CONSTRUCTIVE PROOF OF THE MIN-MAX THEOREM

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1. Introduction. The foundations of a mathematical theory of "games of strategy" were laid by John von Neumann between 1928 and 1941.[1] The publication in 1944 of the book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern climaxed this pioneering effort. The first part of this volume is concerned with games with a finite number of pure strategies with particular emphasis on the "zero-sum two-person" type of game. There it is shown that in most instances a player is at a disadvantage if he always plays the same pure strategy and that it is better to "mix" his pure strategies by some chance device. The starting point of all discussions of this type of game is the celebrated "Main Theorem" or Min-Max Theorem which is concerned with existence and properties of optimal mixed strategies for both players.

The first proofs of this theorem, given by von Neumann, made rather involved use of topology, functional calculus, and fixed point theorems of L. E. J. Brouwer. The first proof of an elementary character was given by J. Ville, 1938. The von Neumann-Morgenstern book, with the purpose of having a proof which is accessible to a less highly trained group, carries the theme of elementarization further [6]. At this late date there still continues to be a need for a truly elementary proof; for example, the recent book of McKinsey on game theory [5] omitted a self-contained proof because none was available.

Kuhn [4] gives a bibliography of some of the better known proofs of the Min-Max Theorem, together with a discussion of their general characteristics which he broadly classifies into (1) those based on separation properties of convex sets and (2) those using some notion of a fixed point of a transformation. Kuhn [4] and McKinsey [5] provide proofs along the lines of von Neumann [6] based on a separation theorem. Dresher [3] gives a self-contained proof along the lines of Ville. As was pointed out in [7], the Min-Max Theorem is completely algebraic and should be given an algebraic proof. The purely algebraic proofs, when made self-contained and elementary, appear to be quite long, [3], [4], [7], and, with the exception of Weyl's proof [7], make use of nonalgebraic concepts as the minimum of a continuous function on a closed bounded set is assumed on the set. All these proofs are either pure existence proofs

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[1] For the contributions of Borel to this field see *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 95-127.

or, from the viewpoint of practical computations, nonconstructive.

The present proof has the following features: It is purely algebraic (in the spirit of Weyl) and elementary in the sense that it used nothing more advanced than the notion of an inverse of a matrix. It is short, self-contained, and noninductive. The very nature of the solution, if desired, could be used to advantage to establish well-known theorems regarding the structure of the class of optimal strategies. It is a special adaptation for games of the simplex method used for solving linear programming problems [1].[2] As such, it provides perhaps the most efficient means currently available for explicitly constructing optimal mixed strategies for both players.

2. The Min-Max Theorem. It has been found convenient in a part of the proof to compare certain vectors "lexicographically." The term is borrowed from an alphabetical ordering of words (as in a dictionary). Thus a vector A is greater than B (written $A > B$) if $(A-B) > 0$; where by $(A-B) > 0$ is meant that $(A-B)$ has nonzero components, the first of which is positive.

Let $[a_{ij}]$ be the payoff matrix of a finite zero-sum two-person game where a_{ij} is the payoff to player I (the maximizing player) when player I plays pure strategy i and player II (the minimizing player) plays pure strategy j. Player I (in order to guard against his strategy being "found out") chooses a mixed strategy $(x_1, x_2, \cdots, x_m)$ where x_i is the probability of playing strategy i; accordingly, player I's expected payoff becomes $(\sum_i a_{ij}x_i)$ if the minimizing player plays pure strategy j. If player I's mixed strategy is found out he can expect that player II will choose j such that $\sum_i a_{ij}x_i$ is minimum. Thus, player I wishes to choose his x_i such that the smallest such sum (which we will denote by x_0) is a maximum. For similar reasons player II chooses a mixed strategy y_1, $y_2, \cdots, y_n$ such that the largest sum $\sum_j a_{ij}y_j$ (denoted by y_0) is minimum. The Min-Max Theorem states that there exists a choice for player I of $x_i = \hat{x}_i$ and a choice for player II of $y_j = \hat{y}_j$ such that the corresponding $x_0 = \hat{x}_0$ is the maximum value for x_0 and the corresponding $y_0 = \hat{y}_0$ is the minimum value for y_0 and, moreover, $\hat{x}_0 = \hat{y}_0$. The common value of $\hat{x}_0$ and $\hat{y}_0$ known as the "value" of the game.

To establish this result we shall consider, as is often done, a related linear inequality problem. Let x_i and y_j satisfy the system of relations

[2] That the simplex method itself could be used to prove the Min-Max Theorem was first pointed out by Dorfman (and H. Rubin) [**2**]. This paper, by incorporating methods for avoiding "degeneracy" and "cycling" in the simplex algorithm [**1**], puts the proof on a completely rigorous foundation.

(1) $x_i \geqq 0, \quad (i=1, \cdots, m);$ (4) $y_j \geqq 0, \quad (j=1, \cdots, n);$

(2) $\sum_{i=1}^{m} x_i = 1 \quad ;$ (5) $\sum_{j=1}^{n} y_j = 1 \quad ;$

(3) $x_0 \leqq \sum_{i=1}^{m} x_i a_{ij}, \quad (j=1, \cdots, n);$ (6) $\sum_j a_{ij} y_j \leqq y_0, \quad (i=1, \cdots, m).$

If we multiply (3) through by any y_j satisfying (4), (5), and (6) and sum with respect to j; similarly multiply through (6) by any x_i satisfying (1), (2), (3) and sum with respect to i, one obtains immediately

$$x_0 = x_0 \sum_j y_j \leqq \sum_i \sum_j x_i a_{ij} y_j \leqq y_0 \sum_i x_i = y_0 \tag{7}$$

so that the lower bounds x_0 never exceed the upper bounds y_0.

We shall, however, construct a solution $x_i = \hat{x}_i$ and $y_j = \hat{y}_j$ with the property that

$$\hat{x}_0 = \hat{y}_0 . \tag{8}$$

In particular (7) holds for $\hat{y}_j$ and any x_0 and also for $\hat{x}_0$ and any y_0. It follows, therefore, that $x_0 \leqq \hat{y}_0 = \hat{x}_0 \leqq y_0$ and

$$\hat{x}_0 = \max x_0 \quad \text{and} \quad \hat{y}_0 = \min y_0 \tag{9}$$

and the Min-Max Theorem would be demonstrated.

3. Proof of the Min-Max Theorem. We shall begin the proof by augmenting the matrix of the game a_{ij} and consider the matrix

$$\begin{pmatrix} 0 & 1 \cdots 1 & 0 \cdots 0 \\ -1 & a_{11} \cdots a_{1n} & 1 \\ -1 & \cdots & \ddots \\ -1 & a_{m1} \cdots a_{mn} & 0 \cdots 1 \end{pmatrix} . \tag{10}$$

The columns of this matrix will be denoted $P_0; P_1, \cdots, P_n; P_{n+1} = U_1, \cdots, P_{n+m} = U_m$ where U_i are unit vectors with 1 as the $(i+1)$st component. It will be convenient to arrange the rows of the matrix such that

$$a_{m1} = \max_i a_{i1} . \tag{11}$$

Let B (which we will call a *basis*) be a subset of $m+1$ columns of (10) (including P_0 as first column) which, considered as an $m+1$ square matrix, is *nonsingular* and let the *rows* of B^{-1} be denoted by β_i $(i=0, 1, \cdots, m)$. We shall further require that B, to be a basis, have the property that each row (except $i=0$) of B^{-1} have its first nonzero component positive. Thus we are assuming in the lexicographic sense that

$$\beta_i > 0 \qquad (i=1, 2, \cdots, m). \tag{12}$$

For example, we may choose $B=B_0$ as consisting of the first two columns of (10) and the unit vectors $U_1, \cdots, U_{m-1}$. This near identity matrix

$$B_0=[P_0, P_1, U_1, \cdots, U_{m-1}]=[P_0, P_1, P_{n+1}, \cdots, P_{n+m-1}]$$

is obviously non-singular and possesses a simple inverse

$$B_0^{-1}=\begin{pmatrix} a_{m1} & 0 & 0 & & 0 & -1 \\ 1 & 0 & 0 & \cdot & 0 & 0 \\ b_1 & 1 & 0 & \cdot & \cdot & -1 \\ b_2 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m-1} & 0 & 0 & & 1 & -1 \end{pmatrix} \tag{13}$$

where $b_i=a_{m1}-a_{i1}$. Because of (11) it follows that $b_i \geqq 0$ and our special lexicographic assumption (12) holds.

Let the columns of a general basis be denoted by

$$B=[P_0, P_{j_1}, \cdots, P_{j_m}] \tag{14}$$

and note that the conditions $\beta_k P_{j_i}=0$ for $i \neq k$ and $\beta_i P_{j_i}=1$ for $i, k=0, 1, \cdots, m$ $(j_0=0)$ must hold between B and its inverse. The 0-row of B^{-1} is used to compute the scalar quantities $\beta_0 P_j$ for $j=1, 2, \cdots, n, \cdots, n+m$. We shall now prove the following.

THEOREM. *If for all $j=1, 2, \cdots, n+m$ we have*

$$\beta_0 P_j \leqq 0 , \tag{15}$$

then the components of the 0-row and 0-column of B^{-1} yield the required optimal strategies.

Proof. Denote the components of the 0-row of B^{-1} by

$$[\hat{x}_0, -\hat{x}_1, \cdots, -\hat{x}_m] ; \tag{16}$$

the components of the 0-column of B^{-1} by

$$\{\hat{y}_0, \hat{y}_{j_1}, \cdots, \hat{y}_{j_m}\} . \tag{17}$$

We shall now show that *an optimum mixed strategy for player* I *is obtained by setting $x_i=\hat{x}_i$ for $i=1, 2, \cdots, m$; and one for player* II, *by setting $y_{j_i}=\hat{y}_{j_i}$ for $j_i \leqq n$ and $y_j=\hat{y}_j=0$ for all other $j \leqq n$.* Moreover, the value of the game is $\hat{x}_0=\hat{y}_0$. Indeed, for player I, it is easy to verify the condition $\beta_0 P_0=1$ is the same as (2); moreover, $\beta_0 P_j \leqq 0$ for $1 \leqq j \leqq n$ are the same as (3), while for $n+1 \leqq j \leqq n+m$ they are the same as (1). For player II, the lexicographic property of the rows of B^{-1}, namely

$\beta_i > 0$ for $i=1, \cdots, m$ implies that the first component of β_i (which by definition is $\hat{y}_{j_i}$) is nonnegative; thus, (4) is satisfied. Multiplying B on the right by 0-column of B^{-1} yields $(m+1)$ linear expressions in $(\hat{y}_0, \hat{y}_{j_1}, \cdots, \hat{y}_{j_m})$ which may be equated to unit vector U_0.

The first of these $(m+1)$ linear equations yields (5) since the 1st components of P_{j_i} are unity for $1 \leqq j_i \leqq n$ and zero otherwise. The remaining m equations yield the inequalities (6) if the terms involving $j_i > n$ are dropped (the latter are nonnegative because $\hat{y}_{j_i} \geqq 0$ and their coefficients are the components of the unit vectors P_{n+i}). Finally, the proof is completed by noting that (8) or $\hat{x}_0 = \hat{y}_0$ holds since both are defined in (16) and (17) as the $(0, 0)$ element of B^{-1}.

Constructing an Optimal Basis. It is clear now that the central problem is one of constructing a basis B with the property that $\beta_0 P_j \leqq 0$ for $j=1, 2, \cdots, n+m$ since this in turn yields an optimal mixed strategy for each player. We shall show that if some basis B, such as B_0, does not have the requisite property (15), then it is easy to construct from B a new basis B^* which differs from B by only one column where 0-row of $[B^*]^{-1}$ (which we denote by β_0^*) has the property that

$$\beta_0 > \beta_0^* \tag{18}$$

that is, the first nonzero component of $(\beta_0 - \beta_0^*)$ is positive. If the new basis B^* does not satisfy (15) then the algorithm just outlined for B is iterated, with B replaced by B^*, etc. This process generates a sequence of bases which terminates when a basis is obtained that has the required property. This must occur in a finite number of steps since the condition (18) is a strict inequality which insures that no basis can be repeated and the number of different bases cannot exceed the number of ways of choosing m columns out of $n+m$ from (10).[3] The 0-column of successive bases of the iterative process may be interpreted as a succession of improved mixed strategies for player II for which his expected loss, y_0, if his opponent is playing optimally, is decreasing to a minimum. Indeed, the components of the first column of *any basis* (as in (17) and sequel) satisfy (4) and (5) independently of condition (15), while y_0, the first component of β_0, is nonincreasing from basis to basis by virtue of (18).

To construct B^* from B let P_s denote the column of (10) which replaces the rth column of B where P_s and P_{j_r} are determined by the following rules: Choose P_s such that

[3] In practical computations with the simplex method, of which this is a variation, the number of iterations is usually very small. In a game case where, say, $m/2$ of pure strategies are used with positive probability in an optimal mixed strategy, something in the order of $m/2$ iterations might be expected before an optimal basis is obtained.

(19) $$\beta_0 P_s = \max_j \beta_0 P_j > 0 \,, \qquad (j=1, \cdots, n+m).$$

In case the choice of s is not unique, then choose s with the smallest index satisfying (23). Next, compute the column vector V satisfying $BV=P_s$. It is clear that components of $V=\{v_0, v_1, \cdots, v_m\}$ are given by

(20) $$v_i = \beta_i P_s \qquad (i=0, \cdots, m)$$

where, in particular $v_0=\beta_0 P_s > 0$ from (19). We now choose to drop from B that column P_{j_r} such that the lexicographic minimum of the vectors $(1/v_i)\beta_i$ for $v_i > 0$ is attained for $i=r$. Thus,

(21) $$\frac{1}{v_r}\beta_r = \min \frac{1}{v_i}\beta_i \qquad (v_r > 0, \; v_i > 0)$$

where i, $r \neq 0$, and where it is assumed for the moment that there is at least one $v_i > 0$. The minimizing vector is easily obtained in practice by finding the vector whose first component is the least; if there is a tie, then one passes to the second components of the tying vectors and selects the least, etc. A relation which will be used later that follows from (21) is

(22) $$\beta_i - \frac{v_i}{v_r}\beta_r > 0 \qquad (v_i > 0).$$

It is clear from the structure of the augmented matrix (10) that the first column P_0 can not be formed as a positive linear combination of the other columns P_j. However, if we assume, contrary to the assumption of (21), that all $v_i \leqq 0$, $(i \neq 0)$ and write

$$P_s = BV = v_0 P_0 + \sum v_i P_{j_i} \,,$$

then, by transposing to the left all terms other than $v_0 P_0$, we obtain a positive linear combination of columns P_s and P_{j_i} that yields $v_0 P_0$, where $v_0 > 0$; a contradiction.

There remains only to show that B^* has the requisite properties (12) and (18). The proof, as well as the efficiency of the computational algorithm, is obtained by constructing $[B^*]^{-1}$ from B^{-1} using the relations

(23) $$\beta_i^* = \beta_i - \frac{v_i}{v_r}\beta_r \,, \qquad (i \neq r),$$

$$\beta_r^* = +\frac{1}{v_r}\beta_r$$

where β_i^* is the ith row of $[B^*]^{-1}$. To verify that (23) is indeed the inverse of B^*, one notes from (23) that for $i \neq r$ the values $\beta_k^* P_{j_i}$ are the same as $\beta_k P_{j_i}=0$ (or 1 if $i=k$); moreover, it follows readily from the definitions of v_i given in (20) that $\beta_r^* P_s=1$ and $\beta_i^* P_s=0$ for $i\neq r$.

The required properties of β_i^* are immediately evident: Thus, the first nonzero component of β_r^* is positive because β_r has this property and $v_r>0$. Next, for all other $i=1, 2, \cdots, m$ the property must hold if $v_i \leqq 0$ since β_i^* is the sum of two vectors with this property. If $v_i>0$ then $\beta_i^*>0$ by (22) and (23). Finally we note that the relation $\beta_0>\beta_0^*$ (and not $\beta_0 \geqq \beta_0^*$) holds because β_r, a row of a nonsingular matrix, possesses at least one nonzero component and β_0^* is formed by subtracting from β_0 a vector $(v_0/v_r)\beta_r$ where $v_0>0$, $v_r>0$; hence, (18) holds and the proof is complete.

4. Example. Solve the 3×6 game matrix M

$$M=\begin{pmatrix} 4 & 3 & 3 & 2 & 2 & 6 \\ 6^* & 0 & 4 & 2 & 6 & 2 \\ 0 & 7 & 3 & 6 & 2 & 2 \end{pmatrix}$$

from [8, Chap. 3, Ex. 10]. Element a_{21} of M has been starred. It will be noted that this is the maximal element in the first column. For convenience, see (11), the second and third rows have been interchanged so that this element appears in the bottom position of this column in forming the augmented matrix, $[P_0, \cdots, P_9]$, given below:

$$\begin{matrix} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{matrix}$$
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ -1 & 4 & 3 & 3 & 2 & 2 & 6 & 1 & & \\ -1 & 0 & 7 & 3 & 6 & 2 & 2 & & 1 & \\ -1 & 6 & 0 & 4 & 2 & 6 & 2 & & & 1 \end{pmatrix}.$$

Initial Iteration. The initial basis, $B=B_0$, consists of P_0, P_1, $P_7=U_1$, $P_8=U_2$. The inverse of B_0 (given below) is determined by formula (13). The entries v_i shown, for the moment, cannot be filled in until P_s is first determined.

$$B_0^{-1}=\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}=\begin{pmatrix} 6 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 \\ 6 & 0 & 1 & -1 \end{pmatrix}; \qquad \begin{matrix} v_0=6 \\ v_1=1 \\ v_2=5 \\ v_3=13. \end{matrix}$$

Next, $P_s=P_2$ is determined by

$$\beta_0 P_s=\beta_0 P_2=\max_{j\neq 0} \beta_0 P_j=6>0$$

so that the entries $v_i=\beta_i P_s$ (given above) can now be computed. The column r to be dropped from the basis is determined by forming the lexicographic minimum of the vectors (See § 2).

$$\frac{1}{v_r}\beta_r=\frac{1}{5}\beta_2=\min_{v_i>0,\ i\neq 0}(\text{lexico.})\frac{1}{v_i}\beta_i$$

Drop col $r=2$; that is, P_7.

1st Iteration. The next basis $B^*=B_1$ is $[P_0, P_1, P_2, P_8]$. To obtain its inverse set: $\beta_i^1=\beta_i-(v_k/v_r)\beta_r$, $(k\neq r)$ and $\beta_r^1=(1/v_r)\beta_r$ where $r=2$ where the superscript (in place of *) refers to the basis $B=B_k$.

$$B_1^{-1}=\begin{pmatrix}\beta_0^1\\ \beta_1^1\\ \beta_2^1\\ \beta_3^1\end{pmatrix}=\begin{pmatrix}\frac{18}{5} & -\frac{6}{5} & 0 & +\frac{1}{5}\\ \frac{3}{5} & -\frac{1}{5} & 0 & +\frac{1}{5}\\ \frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5}\\ \frac{4}{5} & -\frac{13}{5} & \frac{5}{5} & \frac{8}{5}\end{pmatrix}; \qquad \begin{matrix}v_0=\frac{12}{5}\\ v_1=\frac{7}{5}\\ v_2=-\frac{2}{5}\\ v_3=\frac{36}{5}\end{matrix}$$

where $P_s=P_5$ is determined by

$$\beta_0^1 P_s=\beta_0^1 P_5=\max_{j\neq 0}\beta_0^1 P_j=\frac{12}{5}>0$$

and $P_{j_r}=P_{j_3}=P_8$ is determined by

$$\frac{1}{v_r}\beta_r^1=\frac{5}{36}\beta_3^1=\min_{v_i>0,\ i\neq 0}(\text{lexico.})\frac{1}{v_i}\beta_i^1 .$$

2nd (Final) Iteration.

$$B_2=\begin{pmatrix} 0 & 1 & 1 & 1\\ -1 & 4 & 3 & 2\\ -1 & 0 & 7 & 2\\ -1 & 6 & 0 & 6\end{pmatrix}\ (\text{columns } P_0,\ P_1,\ P_2,\ P_5); \quad B_2^{-1}=\begin{pmatrix}\beta_0^2\\ \beta_1^2\\ \beta_2^2\\ \beta_3^2\end{pmatrix}=\begin{pmatrix}\frac{50}{15} & -\frac{5}{15} & -\frac{5}{15} & -\frac{5}{15}\\ \frac{16}{36} & \frac{11}{36} & -\frac{7}{36} & -\frac{4}{36}\\ \frac{16}{36} & \frac{2}{36} & \frac{2}{36} & -\frac{4}{36}\\ \frac{4}{36} & -\frac{13}{36} & \frac{5}{36} & \frac{8}{36}\end{pmatrix};$$

where no P_s can be determined since $\beta_0^2 P_j\leqq 0$ for $j\geqq 1$. Thus an optimal solution has been obtained (from top row) $\hat{x}_1=5/15$, $\hat{x}_2=5/15$, $\hat{x}_3=5/15$ and (from first column) $\hat{y}_1=16/36$, $\hat{y}_2=16/36$, $\hat{y}_5=4/36$ where all other $\hat{y}_i=0$. The "value of the game" (from upper left corner) is $\hat{x}_0=\hat{y}_0=50/15$. It will be noted that actually $\beta_0^2 P_j=0$ for *all* $j \geqq 1$, which means there exist other bases and corresponding solutions. Williams shows in his book, in all, eight such solutions.

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