B that

$$\sum |(T^2a)_n| < \infty \quad \text{and} \quad \sum |(Tb)_n| < \infty,$$

then it follows that $\sum |v_n| < \infty$. These new assumptions enable us to obtain further results about the rate of convergence of v_n and hence of u_n . To this end, we define the operation S on any sequence $\{t_n\}$, $St = \{nt_n\}$. The hypothesis

$$\sum |(T^2a)_n| < \infty \quad \text{and} \quad \sum |(Tb)_n| < \infty$$

or the equivalent assumptions

$$\sum n^2 a_n < \infty \quad \text{and} \quad \sum |nb_n| < \infty,$$

respectively imply easily that STa defines an absolutely convergent series and ST^2a constitutes a bounded sequence which tends to zero as $|n| \rightarrow \infty$. A direct calculation using (5) gives that

$$(6) \quad S(Ta * v) - ST(a) * v = -STb + \frac{\phi_0(b)}{m} ST^2a - STa * v.$$

The left side of (6) is identical componentwise with $Ta * Sv$. Multiply (6) by Ta , then with the aid of (5), we obtain

$$(7) \quad \begin{aligned} Ta * Ta * Sv = & -Ta * STb + \frac{\phi_0(b)}{m} \{Ta * ST^2a\} \\ & -STa * \left[-Tb + \frac{\phi_0(b)}{m} T^2a \right]. \end{aligned}$$

On account of the hypothesis and Lemma 2, we find that the right side is a bounded sequence which tends to zero at $\pm\infty$. Employing Lemma 1, we conclude that Sv is bounded and $\lim_{n \rightarrow \infty} |nv_n| = 0$.

Although it might appear as if the relation (7) is rather fortuitous, a simple method to deduce the formula begins with the Fourier series relation

$$(8) \quad Ta(\theta)v(\theta) = -Tb(\theta) + \frac{f_0(b)}{m} T^2a(\theta)$$

which is well defined and is an alternative way to express (5). Differentiation of (8) with multiplication by $Ta(\theta)$ and use of (8) gives a formal representation of (7). The preceding argument was in essence a justification of this differentiation process.

The preceding analysis extends with the aid of an induction argument. The details are omitted and we sum up the results in the following theorem.

THEOREM 4. *Let $a_n \geq 0$, $\sum_{-\infty}^{\infty} a_n = 1$, satisfying Assumptions A and B. Let u_n represent the unique bounded solution of (2) for which $\lim_{n \rightarrow \infty} u_n = 0$ (see Theorem 3). If*

$$\sum_{n=-\infty}^{\infty} |n^k b_n| < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |n^{k+1} a_n| < \infty,$$

then

$$\sum_{n \geq 0} \left| n^{k-1} \left[u_n - \frac{\phi_0(b)}{m} \right] \right| + \sum_{n < 0} |n^{k-1} u_n| < \infty$$

and

$$\lim_{n \rightarrow \infty} n^k \left[u_n - \frac{\phi_0(b)}{m} \right] = \lim_{n \rightarrow -\infty} n^k u_n = 0.$$

A first classical application of Theorem 1 can be obtained from the theory of Markov chains. Let E represent a recurrent state from an irreducible non-periodic chain. Let u_n represent the probability of starting from E and returning to E in n steps. Let a_n denote the probability that the first return occurs at the n th step ($n > 0$). Put $u_0 = 1$, $u_{-n} = 0$ and $a_{-n} = 0$ (for $n \geq 0$), then

$$u_n = \sum_{k=0}^n a_{n-k} u_k = b_n$$

where $b_n = 0$ for $n \neq 0$ and $b_0 = 1$. Since E describes a recurrent state, $\sum a_i = 1$ and trivially $m = \sum_{i=0}^{\infty} i a_i > 0$. As an immediate consequence of Theorem 4, we infer that if

$$\sum_0^{\infty} n^{k+1} a_n < \infty, \text{ then } \sum_{n=1}^{\infty} n^{k-1} \left| u_n - \frac{1}{m} \right| < \infty \text{ and } \lim_{n \rightarrow \infty} n^k \left[u_n - \frac{1}{m} \right] = 0.$$

A second application deals with the following problem treated by K.L. Chung and J. Wolfowitz [4]. We generalize their result in obtaining stronger rates of convergence by assuming further conditions on the moments. Let X denote a random variable which assumes only integral values and define for all n

$$a_n = \Pr\{X = n\} \quad n = 0, \pm 1, \pm 2, \dots$$

Let $X_i (i = 1, 2, \dots)$ denote an infinite sequence of independent events with the same distribution as X . Define

$$S_j = \sum_{i=1}^j X_i \text{ and } u_n = \sum_{j=1}^{\infty} \Pr\{S_j = n\} = \text{Expected number of sums where } S_j = n.$$

Let $m = E(x)$ be the expectation of X . Suppose the greatest common divisor of the indices n such that $a_n > 0$ is 1 and $0 < m < \infty$. Chung and Wolfowitz in [4] allow $m = \infty$, but the present method does not apply. The restriction on the greatest common divisor is not essential but the requirement that $m \neq 0$ is very crucial and in fact in the contrary case $u_n = \infty$ as is shown by Chung and Fuchs [5]. We obtain that if $\sum_{n=-\infty}^{\infty} |n^{k+1} a_n| < \infty$, then

$$\sum_{n \geq 0} \left| n^{k-1} \left[u_n - \frac{1}{m} \right] \right| + \sum_{n < 0} |n^{k-1} a_n| \text{ and } \lim_{n \rightarrow \infty} n^k \left[u_n - \frac{1}{m} \right] = \lim_{n \rightarrow -\infty} n^k u_n = 0.$$

Indeed, it follows from the definition of u_n that

$$u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k = a_n.$$

It can be seen that the sequence u_n is uniformly bounded and $\lim_{n \rightarrow \infty} u_n = 0$ (see [5]). The conditions of Theorem 4 are met and the conclusion follows from the results of that theorem. Summing up, we have

COROLLARY. *Let X_i be identically distributed independent lattice random variables with distribution given by $\Pr\{x = n\} = a_n$ and $u_n = \sum_{j=1}^n \Pr\{s_j = n\}$ where $s_j = \sum_{i=1}^j x_i$. If the expected value of $x = m > 0$ and g.c.d. $n = 1$, then*

$$\sum_{n=-\infty}^{\infty} |n^{k+1} a_n| < \infty$$

implies

$$\sum_{n \geq 0} \left| n^{k-1} \left[u_n - \frac{1}{m} \right] \right| + \sum_{n < 0} |n^{k-1} u_n| < \infty$$

while

$$\lim_{n \rightarrow \infty} n^k \left[u_n - \frac{1}{m} \right] = \lim_{n \rightarrow -\infty} n^k u_n = 0.$$

3. Continuous renewal equation. This section is devoted to an analysis of the existence and asymptotic properties of solutions for each ξ of the relation

$$(9) \quad u(\xi) - \int_{-\infty}^{\infty} u(\xi - t) df(t) = g(\xi).$$

The convolution of two functions $x(t)$ and $y(t)$ is defined as

$$x * y = \int_{-\infty}^{\infty} x(t - \xi) y(\xi) d\xi$$

which exists if, say, x is integrable and y is bounded. We shall be concerned only with bounded solutions of (9). It is assumed that

$$df(t) \geq 0, \quad \int_{-\infty}^{\infty} df = 1 \quad \text{and} \quad \int |g| < \infty.$$

The following hypotheses are now imposed:

ASSUMPTION A'. The distribution f is a non-lattice distribution, that is, the points of increase of f do not concentrate at the multiples of a fixed value.

ASSUMPTION B'. $\int_{-\infty}^{\infty} |t| df(t) < \infty$ and $\int_{-\infty}^{\infty} t df(t) = m \neq 0$ (say $m > 0$)

These two assumptions constitute the continuous analogues of Assumptions A and B and hereafter we suppose these assumptions satisfied.

We introduce the operation T defined for any function of bounded total variation $h(t)$. Let

$$\sigma(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{and} \quad \phi_0(h) = \int_{-\infty}^{\infty} dh(t)$$

and

$$Th = \phi_0(h) \sigma - \sigma * h$$

or

$$(Th)(t) = \begin{cases} \int_t^{\infty} dh(t) & t \geq 0 \\ -\int_{-\infty}^t dh(t) & t < 0 \end{cases}$$

T is also defined for integrable functions $k(t)$ as follows:

$$Tk = Tk^*(t) \quad \text{where} \quad k^*(t) = \int_{-\infty}^t k(\xi) d\xi.$$

Let

$$u_n(\xi) = n \int_{\xi}^{\xi+1/n} u(t) dt$$

with g_n defined similarly. Equation (9) can be converted to

$$u_n(\xi) - \int_{-\infty}^{\infty} u_n(\xi - t) df(t) = g_n(\xi)$$

Since the derivative of u_n is essentially uniformly bounded, we obtain on integration by parts that

$$(10) \quad Tf * u'_n = \int_{-\infty}^{\infty} u'_n(\xi - t) Tf(t) dt = g_n(\xi).$$

The finiteness of $\int_{-\infty}^{\infty} |t| df(t)$ is equivalent to the integrability of $Tf(t)$ and thus (10) is well defined. Integrating (10) from a to ξ gives

$$\int_{-\infty}^{\infty} [u_n(\xi - t) - u_n(a - t)] Tf(t) dt = \int_a^{\xi} g_n(t) dt.$$

Letting n go to ∞ , we have almost everywhere

$$(11) \quad \int_{-\infty}^{\infty} [u(\xi - t) - u(a - t)] Tf(t) dt = \int_a^{\xi} g(t) dt.$$

Since both the right and left hand sides of (11) are continuous this identity holds everywhere in ξ and a . Allowing $a \rightarrow \infty$, we find from (11) that

$$\lim_{a \rightarrow -\infty} \int_{-\infty}^{\infty} u(a - t) Tf(t) dt = c.$$

Adding to any solution of (9) a constant produces a new solution u of (9). Therefore, we may suppose that $c = 0$. Thus,

$$(12) \quad \int_{-\infty}^{\infty} u(\xi - t) Tf(t) dt = \int_{-\infty}^{\xi} g(t) dt.$$

We define for this u satisfying (12),

$$(12-a) \quad v(\xi) = u(\xi) - \frac{\phi_0(g)}{m} \sigma(\xi),$$

It follows directly that

$$(13) \quad v * Tf = -Tg + \frac{\phi_0(g)}{m} T^2f.$$

We now present a series of lemmas needed in the sequel.

LEMMA 3. *Under the assumptions stated above, the Fourier transform*

$$(Tf)^*(\theta) = \int_{-\infty}^{\infty} e^{it\theta} Tf(t) dt$$

vanishes nowhere.

The proof is similar to that of Theorem 1, and is based on the identity

$$i\theta(Tf)^*(\theta) = -1 + \int_{-\infty}^{\infty} e^{it\theta} df(t).$$

LEMMA 4. *Any two bounded solutions of (9) differ by a constant.*

Proof. It is enough to show that the only bounded solution of

$$u(\xi) - \int_{-\infty}^{\infty} u(\xi - t) df(t) = 0$$

are constants. Using a reasoning similar to that of deducing (12), we get

$$(13a) \quad \int_{-\infty}^{\infty} u(\xi - t) Tf(t) dt = c.$$

By subtracting an appropriate constant from (13a), we have for $v = u - c'$ that $v * Tf = 0$. Lemma 3 and the general Wiener's Tauberian theorem yields that $v * r = 0$ for every integrable $r(t)$. It follows readily from this last fact that $v = 0$ almost everywhere or $u = c'$ a.e.

LEMMA 5. *If $r(t)$ is integrable and $w(t) \rightarrow 0$ as $|t| \rightarrow \infty$, then*

$$\lim_{|\xi| \rightarrow \infty} \int_{-\infty}^{\infty} w(\xi - t) r(t) dt = 0.$$

This last Abelian theorem is well known and straightforward.

LEMMA 6. *If v is bounded and satisfies (13), then*

$$\lim_{|\xi| \rightarrow \infty} \int_{-\infty}^{\infty} r(\xi - t) v(t) dt = 0$$

for any integrable function r .

Proof. The hypothesis and the character of the operation T imply that

$$\lim_{|\xi| \rightarrow \infty} Tg(\xi) = \lim_{|\xi| \rightarrow \infty} T^2f(\xi) = 0.$$

Consequently,

$$\lim_{|\xi| \rightarrow \infty} \int_{-\infty}^{\infty} v(\xi - t) Tf(t) dt = 0.$$

An application of the general Wiener Tauberian theorem leads to the conclusion of the lemma.

COROLLARY. *Under the assumptions of Lemma 6 we have*

$$\lim_{|x| \rightarrow \infty} \int_x^{x+\Delta} v(t) dt = 0.$$

Indeed, choose

$$r(\xi) = \begin{cases} \frac{1}{\Delta} & \text{for } 0 \leq \xi \leq \Delta \\ 0 & \text{elsewhere} \end{cases}.$$

We now establish the fundamental asymptotic limit theorem for bounded solutions of (9). The basic Tauberian theorem used is the Wiener theorem coupled with the properties of slowly oscillating sequences.

THEOREM 5. *If u is a bounded solution of (9), and f has a decomposition $f = f_1 + f_2$ where f_1 is absolutely continuous and the total variation of $f_2 = \lambda < 1$, and $\lim_{|\xi| \rightarrow \infty} g(\xi) = 0$, then $\lim_{t \rightarrow \infty} u(t)$ and $\lim_{t \rightarrow -\infty} u(t)$ both exist. If $\lim_{t \rightarrow -\infty} u(t) = 0$, then $\lim_{t \rightarrow \infty} u(t) = \phi(g)/m$.*

Proof. It is enough to assume that v defined by (12-a) from u satisfies (13). This can be achieved if necessary by altering u by a fixed constant (see the discussion preceding Lemma 3). As before, we find that

$$(14) \quad \lim_{|\xi| \rightarrow \infty} \int_{-\infty}^{\infty} v(\xi - t) Tf(t) dt = 0$$

It will now be shown that $v(t)$ is slowly oscillating as $|t| \rightarrow \infty$ ($v(t)$ is

said to be slowly oscillating (s.o.) if

$$\lim_{\substack{\xi \rightarrow \infty \\ \eta \rightarrow 0}} |v(\xi + \eta) - v(\xi)| = 0.$$

A similar definition applies at $t = -\infty$. The general Wiener theorem and the s.o. character of $v(t)$ implies the stronger conclusion over Lemma 6 that

$$\lim_{|t| \rightarrow \infty} v(t) = 0$$

which is our assertion. It thus remains to establish that $v(\xi)$ is s.o. and we confine our argument to the situation where $\xi \rightarrow \infty$. A similar analysis applies at $-\infty$. Remembering that the convolution of an absolutely continuous distribution and any other distribution remains absolutely continuous, we obtain upon n fold iteration of (9) that

$$\begin{aligned} u(\xi) &= \int_{-\infty}^{\infty} u(\xi - t) dk_1(t) + \int_{-\infty}^{\infty} u(\xi - t) dk_2(t) + \int_{-\infty}^{\infty} g(\xi - t) dk_3(t) \\ &= I_1(\xi) + I_2(\xi) + I_3(\xi) \end{aligned}$$

where k_1 is absolutely continuous, k_2 is the n fold convolution of f_2 with itself and $k_3(t)$ is of bounded total variation. Since $g(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ by Lemma 5

$$\lim_{|\xi| \rightarrow \infty} I_3(\xi) = 0.$$

Next, we observe that $|I_2(\xi)| \leq \lambda^n c$ where c is the upper bound of u . Finally,

$$\begin{aligned} |I_1(\xi + \eta) - I_1(\xi)| &\leq \int |u(\xi - t)| |k_1'(t + \xi) - k_1'(t)| dt \\ &\leq c \int |k_1'(t + \eta) - k_1'(t)| dt \rightarrow 0 \quad \text{as } \eta \rightarrow 0 \end{aligned}$$

by virtue of a well-known theorem of Lebesgue. Combining these estimates, we get that

$$\overline{\lim}_{\substack{\xi \rightarrow \infty \\ \eta \rightarrow 0}} |v(\xi + \eta) - v(\xi)| \leq 2c \lambda^n$$

which by proper choice of n can be made as small as one pleases. This completes the proof.

REMARK. Theorem 5 is valid if we merely assume that some iterate of f has an absolutely continuous part.

COROLLARY. Under the conditions of Lemma 6, if $v(t)$ is uniformly continuous for $t > 0$ and $t < 0$, then

$$\lim_{|t| \rightarrow \infty} v(t) = 0.$$

Proof. The function $v(t)$ is s.o. from which the conclusion follows as in Theorem 5.

In many examples, we deal with a solution u of (9) which is by physical considerations bounded while in other cases boundedness for certain solutions has to be verified. Our next object is to give sufficient conditions so that we can establish the existence of bounded solutions of (9). From now on we assume that f is absolutely continuous and let

$$f(\xi) = \int_{-\infty}^{\xi} a(t) dt.$$

LEMMA 7. If u is a solution of (9) which belongs to L^p ($p \geq 1$), $a \in L^1$ also belongs to $L^{p'}$ where p' is the conjugate exponent to p and g is bounded, then u is bounded.

Proof. Applying Hölder's inequality to (9) and an obvious change of variable, we obtain

$$|u(\xi)| \leq \left(\int_{-\infty}^{\infty} |u|^p \right)^{1/p} \left(\int_{-\infty}^{\infty} |a|^{p'} \right)^{1/p'} + c.$$

THEOREM 6. If $a(t)$ belongs to L^1 and L^2 , $g(t)$ is bounded,

$$\int |x|^2 a(x) dx < \infty$$

and

$$\int |x| g(x) dx < \infty,$$

then a bounded solution $u(t)$ of (9) exists.

Proof. The Fourier transform of any integrable $h(t)$ is denoted by $h^*(\theta)$. Consider the expression

$$(14) \quad w^*(\theta) = \frac{g^*(\theta) - (Ta)^*(\theta) [\phi(g)/m]}{1 - a^*(\theta)}$$

It will now be shown that (14) is the Fourier transform of a function in L^2 . To this end, by the Riemann Lebesgue lemma $a^*(\theta) \rightarrow 0$ as $|\theta| \rightarrow \infty$ and $|a^*(\theta)| < 1$ for $\theta \neq 0$ with $a^*(\theta)$ continuous. Since the first moment of a exists, Ta is bounded and in L^1 . Hence, Ta belongs to L^2 and $Ta^*(\theta) \in L^2$. A similar argument shows that $g^*(\theta) \in L^2$. Thus for $|\theta| \geq \alpha > 0$, $w^*(\theta)$ is in L^2 for any fixed positive constant α . But,

$$(15) \quad w^*(\theta) = \frac{\{g^*(\theta) - (Ta)^*(\theta)[\phi(g)/m]\}/i\theta}{(1 - a^*(\theta))/i\theta}$$

$$= \frac{-(Tg)^*(\theta) + [\phi(g)/m](T^2a)^*(\theta)}{(Ta)^*(\theta)}$$

The existence of the second moment of a implies that $(T^2a)^*(\theta)$ is continuous. Analogously, $(Tg)^*(\theta)$ is continuous by virtue of $\int |x|g(x) < \infty$. Since $Ta^*(0) = m > 0$, we find that $w^*(\theta)$ is continuous in the neighborhood of zero and hence $w^*(\theta)$ is in L^2 . Consequently, $w(t)$ in L^2 exists which is the Fourier transform of $w^*(\theta)$ and conversely. Moreover, (14) yields

$$w(\xi) - \int_{-\infty}^{\infty} w(t) a(\xi - t) dt = g(\xi) - Ta(\xi) \frac{\phi_0(g)}{m}$$

for almost all ξ .

As a convolution of two elements of L^2 the integral on the right is bounded and continuous. Hence the right side is bounded and remains unaltered, if w is changed on a set of measure zero.

As in Lemma 7, it follows that $w(\xi)$ is bounded. Putting

$$u(\xi) = w(\xi) + \frac{\phi_0(g)}{m} \sigma(\xi),$$

we find that u is bounded and satisfies (9).

REMARK. Theorem 6 can be established under the weaker conditions that

$$\int |x|^{1+\alpha} a(x) < \infty$$

and

$$\int |\xi|^\alpha g(x) < \infty$$

for some $\alpha > 0$. These assumptions are sufficient to imply the boundedness of $w^*(\theta)$ in the neighborhood of zero.

Other sufficient criteria can be obtained for the existence of bounded solutions to (9) involving use of the Hausdorff-Young inequalities in place of the Plancherel theorem.

THEOREM 7. If $a(t)$ belongs to L^1 and L^p ($1 < p < 2$),

$$\int |t|^{1+\alpha} a(t) dt < \infty$$

with $\alpha > 0$,

$$\int |g^*(\theta)|^p d\theta < \infty$$

and g is bounded, then a bounded solution of (9) exists.

It is worth noting that the solutions u guaranteed by Theorems 6 and 7 have the property on account of Theorem 5 that $\lim_{|t| \rightarrow \infty} u(t)$ exist.

Our next objective is to find conditions which imply conclusions about the rate of convergence of $w(\xi)$ of Theorem 6 as $|\xi| \rightarrow \infty$ and thus of $u(\xi)$. To this end, we differentiate (14) and (15), we get

$$(16) \quad w^{*'}(\theta) = \frac{a^{*'}(\theta)w^*(\theta) + g^{*'}(\theta) - Ta^{*'}(\theta) [\phi(g)/m]}{1 - a^*(\theta)}$$

$$(17) \quad w^{*'}(\theta) = \frac{Ta^{*'}(\theta)w^*(\theta) - Tg^{*'}(\theta) + [\phi(g)/m] (T^2a)^{*'}(\theta)}{Ta^*(\theta)}$$

Relation (17) can be derived from (16) by dividing numerator and denominator by $i\theta$ similar to the method of obtaining (15) from (14).

Under the assumptions that

$$\int |t|^3 a(t) dt < \infty$$

and

$$\int t^2 g(t) dt < \infty$$

with g bounded and monotone decreasing as $|t| \rightarrow \infty$ we now show that $w^{*\prime}(\theta)$ belongs to L_2 . Indeed, for $|\theta| \geq \alpha > 0$ we use (16) to estimate $w^{*\prime}(\theta)$ and we use (17) to analyze $w^{*\prime}(\theta)$ in the neighborhood of the origin.

For $\xi > 0$

$$\xi Ta(\xi) \leq \int_{\xi}^{\infty} ta(t) dt \leq c$$

and similarly $|\xi Ta(\xi)| \leq c$ for ξ negative. Also,

$$\int_{-\infty}^{\infty} t^2 Ta^2(t) \leq c \int_{-\infty}^{\infty} |t Ta(t)| \leq c' \int t^2 a(t) dt < \infty$$

and

$$\int_{-\infty}^{\infty} t^2 g^2(t) \leq c \int t^2 g(t) < \infty.$$

Since $g(t)$ is monotone decreasing as $|t| \rightarrow \infty$, we obtain easily that $|tg(t)| \leq c$. As $a^{*\prime}(\theta)$ is the Fourier transform of $t a(t)$ in L^1 (except for a fixed constant factor) we know that $a^{*\prime}(\theta)$ is uniformly bounded. By Theorem 6, $w^*(\theta)$ is in L^2 and therefore $a^{*\prime}(\theta) w^*(\theta)$ is in L^2 . $g^{*\prime}(\theta)$ is in L^2 by virtue of $t g(t) \in L^2$ and $Ta^{*\prime}(\theta)$ is in L^2 as a consequence of $t Ta(t)$ in L^2 which were established above. Since $|a^*(\theta)| < 1$ for $\theta \neq 0$ and tends to zero as $|\theta| \rightarrow \infty$, we find, collecting all these cited facts, that $w^{*\prime}(\theta)$ is in L^2 for $|\theta| \geq \alpha > 0$. The assumptions of the existence of the third and second moments of a and g respectively yield as in the proof of Theorem 6 using (17) that $w^{*\prime}(\theta)$ is continuous at zero. Thus $w^{*\prime}(\theta)$ is square integrable throughout and as a result of standard Fourier analysis is the Fourier transform of $tw(t)$ in L^2 . Relation (16) gives

$$\begin{aligned} (18) \quad tw(t) &= \int_{-\infty}^{\infty} a(t-\xi) \xi w(\xi) d\xi \\ &= \int_{-\infty}^{\infty} w(t-\xi) \xi a(\xi) + tg(t) - \frac{\phi(g)}{m} tTa(t) \end{aligned}$$

The fact that $tg(t)$ and $t(Ta)(t)$ are bounded imply by an argument completely analogous to the proof of Lemma 7 that $tw(t)$ is bounded. It follows as before that $tw(t)$ is s.o. (see Theorem 5). The relation (17) leads to

$$(19) \quad \int_{-\infty}^{\infty} Ta(\xi - t) tw(t) dt \\ = \int_{-\infty}^{\infty} (\xi - t) Ta(\xi - t) w(t) dt - \xi Tg(\xi) + \frac{\phi(g)}{m} \xi T^2a(\xi).$$

Since $w(t) \rightarrow 0$, $\xi Tg(\xi) \rightarrow 0$ and $\xi(T^2a)(\xi) \rightarrow 0$ as $|t| \rightarrow \infty$, we obtain by Lemma 5 that the right side of (19) tends to zero as $|\xi| \rightarrow \infty$. Combining the s.o. character of $tw(t)$, its boundedness and the Wiener Tauberian theorem leads to the conclusion that

$$\lim_{|t| \rightarrow \infty} tw(t) = 0.$$

Proceeding inductively we can obtain higher rates of convergence by imposing the requirement of the existence of higher moments using this same method. We sum up the discussion in the following theorem.

THEOREM 8. *Let*

$$\int_{-\infty}^{\infty} |t|^{n+2} a(t) dt < \infty$$

with a in L^1 and L^2 . Let $g(t)$ be bounded monotone decreasing for $t \geq t_0 > 0$ and nondecreasing for $t \leq -t_0 < 0$ with

$$\int |t|^{n+1} g(t) dt < \infty,$$

then

$$\lim_{|t| \rightarrow \infty} t^n w(t) = 0$$

where

$$u = w + \frac{\phi_0(g)}{m} \sigma$$

is a solution of (9), and $w(t)$ is the Fourier transform of $w^*(\theta)$ (see (14)). (We recall that Lemma 4 shows that $u(t)$ as given above is the only bounded solution for which $u(t) \rightarrow 0$ as $t \rightarrow -\infty$.)

We now append some remarks about the classical renewal equation

$$(20) \quad u(x) = g(x) + \int_0^x u(x-\xi) df(\xi) \quad x \geq 0.$$

The assumptions made are that

$$u \geq 0, g \geq 0, \quad \int_0^\infty g(\xi) d\xi = b < \infty$$

and f is the distribution of a non-lattice random variable. The function $Tf(\xi)$ is introduced as before. If the first moment of f exists, then $Tf \in L^1$ and

$$m = \int_0^\infty x df(x) > 0.$$

Thus, we deduce as before that Tf possesses a Fourier transform which is never zero. Throughout the discussion of this case it is no longer necessary to assume any boundedness condition on $u(\xi)$, the nonnegativeness of u suffices to enable us to obtain all the results of Theorems 5-8.

To indicate the simplicity of our methods we now show how Wiener's Tauberian theorem can be used directly to establish a slight generalization of one of the fundamental results of Täcklind on the classical renewal equation. His procedure involves complicated estimates.

THEOREM 9. *Let $\Phi(x)$ denote a monotonic solution to the integral equation*

$$(21) \quad \Phi(x) = Q(x) + \int_0^x \Phi(x-y) df(y) \quad x \geq 0$$

where $\Phi(x)$ is continuous and $\Phi(0) = 0$, $Q(x)$ is a distribution on $(0, \infty)$ with finite first moment and f is a non-lattice distribution continuous at zero with finite second moments, then

$$\lim_{x \rightarrow \infty} \left[\Phi(x) - \frac{1}{m} x + \frac{\mu}{m} - \frac{a}{2m^2} \right] = 0$$

where

$$m = \int_0^\infty x df(x), \quad a = \int_0^\infty x^2 df(x) \quad \text{and} \quad \mu = \int_0^\infty x dQ(x).$$

Proof. Define

$$\Psi(x) = \Phi(x) - \frac{1}{m}x,$$

then it follows from (21) that

$$(22) \quad \Psi(x) - \int_0^x \Psi(x-t)df = Q(x) - \frac{1}{m} \int_0^x Tf(\xi)d\xi.$$

Integrating (22) over the interval $(0, y)$ and then performing an integration by parts we obtain

$$(23) \quad \int_0^y Tf(y-t)\Phi(t)dt = - \int_0^y [1-Q(\xi)]d + \frac{1}{m} \int_0^y T^2f(\xi)d\xi.$$

By an elementary calculation as $y \rightarrow \infty$ the limit of the right side tends to

$$\left(-\frac{\mu}{m} + \frac{\sigma^2}{2m^2} \right) \int_0^\infty Tf(t)dt.$$

We now collect the facts needed to employ the Wiener theorem. That the Fourier transform of Tf never vanishes has been shown previously. It is easy to show that $\Psi(t) = O(1)$ see [12]. Finally, we verify that $\Psi(t)$ is slowly decreasing (s.d.) that is,

$$\lim_{\substack{\xi \rightarrow \infty \\ \eta \rightarrow 0}} [\Psi(\xi + \eta) - \Psi(\xi)] \geq 0.$$

In fact,

$$\Psi(\xi + \eta) - \Psi(\xi) = \Phi(\xi + \eta) - \Phi(\xi) + \frac{\eta}{m} \geq \frac{\eta}{m}.$$

Thus,

$$\lim_{\substack{\xi \rightarrow \infty \\ \eta \rightarrow 0}} [\Psi(\xi + \eta) - \Psi(\xi)] \geq 0.$$

As Tf is nonnegative and $\Psi(\xi) \geq -C$ a sharp form of the Wiener theorem because of the (s.d.) character of Ψ implies that

$$\lim_{t \rightarrow \infty} \Psi(t) = -\frac{\mu}{m} + \frac{a}{2m^2}.$$

We continue with a brief examination of the example discussed in the introduction. Let X_i denote independent identically distributed non-lattice random variables with cumulative distribution f which has an absolutely continuous component. We assume the first moment exists and

$$\int x df = m > 0.$$

Put

$$s_j = \sum_{i=1}^j X_i \quad \text{and} \quad u(\xi) = \sum_{j=1}^n \Pr\{\xi \leq s_j \leq \xi + h\}$$

where h is a fixed positive number. The intuitive fact that $u(\xi)$ is bounded can be proved directly from probability considerations. We do not present the details. The function u is readily seen to satisfy the renewal equation (1).

$$u(x) - \int_{-\infty}^{\infty} u(x - \xi) df(\xi) = g(x) = \int_x^{x+h} df(\xi).$$

The hypothesis of the corollary to Theorem 5 can be shown to be satisfied by probability analysis and we obtain $\lim_{t \rightarrow \infty} u(t) = h/m$ and $\lim_{t \rightarrow -\infty} u(t) = 0$, the result obtained by Chung and Pollard by other methods [3]. We close this section by presenting some extensions of these results by imposing further conditions of the existence of higher moments of f to secure some results about the rate at which $u(t)$ converges.

THEOREM 10. *If X_i are independent identically distributed non-lattice random variables with density function $a(t)dt$ such that $a \in L_2$ and*

$$\int |t|^{n+2} a(t) dt < \infty$$

and

$$\int_{-\infty}^{\infty} ta(t) dt = m > 0, \quad u(\xi) = \sum_{j=1}^{\infty} \Pr\{\xi \leq s_j \leq \xi + h\}$$

where

$$s_j = \sum_{i=1}^j X_i,$$

then

$$\lim_{t \rightarrow \infty} t^n \left| u(t) - \frac{h}{m} \right| = \lim_{t \rightarrow -\infty} t^n u(t) = 0.$$

Proof. This is an immediate consequence of Theorem 8.

4. Abstract renewal equation. The purpose of the subsequent analysis is to present an abstract approach to some of the fundamental ideas involved in the analysis of the renewal equation. Although some of the results are formal and simple, it is felt that this study sheds some light on the real nature of the renewal equation.

Let T denote a linear operator which can be viewed as a bounded operator from (m) into (m) or from (l) into (l) . The spaces (m) and (l) designate the Banach spaces of bounded sequences and absolutely convergent series respectively. Suppose furthermore that the operator T is of norm one viewed in either space. Let $a_r \geq 0$, $\sum a_r = 1$ ($r = 0, \pm 1, \dots$) and we assume that the g.c.d. of the indices r for which $a_r > 0$ is 1. Suppose also that $\sum |n| a_n$ exists with $\sum n a_n = m \neq 0$. If $a_r = 0$ for $r < 0$, then automatically $\sum n a_n$ is not zero provided $a_1 \neq 1$. In this case we consider the operator $\sum_{r=0}^{\infty} a_r T^r$ where $T^0 = I$. This operator is linear and has norm bounded by 1 as $\|T^r\| \leq 1$ and $\sum a_r = 1$. If T^{-1} exists and is of norm 1, then we can deal with the general case where a_r is given not necessarily zero for both r positive and negative. We consider then the operator $\sum_{n=-\infty}^{\infty} a_n T^n$. As a generalization of the renewal equation, we set

$$(+)\quad Su = \left[I - \sum_{r=-\infty}^{\infty} a_r T^r \right] u = v.$$

It is given that the operator S applied to u produces the element v . In many examples, u is a bounded sequence, that is, an element of (m) while v is an element in (l) . Put,

$$r_n = \sum_{i=n+1}^{\infty} a_i \text{ for } n \geq 0 \text{ and } r_n = - \sum_{i=-\infty}^n a_i \text{ for } n < 0,$$

then $\sum |r_n| < \infty$ as $\sum |na_n| < \infty$. It is important to note on account of $\sum |r_n| < \infty$, the series

$$\sum_{n=-\infty}^{\infty} r_n T^n$$

defines a bounded linear operator which can be viewed acting either on (m) or (l) into itself. By a summation by parts, we obtain that

$$\left(\sum_{n=-\infty}^{\infty} r_n T^n \right) (I - T)u = v.$$

Since $\sum_{n=-\infty}^{\infty} r_n s^n$ with $|s| = 1$ has an absolute convergent reciprocal (Wiener's theorem is used here analogously to the analysis of section 1), we secure that $(\sum r_n T^n)^{-1}$ exists as a bounded operator over (m) and (l) and that

$$(I - T)u = (\sum r_n T^n)^{-1} v.$$

Since $v \in (l)$ we conclude that $(I - T)u \in (l)$ although u itself might only be an element of (m) . This represents the basic abstract conclusion obtained from (+). Further results are obtained by specializing T . A particular example is obtained by $(m) =$ the set of all bounded sequences $u = \{u_n\} n = 0, \pm 1, \pm 2, \dots$, where T is the shift operator which moves each component one unit to the right. Whence, (+) reduces to

$$(I - \sum a_n T^n)u = \left\{ u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k \right\}.$$

If all the hypothesis on a_n are met and $v_n \in (l)$, then the abstract theorem tells us that $(I - T)u \in (l)$ or

$$\sum_{n=-\infty}^{\infty} |u_n - u_{n+1}| < \infty.$$

This implies that both $\lim_{n \rightarrow \infty} u_n$ and $\lim_{n \rightarrow -\infty} u_n$ exist. Similar results are valid for the circumstance where T^{-1} does not exist. Then we deal only with the case where $a_n = 0$ for $n < 0$. Considering the same shift operator leads to

$$\left(I - \sum_{n=0}^{\infty} a_n T^n \right) u = u_n - \sum_{k=0}^n a_{n-k} u_k$$

and we deduce as above that $\lim_{n \rightarrow \infty} u_n$ exists.

We turn now to examine some continuous analogues of (+). Let $T(t)$ denote for $\infty \geq t \geq 0$ a strongly continuous semi-group of operators acting either on the space of bounded functions (M) or integrable functions (L) with $\|T(t)\| \leq 1$. Let A denote the infinitesimal generator of $T(t)$ and let $df(t)$ define a non-lattice distribution with finite first moment on $[0, \infty]$. If u belongs to (M) we consider

$$u(t) - \left[\int_0^\infty T(t) df(t) \right] u = v$$

where v belongs to (L). The linear operator

$$\int_0^\infty T(t) df(t)$$

is well defined either over (M) or (L) into itself. Put $r(t) = 1 - f(t)$, then $r \in L$ and the Fourier transform of r never vanishes. Since r is monotonic decreasing and in L it can be easily shown that

$$\left[\int_0^\infty r(t) T(t) dt \right] u$$

belongs to the domain of the infinitesimal generator A and

$$A \int_0^\infty r(t) T(t) dt u = v.$$

Formally, we also obtain upon commuting A and the integral operator

$$\left[\int_0^\infty r(t) T(t) dt \right] Au = v.$$

We note that if

$$\int_0^\infty r(t) T(t) dt$$

is multiplied by any other operator of the form

$$\int_0^\infty s(t) T(t) dt,$$

we obtain the operator

$$\int_0^{\infty} p(t)T(t)dt$$

where

$$p(t) = \int_0^t r(t-\xi)s(\xi)d\xi.$$

Since the Fourier transform of r does not vanish, then Wiener's theorem in a formal sense, furnishes an inverse to

$$\int_0^{\infty} r(t)T(t)dt$$

which takes $v \in L$ into L . Thus, Au belongs to L . Specializing $T(t)$ to the translation semi-group $T(t)u(x) = u(x-t)$, then $Au = du(x)/dx$ whenever the derivative exists and belongs to the proper space. The fact that $Au \in L$ yields $\int |du/dx|$ exists from which we infer that $\lim_{t \rightarrow \infty} u(t)$ exists. Thus we obtain the limit behavior of Theorem 5 for the one-sided case. The justification of these last formal considerations is very difficult and can only be carried through in certain cases as is shown in § 2. The full renewal equation is generalized by taking $T(t)$ a group and proceeding as above.

REFERENCES

1. D. Blackwell, *A renewal theorem*, Duke Math. J., **15** (1948), 145-150.
2. ———, *Extension of a renewal theorem*, Pacific J. Math. **3** (1953), 315-332.
3. K. L. Chung and H. Pollard, *An extension of renewal theory*, Proc. Amer. Math. Soc., **3** (1952), 303-309.
4. K. L. Chung and J. Wolfowitz, *On a limit theorem in renewal theory*, Ann. of Math. **55-56** (1952), 1-6.
5. K. L. Chung and W. H. J. Fuchs, *On the distribution of values of sums of random variables*, Memoirs of the Amer. Math. Soc., No. 6.
6. J. L. Doob, *Renewal theory from the point of view of the theory of probability*, Trans. Amer. Math. Soc., **63** (1948), 422-438.
7. P. Erdős, H. Pollard and W. Feller, *A property of power series with positive coefficients*, Bull. Amer. Math. Soc., **55** (1949), 201-204.
8. W. Feller, *On the integral equation of renewal theory*, Ann. Math. Stat., **12** (1941), 243-267.
9. ———, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc., **67** (1949), 98-119.

10. W. Feller, *An introduction to the theory of probability and its applications*, J. Wiley and Sons, 1950.
11. W. L. Smith, *Asymptotic renewal theorems*, Proc. of the Royal Soc. of Edinburgh, LXIV (1953-54), 9-48.
12. S. Täcklind, *Fourieranalytische Behandlung vom Erneuerungsproblem*, Skandinavisk Aktuarietidskrift, **28** (1945), 68-105.

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