ON THE UNIFORM CONVERGENCE OF A CERTAIN EIGENFUNCTION SERIES

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1. Introduction. In the attempt to solve certain problems in mathematical physics, such as diffraction of an arbitrary pulse by a wedge as considered by Irvin Kay [1], one encounters a hyperbolic differential equation of the type

$$u_{xx}-q(x)u=u_{xt}-p(x)u_t \tag{a}$$

where $u(x,t)$ must satisfy the boundary conditions $u(1,t)=u(0,t)=0$ and $u(x,0)=F(x)$. In attempting to solve equation (a) by separation of variables, one is led to the consideration of expanding an arbitrary function $F(x)$ in terms of the eigenfunctions $u_n(x)$ of the equation

$$u''+q(x)u+\lambda(p(x)u-u')=0$$

satisfying the boundary conditions $u(0)=u(1)=0$.

In the previous paper [2] by B. Friedman and L. I. Mishoe, it was proved that a function $F(x)$ of bounded variation for $0\leqq x\leqq 1$ could be expanded in terms of the eigenfunctions $u_n(x)$ of the system $u''+qu+\lambda(pu-u')=0$, $u(0)=u(1)=0$, provided $F(0^+)+F(1^-)\exp\left(-\int_0^1 pdt\right)=0$. However, the question of uniform convergence of the series $\sum_{-\infty}^{\infty}a_nu_n(x)$ to $F(x)$ was not considered. In this paper we establish sufficient conditions for the series $\sum_{-\infty}^{\infty}a_nu_n(x)$ to converge uniformly to $F(x)$ for $0<x<1$.

The following theorem has already been proved [2]:

THEOREM 1. *Let $F(x)$ be a function of bounded variation for $0\leqq x\leqq 1$. Let $u_n(x)$ be the eigenfunctions of the system*

$$(A+\lambda B)u=0\,;\qquad u(0)=u(1)=0\,, \tag{1}$$

where A is the operator $d^2/dx^2+q(x)$, and where B is the operator $-d/dx+p(x)$.

Let $q(x)$ be continuous and $p(x)$ have a continuous second derivative. Furthermore, let $v_n(x)$ be the eigenfunctions of the system adjoint to (1). If

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$$(2)\qquad F(0^+)+F(1^-)\exp\left(-\int_0^1 p(t)dt\right)=0\ ,$$

then the series

$$(3)\qquad \sum_{-\infty}^{\infty} a_n u_n(x)\ ,$$

where

$$(4)\qquad a_n=\int_0^1 F(\xi)\left[\frac{p(\xi)v_n(\xi)+v_n'(\xi)}{C'(\lambda_n)}\right]d\xi\ ,$$

and where the Wronskian $\omega(x)$ of the two independent solutions $u_1(x)$ and $u_2(x)$ has the form

$$(5)\qquad \omega(x)=u_1u_2'-u_2u_1'=C(\lambda)e^{\lambda x}$$

with

$$(6)\qquad C(\lambda)=\lambda^{-1}\exp\left(-\int_0^1 p(t)dt\right)-\exp\left(-\lambda+\int_0^1 p(t)dt\right)+O(\lambda^{-2})\ ,$$

converges to $F(x)$ at every point where $F(x)$ is continuous in $0<x<1$. At all other points, the series converges to $\frac{1}{2}(F(x+0)+F(x-0))$. If $F(x)$ does not satisfy the boundary conditions (2), then the series (3) converges to

$$(7)\ \frac{1}{2}\left[F(x+0)+F(x-0)-\left\{F(0^+)+F(1^-)\exp\left(-\int_0^1 p(t)dt\right)\right\}\exp\left(\int_0^x p(t)dt\right)\right].$$

In this paper, we prove:

THEOREM 2. *If $F'(x)$ exists and is of bounded variation for $0\leqq x\leqq 1$, then a sufficient condition for the series $\sum_{-\infty}^{\infty} a_n u_n(x)$ to converge uniformly to $F(x)$ for $0<x<1$ is that $F(0)=F(1)=0$.*

2. An asymptotic form for $C'(\lambda_n)$. Using (5) and the boundary conditions $u(0)=u(1)=0$ and $u'(0)=u'(1)=1$, we have

$$(8)\qquad C(\lambda)=e^{-\lambda}u_1(1,\lambda)\ .$$

Then it follows that

$$(9)\qquad C'(\lambda)=\frac{d}{d\lambda}C(\lambda)=-C(\lambda)+\tfrac{1}{2}e^{-\lambda}u_1(1,\lambda)+e^{-\lambda/2}\frac{d}{d\lambda}w_1(x,\lambda)\qquad \text{at } x=1$$

where

$$(10)\qquad u_1=e^{\lambda x/2}w_1\ .$$

Now (10) transforms the equation $(A+\lambda B)u=0$ into $w_1''+\left(q+\lambda p-\frac{\lambda^2}{4}\right)w_1=0$. It can be verified [2] that w_1 satisfies the equation

$$w_1=\frac{\sinh R(0,x)}{[r(x)r(0)]^{1/2}}-\int_0^x \frac{\sinh R(\xi,t)}{[r(x)r(\xi)]^{1/2}}g(\xi)w_1(\xi)d\xi \tag{11}$$

where

$$g(x)=p^2+\frac{p''}{\lambda-2p}+\frac{3p'^2}{(\lambda-2p)^2}+q \tag{12}$$

and

$$r(x)=\frac{\lambda}{2}-p(x)\ , \qquad R(\xi,x)=\int_\xi^x r(t)dt\ . \tag{13}$$

We note that $g(x)$ and $g'(x)=\frac{d}{d\lambda}g(x)$ are bounded for $|\lambda|$ sufficiently large. Also, if in (11) we make the substitution

$$w_1=\lambda^{-1}\exp\left(\tfrac{1}{2}|\sigma|x\right)Z_1(x) \tag{14}$$

where $\sigma=\mathscr{R}\lambda$, we note that $Z_1(x)$ is bounded [2] for $|\lambda|$ sufficiently large. Differentiating (11) with respect to λ, we obtain

$$\begin{aligned} w_1' = {} & \frac{x\cosh R(0,x)}{2[r(x)r(0)]^{1/2}}-\frac{[r(x)+r(0)]\sinh R(0,x)}{4[r(x)r(0)]^{3/2}} \\ & -\int_0^x \frac{\sinh R(\xi,x)}{[r(x)r(\xi)]^{1/2}}g(\xi)w_1'(\xi)d\xi-\int_0^x \frac{\sinh R(\xi,x)}{[r(x)r(\xi)]^{1/2}}g'(\xi)w_1(\xi)d\xi \\ & -\int_0^x (x-\xi)\frac{\cosh R(\xi,x)}{[r(x)r(\xi)]^{1/2}}g(\xi)w_1(\xi)d\xi \\ & +\frac{1}{4}\int_0^x \frac{[r(x)+r(\xi)]\sinh R(\xi,x)}{[r(x)r(\xi)]^{3/2}}g(\xi)w_1(\xi)d\xi\ . \end{aligned} \tag{15}$$

If we substitute

$$w_1=\lambda^{-1}\rho(x)\exp\left(\tfrac{1}{2}|\sigma|x\right)\ , \tag{16}$$

we obtain that

$$\begin{aligned} \rho(x)= {} & \frac{\lambda x\exp\left(-\tfrac{1}{2}|\sigma|x\right)\cosh R(0,x)}{[r(x)r(0)]^{1/2}} \\ & -\frac{\lambda[r(x)+r(0)]\exp\left(-\tfrac{1}{2}|\sigma|x\right)\sinh R(0,x)}{4[r(x)r(0)]^{3/2}} \end{aligned} \tag{17}$$

$$-\int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)\rho(\xi)d\xi$$

$$-\int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g'(\xi)Z_1(\xi)d\xi$$

$$-\int_0^x \frac{1}{2}(x-\xi)\frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\cosh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)Z_1(\xi)d\xi$$

$$+\int_0^x \frac{[r(x)+r(\xi)]\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{4[r(x)r(\xi)]^{3/2}} g(\xi)Z_1(\xi)d\xi$$

where $g'(\xi)=\dfrac{d}{d\lambda}g(\xi)$. Now $\lambda[r(x)r(\xi)]^{-1/2}$ and hence $\lambda[r(x)+r(\xi)][r(x)r(\xi)]^{-3/2}$ are both bounded by some constant C as $|\lambda|\to\infty$. Also, $\exp[-\frac{1}{2}|\sigma|(x-\xi)]\times\cosh R(\xi, x)$, and $\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)$ are both bounded by some constant C as $|\lambda|\to\infty$ and $0\leqq\xi\leqq x$. Using these results we obtain from equation (17) that

$$\begin{aligned}(18)\qquad |\rho(x)|\leqq 2C^2+\frac{C^2}{|\lambda|}\int_0^x|g(\xi)\rho(\xi)|d\xi+\frac{C^2}{|\lambda|}\int_0^x|g'(\xi)Z_1(\xi)|d\xi\\ +\frac{C^2}{|\lambda|}\int_0^x\frac{1}{2}|x-\xi||g(\xi)Z_1(\xi)|d\xi+\frac{C^2}{|\lambda|}\int_0^x|g(\xi)Z_1(\xi)|d\xi\ .\end{aligned}$$

If we set $\mu(\lambda)$ equal to the maximum of $|\rho(x)|$ in $0\leqq x\leqq 1$, then we certainly have that

$$\mu\leqq\frac{2C^2}{1-\dfrac{C^2}{|\lambda|}\displaystyle\int_0^x|g(\xi)|d\xi}+\frac{\dfrac{C^2}{|\lambda|}\displaystyle\int_0^x(|g'(\xi)Z_1(\xi)|+\frac{1}{2}|x-\xi|\,|g(\xi)Z_1(\xi)|+|g(\xi)Z_1(\xi)|)d\xi}{1-\dfrac{C^2}{|\lambda|}\displaystyle\int_0^x|g(\xi)|d\xi}\ .$$

Therefore, μ, and consequently $\rho(x)$ are bounded as $|\lambda|\to\infty$. Rewrite equation (15) as follows:

$$(19)\quad w_1'(x,\lambda)=\frac{x\cosh R(0, x)}{2[r(x)r(0)]^{1/2}}-\frac{[r(x)+r(0)]\sinh R(0, x)}{4[r(x)r(0)]^{3/2}}$$

$$-\lambda^{-1}\exp(\tfrac{1}{2}|\sigma|x)\int_0^x\frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}}g(\xi)\rho(\xi)d\xi$$

$$-\lambda^{-1}\exp(\tfrac{1}{2}|\sigma|x)\int_0^x\frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}}g(\xi)Z_1(\xi)d\xi$$

$$-\lambda^{-1}\exp(\tfrac{1}{2}|\sigma|x)\int_0^x(x-\xi)\frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)]\sinh R(\xi, x)}{2[r(x)r(\xi)]^{1/2}}g(\xi)Z_1(\xi)d\xi$$

$$+\lambda^{-1}\exp(\tfrac{1}{2}|\sigma|x)\int_0^x\frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)][r(x)+r(\xi)]\sinh R(\xi, x)}{4[r(x)r(\xi)]^{3/2}}g(\xi)Z_1(\xi)d\xi\ .$$

The above four integrals are all at least $O(\lambda^{-2}\exp[\frac{1}{2}|\sigma|x])$. Also,

$$[r(x)r(0)]^{-1/2}=\lambda^{-1}+O(\lambda^{-2}) ,$$

and

$$[r(x)]^{-1/2}[r(0)]^{-3/2}=\lambda^{-2}+O(\lambda^{-3}) .$$

So it follows that

$$(20)\quad w_1'(1,\lambda)=\lambda^{-1}\cosh\left(\int_0^1 r(t)dt\right)+O(\lambda^{-2}\exp[\tfrac{1}{2}|\sigma|])-\frac{\lambda^{-2}}{2}\sinh\left(\int_0^1 r(t)dt\right)$$
$$+O(\lambda^{-3}\exp[\tfrac{1}{2}|\sigma|]) .$$

Using this result for $w_1'(1,\lambda)$ in equation (9), we have, for $\mathscr{R}\lambda>0$,

$$(21)\quad C'(\lambda)=\lambda^{-1}\left[\exp\left(-\lambda+\int_0^1 p(t)dt\right)\right]$$
$$-\frac{1}{2}\lambda^{-2}\left[\exp\left(-\int_0^1 p(t)dt\right)-\exp\left(-\lambda+\int_0^1 p(t)dt\right)\right]+O(\lambda^{-3})+O(\lambda^{-2}) ,$$

and for $\mathscr{R}\lambda<0$,

$$(22)\quad C'(\lambda)=\lambda^{-1}\left[\exp\left(-\lambda+\int_0^1 p(t)dt\right)\right]$$
$$-\frac{1}{2}\lambda^{-2}\left[\exp\left(-\int_0^1 p(t)dt\right)-\exp\left(-\lambda+\int_0^1 p(t)dt\right)\right]+O(\lambda^{-3}e^{\lambda}) .$$

3. Distribution of the eigenvalues. Since by [2]

$$C(\lambda)=\lambda^{-1}\exp[-\lambda a]\left(\exp\left[-\int_a^b p(t)dt\right]-\exp\left[-\lambda(b-a)+\int_a^b p(t)dt\right]+O(\lambda^{-1})\right)$$
$$=\lambda^{-1}\exp[-\lambda a]C_1(\lambda)\qquad\text{for }\mathscr{R}\lambda\geqq 0,$$
$$C(\lambda)=\lambda^{-1}\exp[-\lambda b]\left(\exp\left[-\lambda(a-b)-\int_a^b p(t)dt\right]-\exp\left[\int_a^b p(t)dt\right]+O(\lambda^{-1})\right)$$
$$=\lambda^{-1}\exp[-\lambda b]C_2(\lambda)\qquad\text{for }\mathscr{R}\lambda\leqq 0,$$

and where a and b equal 0 and 1 respectively.
The condition that λ be an eigenvalue is that $C(\lambda)$ and hence either $C_1(\lambda)$ or $C_2(\lambda)$ be zero. Equating $C_1(\lambda)$ to zero we obtain

$$(23)\quad \exp\left[-\lambda(b-a)+\int_a^b p(t)dt\right]=\exp\left[-\int_a^b p(t)dt\right]+O(\lambda^{-1})$$
$$=\exp\left[-\int_a^b p(t)dt\right](1+O(\lambda^{-1})) .$$

By taking the logarithm of both sides of the above equation (23) and expanding the term $\log(1+O(\lambda^{-1}))$ we obtain that the large eigenvalues satisfy the equation

$$-\lambda_n=-2\int_0^1 p(t)dt+2n\pi i+O(\lambda_n^{-1}), \quad n=\pm N, \pm N+1, \cdots .$$

Hence the eigenvalues with positive real parts, if they exist, are given by

$$\lambda_n=2n\pi i+2\int_0^1 p(t)dt+O\left(\frac{1}{n}\right). \tag{24}$$

The equation $C_2(\lambda)=0$ leads to the same result for those eigenvalues with negative real parts. Consequently, all the eigenvalues are represented by equation (24).

4. On the uniform convergence of series (3). Consider equation (4). In [2] it was shown that

$$B^* V_1(x)=\exp\left[-\lambda x+\int_0^x p(t)dt\right]+\Omega_1 , \tag{25}$$

where

$$\Omega_1=\begin{cases} O(\lambda^{-2})+O(\lambda^{-1}\exp[-\lambda x]) & \text{for } \mathscr{R}\lambda\geqq 0 \\ O(\lambda^{-1}\exp[-\lambda x]) & \text{for } \mathscr{R}\lambda\leqq 0. \end{cases}$$

Similarly,

$$B^* V_2(x)=\exp\left[-\lambda x+\int_1^x p(t)dt\right]+\Omega_2 , \tag{26}$$

where

$$\Omega_2=\begin{cases} O(\lambda^{-1}\exp[-\lambda x]) & \text{for } \mathscr{R}\lambda\geqq 0 \\ O(\lambda^{-1}\exp[-\lambda x])+O(\lambda^{-1}\exp[-\lambda b]) & \text{for } \mathscr{R}\lambda\leqq 0. \end{cases}$$

Also from [2], we have that

$$u_1(x)=\lambda^{-1}\left\{\exp\left[\lambda x-\int_0^x p(t)dt\right]-\exp\int_0^x p(t)dt\right]\right\}+O(\lambda^{-2}e^{\lambda x}) \tag{27}$$

and

$$u_2(x)=\lambda^{-1}\left\{\exp\left[\lambda(x-1)-\int_1^x p(t)dt\right]-\exp\int_1^x p(t)dt\right]\right\}+O(\lambda^{-2}) . \tag{28}$$

Using equations (26) and (27), for $\mathscr{R}\lambda>0$, we have

$$a_n u_1 = u_1 \int_0^1 F(\xi) \frac{B^* V_2(\xi)}{C'(\lambda_n)} d\xi$$

$$= O\left(\frac{e^{\lambda_n x}}{\lambda_n}\right) \int_0^1 \frac{F(\xi)\left[\exp\left(-\lambda_n \xi + \int_1^\xi p(t)dt\right)\right] d\xi}{C'(\lambda_n)}$$

$$+ O(\lambda_n^{-1} e^{\lambda_n x}) \int_0^1 \frac{F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi})}{C'(\lambda_n)} d\xi$$

$$= \frac{A}{\lambda_n} \int_0^1 \frac{F(\xi) \exp\left(-\lambda_n \xi + \int_1^\xi p(t)dt\right)}{C'(\lambda_n)} d\xi + \frac{A}{\lambda_n} \int_0^1 \frac{F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi})}{C'(\lambda_n)} d\xi$$

where A is bounded.

By equation (22), $C'(\lambda_n) = O(\lambda_n^{-1})$, therefore $\dfrac{1}{C'(\lambda_n)} = O(\lambda_n)$.

Hence

$$a_n u_1 = B_n \int_0^1 F(\xi) \exp\left(-\lambda_n \xi + \int_0^\xi p(t)dt\right) d\xi + B_n \int_0^1 F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi}) d\xi \tag{29}$$

where $B_n = \lambda^{-1} O(\lambda_n) A$ is also bounded. Using equation (26) for $\mathscr{R}\lambda > 0$, and observing that $O(\lambda^{-1} \exp(-\lambda_n \xi))$ is the indefinite integral of a bounded function, it can be easily shown that

$$\int_0^1 F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi}) d\xi = O(\lambda_n^{-2}) . \tag{30}$$

Consider now the first integral in equation (29). Setting $H(\xi) =$ $F(\xi) \exp\left(\int_1^\xi p(t)dt\right)$ and integrating by parts, we obtain

$$\int_0^1 H(\xi) \exp(-\lambda_n \xi) d\xi = -\lambda_n^{-1} H(\xi) \exp(-\lambda_n \xi) \Big]_0^1 + \int_0^1 \lambda_n^{-1} H'(\xi) \exp(-\lambda_n \xi) d\xi . \tag{31}$$

Since $F(1) = F(0) = 0$, then $H(1) = H(0) = 0$, and the first term on the right hand side of equation (31) vanishes.

Now

$$H'(\xi) = p(\xi) F(\xi) \exp\left(\int_1^\xi p(t)dt\right) + F'(\xi) \exp\left(\int_1^\xi p(t)dt\right) . \tag{32}$$

$F'(\xi)$ is of bounded variation on $(0, 1)$ and $p'(\xi)$ is continuous on $(0, 1)$. Therefore, $H'(\xi)$ is of bounded variation on $(0, 1)$. Hence,

$$H'(\xi) = \varphi_1(\xi) - \varphi_2(\xi)$$

where $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are two bounded, positive, monotone functions, either both nonincreasing or both nondecreasing. Now $\lambda_n^{-1}\exp(-\lambda_n\xi)$ is bounded and integrable for $0\leq\xi\leq 1$. Assume $\varphi_1(\xi)$ to be a monotone decreasing function, then

$$\int_0^1 \lambda_n^{-1}H'(\xi)\exp(-\lambda_n\xi)d\xi \tag{33}$$

$$=\varphi_1(0)\int_0^{\xi_0}\lambda_n^{-1}\exp(-\lambda_n)d\xi-\varphi_2(0)\int_0^{\xi_1}\lambda_n^{-1}\exp(-\lambda_n\xi)d\xi=O(\lambda_n^{-2})$$

where ξ_0 and ξ_1 are on the interval (0,1).

Combining the results of (30) and (33) we have

$$\sum_{-\infty}^{\infty}a_nu_n(x)=\sum_{-\infty}^{-(N+1)}O(\lambda_n^{-2})+\sum_{-N}^{N-1}a_nu_n(x)+\sum_{N}^{\infty}O(\lambda_n^{-2})$$

where $\sum_{-N}^{N-1}a_nu_n(x)$ is finite for $0<x<1$. From (24) it is clear that $\lambda_n=O(n)$ for $n=\pm N,\ \pm N+1,\cdots$ Therefore

$$\sum_{-\infty}^{\infty}a_nu_n(x)=\sum_{-\infty}^{-(N+1)}\frac{O(1)}{n^2}+\sum_{-N}^{N-1}a_nu_n(x)+\sum_{N}^{\infty}\frac{O(1)}{n^2}$$

where $O(1)$ is a bounded function.

Since $\left|\frac{O(1)}{n^2}\right|\leq\frac{M}{n^2}$, $M>0$ and the series $M\sum_{N}^{\infty}\frac{1}{n^2}$ converges, it is clear that $\sum_{-\infty}^{\infty}a_nu_n(x)$ converges uniformly to $F(x)$ for $0<x<1$. And our theorem is proved.

We note, however, that while Theorem 2 is sufficient, it is not a necessary condition for uniform convergence. For suppose $F(0)$ and $F(1)$ differ from zero, then by equations (31) and (33) we have $\sum_{-\infty}^{\infty}a_nu_n=\sum_{-\infty}^{\infty}O\left(\frac{1}{n}\right)$ which may or may not converge uniformly.

In fact, *a necessary and sufficient condition* for the uniform convergence of this series does not seem to be known.

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References

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