AN EXTENSION OF A THEOREM OF JORDAN AND VON NEUMANN

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1. Introduction. Let $\{E\}$ denote the class of generalized euclidean spaces E (that is, $E \subseteq \{E\}$ provided all finite dimensional subspaces of E are euclidean spaces). The problem of characterizing metrically the class $\{E\}$ with respect to the class $\{B\}$ of all Banach spaces has been solved in many different ways. [1] Fréchet's characteristic conditions [5]

$$(*) \qquad \frac{1}{2} \sum_{i,j=1}^{3} [\|p_i\|^2 + \|p_j\|^2 - \|p_i - p_j\|^2] x_i x_j \geq 0, \qquad (p_1, p_2, p_3 \in \mathbf{B})$$

was immediately weakened by Jordan and von Neumann [6] to

$$(**) \qquad \|p_1 + p_2\|^2 + \|p_1 - p_2\|^2 = 2(\|p_1\|^2 + \|p_2\|^2) \qquad (p_1, p_2 \in \mathbf{B}).$$

This relation has now become a kind of standard to which others repair by showing that it is implied by newly postulated conditions [3, 4, 10], and it has been, apparently, the motivation of work in which it does not enter directly [7, 9]. Perhaps the best possible result in this direction, however, is due to Aronszajn [1] who assumed merely that

$$\|(x + y)/2\| = \frac{1}{2} \phi(\|x\|, \|y\|, \|x - y\|) \qquad (x, y \in \mathbf{B}),$$

with ϕ *unrestricted* except for being nonnegative and $\phi(r, 0, r) = r$, $r \geq 0$.

These conditions, and others like them, are all equivalent in a Banach space, for each is necessary and sufficient to insure the euclidean character of all subspaces. In a more general environment, however, this is not the case, and so the desirability of making a comparative study of such conditions in more general spaces is suggested. In this note the larger environment is furnished by the class $\{M\}$ of complete, metrically convex and externally convex,

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[1] This note deals exclusively with normed linear spaces over the field of reals.

metric spaces, of which the class of Banach spaces is a very special sub-class.[2] After giving purely metric interpretations of those conditions that we shall discuss (in order that they might be meaningful in spaces of class $\{M\}$) we are chiefly concerned with showing that the Jordan-von Neumann relation (**) characterizes class $\{E\}$ among the class $\{M\}$.[3] This is true, *a fortiori*, for Fréchet's condition (*) also, but an easy example shows that the inequality used in Schoenberg [10] is *not* so extensible.

2. Metrization of the Jordan-von Neumann relation and comparison with other four-point conditions. Writing (**) in the form

$$\|(p_1 + p_2)/2\| = \frac{1}{2}[2\|p_1\|^2 + 2\|p_2\|^2 - \|p_1 - p_2\|^2]^{\frac{1}{2}}$$

we see that the length $\|(p_1 + p_2)/2\|$ of the median of the triangle with ver-tices θ, p_1, p_2 (θ denotes the null element of $\mathbf{B}$) is the same function of the lengths $\|p_1\|$, $\|p_2\|$, $\|p_1 - p_2\|$ of the sides of the triangle that it is in euclidean space. Since any three elements x, y, z of $\mathbf{B}$ are superposable with θ, $p_1 = y - x$, $p_2 = z - x$ (the middle-element $(y + z)/2$ of y, z being carried into $(p_1 + p_2)/2$, the middle-element of p_1, p_2) we have the following metric interpretation of (**): (†) *every four elements p, q, r, s of $\mathbf{B}$ with q a middle-element of p, r (that is, $pq = qr = pr/2$) are congruently imbeddable in the euclidean plane E_2.*

In this formulation, the Jordan-von Neumann criterion is meaningful in every metric space and may, therefore, be compared with other so-called *four-point conditions* that antedated it.

A metric space has the *euclidean k-point property* provided each k-tuple of its elements is congruently contained in a euclidean space (and hence in an E_{k-1}). Observing that every metric space has the euclidean three-point property, W. A. Wilson [11] investigated in 1932 the consequences of assuming that a space has the euclidean *four-point* property. It follows from a result due to the writer [2, p. 131] that if M is any metric space whatever, and $M^{\frac{1}{2}}$ denotes the space obtained by taking the positive square root of the metric of M, then $M^{\frac{1}{2}}$ has the euclidean four-point property. Thus the special class $\{M^{\frac{1}{2}}\}$ of spaces with the euclidean four-point property has the same cardinality as the class of

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[2] For definitions of these and other metric concepts used in this paper see [2].

[3] The abstract of [8] given in *Math. Rev.* vol. 13 (1952) p. 850 indicates a con-nection between that paper (which the writer has not seen) and the present note.

all metric spaces (of which it is a proper subclass) and consequently the same is true of the class of all spaces with the euclidean four-point property.

But none of the spaces $\{M^{\frac{1}{2}}\}$ is metrically convex, and Wilson proved that if a complete, metrically convex space has the euclidean four-point property, it is congruent with a subset of a generalized euclidean space. If also external convexity is assumed, then congruence with a generalized euclidean space results.

The *weak euclidean four-point property*, introduced by the writer in 1933, assumes the congruent imbedding in euclidean space (and hence in E_2) of only those quadruples that contain a linear triple (that is, a triple which is imbeddable in E_1), and it was shown that the weak euclidean four-point property suffices to obtain all of the results that Wilson had proved by use of the stronger assumption [2, pp. 123-128]. But the Jordan-von Neumann condition, as metrized in (†), restricts the class of quadruples assumed to be imbeddable in euclidean space even more than does the weak euclidean four-point property, and consequently is a weaker assumption. We shall refer to it as the *feeble euclidean four-point property*.

3. Equivalence in $\{M\}$ of the feeble and the weak euclidean four-point properties. We prove in this section that in complete, metrically convex and externally convex metric spaces, the feeble euclidean four-point property implies (and hence is equivalent to) the weak euclidean four-point property. Some elementary consequences of the feeble property in such a space are first set down.

I. *Middle-elements are unique*; for if p, $r \in M(p \neq r)$ and q_1, q_2 are middle-elements of p, r then

$$p, q_1, q_2, r \approx p', q_1', q_2', r'$$

where the "primed" points are in E_2 and "$\approx$" denotes the congruence relation. But then q_1' and q_2' are middle-points of p', r' and consequently

$$q_1' = q_2', \; q_1 q_2 = q_1' q_2' = 0, \; q_1 = q_2 .$$

II. *Each two distinct elements are joined by exactly one metric segment.* Since M is complete, metrically convex and metric, each two of its distinct points are joined by at least one metric segment. If p, $r \in M(p \neq r)$ and $S_{p,r}$, $S_{p,r}^*$ are two segments with end-elements p, r, suppose q^* belongs to the second

segment and not to the first. Then $p \neq q^* \neq r$, and traversing $S^*_{p,r}$ from q^* to p a first point p^* of $S_{p,r}$ is encountered. Similarly, traversing $S^*_{p,r}$ from q^* to r a first point r^* of $S_{p,r}$ is obtained. The sub-segments S_{p^*,r^*}, $S^*_{p^*,r^*}$ have only their end-elements in common, but each obviously contains a middle-element of p^*, r^*, contrary to I.

III. *Segments admit unique prolongations.* Since M is externally convex, each segment may be prolonged beyond its end-elements. But if $S_{p,q}$ admits two prolongations beyond q, then clearly elements r, r^* of different prolongations exist $(r \neq r^*)$ such that q is a middle-element of p^*, r as well as a middle-element of p^*, r^* for some element p^* of $S_{p,q}$. The congruent imbedding in E_2 of p^*, q, r, r^* shows this to be impossible.

IV. *Each two distinct elements of M are on exactly one metric line.* Since M is metric, complete, metrically convex and externally convex, each two of its distinct points p, q are on at least one metric line $L(p,q)$ [2, p. 56]. It follows at once from II and III that $L(p,q)$ is unique.

THEOREM 3.1. *If p is a point and L a metric line of M, then $L + (p)$ is congruently imbeddable in E_2.*

Proof. If $p \in L$ then

$$L + (p) = L \approx E_1 \subset E_2 ,$$

by the definition of a metric line. Suppose $p \notin L$, select points r_0, r_1 on L with $r_0 r_1 = 1$, and let p', r_0', r_1' be points of E_2 such that $p, r_0, r_1 \approx p', r_0', r_1'$. Let L' denote the straight line of E_2 determined by r_0', r_1', and consider the one-to-one correspondence

$$\Gamma: p \leftrightarrow p', \ L(r_0, r_1) \approx L'(r_0', r_1') ,$$

where the congruence of the two lines is the unique extension of the congruence $r_0, r_1 \approx r_0', r_1'$. We shall show that Γ is a congruence.

If $r_{1/2}$ denotes the unique middle-point of r_0, r_1, and $r_{1/2}' = \Gamma(r_{1/2})$, then $r_{1/2}'$ is the middle-point of r_0', r_1'. By the feeble euclidean four-point property

$$p, r_0, r_{1/2}, r_1 \approx \overline{p}, \overline{r_0}, \overline{r_{1/2}}, \overline{r_1} .$$

with the "barred" points in E_2, and since $p', r_0', r_1' \approx p, r_0, r_1$, a motion of E_2 exists that carries p, r_0, r_1 into p', r_0', r_1', respectively. This motion evidently

sends $\overline{r}_{1/2}$ into $r'_{1/2}$, and we have

$$p, r_0, r_{1/2}, r_1 \approx p', r'_0, r'_{1/2}, r'_1 \; ;$$

that is, $pr_{1/2} = p'r'_{1/2}$.

If $r_{3/4}$ denotes the middle-point of $r_{1/2}$, r_1, and $r'_{3/4} = \Gamma(r_{3/4})$, the feeble four-point property, applied to the quadruple p, $r_{1/2}$, $r_{3/4}$, r_1 gives $pr_{3/4} = p'r'_{3/4}$. Continuing in this manner, we obtain $pr_{i/2^n} = p'r'_{i/2^n}$ for each dyadically rational fraction $i/2^n$. Since the points $r_{i/2^n}$ are dense in $\mathrm{seg}[r_0, r_1]$, continuity of the metric (and continuity of the congruence $L(r_0, r_1) \approx L'(r'_0, r'_1)$) yields $px = p'x'$, $x' = \Gamma(x)$, for every $x \in \mathrm{seg}[r_0, r_1]$.

Let r_2 be a point of L such that r_1 is the middle-point of r_0, r_2. The feeble four-point property gives (in the manner employed above) $p, r_0, r_1, r_2 \approx p', r'_0, r'_1, r'_2$, where $r'_2 = \Gamma(r_2)$, and consequently $pr_2 = p'r'_2$. Then from $p, r_1, r_2 \approx p', r'_1, r'_2$ we obtain $px = p'x'$, $x \in \mathrm{seg}[r_1, r_2]$ in the same manner as described above for $\mathrm{seg}[r_0, r_1]$. It is clear that a continuation of the procedure establishes $px = p'x'$ for every x of L and $x' = \Gamma(x)$.

THEOREM 3.2. *In a complete, metrically convex and externally convex metric space* M, *the feeble and the weak euclidean four-point properties are equivalent.*

Proof. The weak property obviously implies the feeble one in any metric space. Suppose M has the feeble property, and p, q, r, $s \in M$ (pairwise distinct) with q, r, s congruent with a triple of E_1. Then the line $L(q, r)$ contains s, and $L(q, r) + (p)$ is congruently imbeddable in E_2. Hence p, q, r, s are imbeddable in E_2.

4. Extension of the Jordan-von Neumann theorem. The writer has shown [2, p. 127] that a complete, metrically convex and externally convex semimetric space with the weak euclidean four-point property has the euclidean k-point property for every positive integer k. It follows easily that such a space is generalized euclidean. Use of Theorem 3.2 now yields the following result:

THEOREM 4.1. *A complete, metrically convex and externally convex metric space with the feeble euclidean four-point property is generalized euclidean.*

This is the desired extension of the Jordan-von Neumann theorem for real normed linear spaces. For if L is such a space, and L satisfies the Jordan-von Neumann condition (**), then the Banach space that arises by completing

L in the Hausdorff manner is a complete, metrically convex and externally con-
vex metric space with the feeble euclidean four-point property. According to
Theorem 4.1, it is generalized euclidean and so an inner product is definable
in it. Hence an inner product is definable in L, and the Jordan-von Neumann
theorem for real normed linear spaces is obtained. Thus the metric essence
of (**) determines the euclidean character of L *by use of the purely metric
features of the space,* without regard, for example, for its very special proper-
ties due to linearity.

5. Concluding remarks. Condition (*) of Fréchet is equivalent to Wilson's
euclidean four-point condition [2, p. 106] and consequently his theorem of 1935
had already been proved in more general form by Wilson in 1932.

A semimetric space is *ptolemaic* provided for any four of its elements p,
q, r, s, the three products $pq \cdot rs$, $ps \cdot qr$, $pr \cdot qs$ of "opposite" distances satis-
fy the triangle inequality. Schoenberg [10] showed that in a real linear semi-
normed ptolemaic space, the semi-norm satisfies the triangle inequality (and
so is actually a norm) and an inner product is definable which is related to
the norm in the usual way.

Schoenberg's ptolemaic condition which (as a norm postulate in L has the
form

$$\|f\| \cdot \|g - h\| + \|g\| \cdot \|h - f\| \geq \|h\| \cdot \|f - g\| \qquad (f, g, h \in \mathbf{L})$$

is not extensible to the class $\{\mathbf{M}\}$. For if three pairwise distinct rays of E_2,
with a common initial point, be metrized convexly (that is, if p, q are points
of different rays, then $pq = e(p, o) + e(o, q)$, where $e(,)$ denotes euclidean
distance and o is the common point of the rays, while $pq = e(p, q)$ if p, q belong
to the same ray) the resulting space is easily shown to be metric, complete,
convex and externally convex, and ptolemaic. But it is not, of course, general-
ized euclidean. It would be interesting to know whether or not this "tripod"
is present in every such example.

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