INVARIANT FUNCTIONALS

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1. Introduction. Let $E$ be a normed linear space and $G$ a solvable group of bounded linear operators on $E$. If there exists a non-trivial bounded linear functional invariant under $G$ then there exists $x_0 \in E$ such that $\inf \|T(x_0)\| > 0$, $T \in G$, the convex envelope of $G$. Assume that such an $x_0$ exists. If $G$ is bounded then there exists an invariant functional [7]. If $G$ is unbounded, however, such a functional may or may not exist.

For simplicity we discuss here the abelian case. In a previous work [7] it was shown that the invariant functional exists if there is a constant $K > 0$ such that to each $U \in G$, there corresponds $V \in G$ where $\|V\| \leq K$ and $\|VU\| \leq K$. A consequence of this condition is that for each $x \in E$

\[(1) \quad \inf_{\|T\| \leq K} \inf_{T \in G} \|T(x)\| \leq K \inf_{T \in G} \|T(x)\|.\]

Now call an element $y$ stable if (1) holds for some $K = K(y)$ for all $x$ of the form $U(y)$, $U \in G$. We show here that the invariant functional exists if $E$ is complete and if there exists an open set $S$ in $E$ such that for all $x \in S$, $T \in G$, $x$ and $T(x) - x$ are stable. An analogous result is shown to hold if $G$ is solvable.

The problem of the existence and extension of functionals invariant under solvable groups of operators has been considered by Agnew and Morse and by Klee (see [3] for references). These authors use for $E$ any real linear space while we take $E$ to be a Banach space in order to utilize category arguments.

2. Notations. Let $E$ be a Banach space and $\mathcal{G}(E)$ be the set of all bounded operators on $E$. Let $H$ be a (multiplicative) semi-group in $\mathcal{G}(E)$. By $H$ we mean the convex envelope of $H$ (the smallest convex subset of $\mathcal{G}(E)$ which contains $H$). As in [7] we adopt the following notation. By $B(H)$ we mean the linear manifold generated by elements of the form $T(x) - x$, $x \in E$, $T \in H$. By $Z(H)$ we mean $\{x \in E | \inf_{T \in H} \|T(x)\| = 0, T \in H\}$.

We introduce the following notation. An element $x \in E$ is stable with respect to $H$ if there exist positive numbers $K, L$ such that

$$\inf_{\|T\| \leq K} \inf_{T \in H} \|T(y)\| \leq L \inf_{T \in H} \|T(y)\|$$

for all $y$ of the form $U(x)$, $U \in H$.

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We use the following symbolism,
\[ \delta(y, H) = \inf_{T \in H} \| T(y) \|, \]
\[ \delta(y, H, r) = \inf_{T \in H} \| T(y) \| + r. \]

It is readily seen that \( x \) is stable with respect to \( H \) if and only if there exists a constant \( r > 0 \) such that
\[ \delta(y, H, r) \leq r \delta(y, H) \]
for all \( y \) of the form \( U(x), U \in H \). Such an \( r \) is called a constant connected with the stability of \( x \) with respect to \( H \). If \( x \) is stable with respect to \( H \) and if the right-hand side of (2) is zero for all \( y \) of the form \( U(x), U \in H \), we say that \( x \) is null-stable with respect to \( H \).

If \( G \) is a solvable group, then \( G^{(i)} \) will represent the \( i \)th derived subgroup.

3. Invariant functionals for solvable groups.

3.1 LEMMA. If \( y_1, \ldots, y_n \) are null-stable with respect to \( H \) then so is \( y_1 + \cdots + y_n \).

Proof. It is enough to show this for \( y_1 + y_2 \). Let \( M \) be the maximum of the constants in the definition of the null-stability of \( y_1 \) and \( y_2 \). Take \( U \in H, \varepsilon > 0 \). There exist \( V_i \in H, \| V_i \| \leq M, i = 1, 2 \) such that \( \| V_i U(y_i) \| < \varepsilon/(2M) \) and \( \| V_1 V_2 U(y_2) \| < \varepsilon/2 \). Then \( \| V_1 V_2 U(y_1 + y_2) \| < \varepsilon \) with \( \| V_1 V_2 \| \leq M^2 \). Similarly we see that \( y_1 + \cdots + y_n \) is null-stable with constant \( M^a \) if \( M \) is the maximum of the constants connected with the \( y_i \).

3.2 LEMMA. Let \( E \) be a Banach space and \( G \) a solvable group of bounded linear operators on \( E \). Then either (a) every element of \( E \) is null-stable with respect to \( G \), or (b) there exists a non-void open set of \( E \) containing only elements not null-stable with respect to \( G \), or (c) the set of elements not stable with respect to every \( G^{(i)} \) is dense.

Proof. Let \( Q_n = \{ x \in E | x \) is stable with respect to each \( G^{(i)} \) with constant \( n \}, n = 1, 2, \ldots \). We show that \( Q_n \) is closed. Let \( x_m \in Q_n, x_m \rightarrow y \). Then for each \( i \) and each \( x_m \) we have
\[ \delta(x_m, G^{(i)} n) \leq \delta(x_m, G^{(i)}). \]
We show that (1) also holds for \( y \). If \( \delta(y, G^{(i)} n) = 0 \) this is clear. Otherwise set \( \delta = \delta(y, G^{(i)} n) \) and take \( 0 < 2\varepsilon < \delta \). Select \( T \in G^{(i)} \). Choose \( m \).
so large that
\[
\|T(y - x_m)\| < \varepsilon/n, \quad \|y - x_m\| < \varepsilon/n.
\]
Then from (1) and (2) we obtain
\[
n\|T(y)\| \geq n\|T(x_m)\| - \varepsilon \geq \delta(x_m, G^{(i)}, n) - \varepsilon \geq \delta(y, G^{(i)}, n) - 2\varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary in (3),
\[
n\|T(y)\| \geq \delta(y, G^{(i)}, n).
\]
Since the \(T\) of (4) is arbitrary in \(G^{(i)}\), (1) holds for \(y\). Since the same argument is applicable to every \(V(y), V \in G^{(i)}\) as well as for \(y\) and for each \(i, y \in Q_n\).

Suppose that some \(Q_n\) contains an open sphere \(S\). Let \(\Sigma\) be the collection of elements of \(S\) which are null-stable with respect to \(G_i\). If \(\Sigma\) is dense in \(S\) we show that \(\Sigma = S\). For let \(y_m \in \Sigma, m = 1, \ldots, y_m \rightarrow z \in S\). For each \(m, U \in G_i\), we have \(\delta(U(y_m), G_i, n) = 0\). This implies that \(\delta(U(z), G_i, n) = 0\) which in turn shows that \(z \in \Sigma\). In this case by Lemma 3.1, (a) holds since the set of elements which are null-stable with respect to \(G_i\) forms a linear manifold with interior. If \(\Sigma\) is not dense in \(S\) then there is an open subset \(S_i\) of \(S\) on which (b) holds.

Suppose next that no \(Q_n\) contains a sphere. By a theorem of Baire, the intersection \(P\) of the sets \(E - Q_n\) is dense. If \(x \in P\), then \(x\) fails to be stable with respect to at least one of the semi-groups \(G^{(i)}\), for otherwise \(x \in Q_n\) for all sufficiently large \(n\).

3.3 Lemma. Let \(G\) be a solvable group in \(\mathfrak{S}(E)\). If \(S \in G^{(i)}, T \in G^{(i)}, x \in E\) then \(S[T(x) - x]\) can be expressed in the form \(z + TS(x) - S(x)\) where \(z \in B(G^{(i+1)}), i = 0, \ldots, n - 1\).

Proof. Let
\[
S = \sum_{j=1}^{m} \alpha_j S_j, \quad \alpha_j \geq 0, \quad \sum_{j=1}^{m} \alpha_j = 1, \quad S_j \in G^{(i)}.
\]
For each \(j = 1, \ldots, m\) there exists \(U_j \in G^{(i+1)}\) such that \(S_j T = U_j T S_j\). Then
\[
S[T(x) - x] = \sum_{j=1}^{m} \alpha_j [U_j T S_j(x) - S_j(x)]
\]
\[
= \sum_{j=1}^{m} [U_j T S_j(\alpha, x) - T S_j(\alpha, x)] + TS(x) - S(x)
\]
which is in the required form.

3.4 Lemma. If \(S \in G^{(i)}, T \in G^{(i+1)}, x \in E\) then \(S[T(x) - x] \in B(G^{(i+1)})\).
This follows from Lemma 3.3.

3.5 Lemma. Let \( H \) be a semi-group in \( \mathcal{S}(E) \). Suppose that \( x \) is stable with respect to \( H \) and that \( U(x) \in Z(H) \) for all \( U \in H \). Then \( x \) is null-stable with respect to \( H \).

This follows directly from the definitions.

3.6 Lemma. Let \( G \) be a group in \( \mathcal{S}(E) \), \( x \in E \) where \( x \) is stable with respect to \( G \). Then \( TW(x) - W(x) \in Z(G) \) for all \( T \in G, W \in G \).

Proof. Set \( V = (I + T + \cdots + T^{n-1})/s \). Then \( V[TW(x) - W(x)] = [T^nW(x) - W(x)]/s \). Let \( r \) be the constant connected with the stability of \( x \). Then since \( T^{-1} \in G \),

\[
\delta(T^nW(x), G, r) \leq r\delta(T^nW(x), G) \leq r\|W(x)\|.
\]

Pick \( U \in G \), \( \|U\| \leq r \) where \( \|UT^nW(x)\| < r\|W(x)\| + 1 \). Then

\[
\|UV[TW(x) - W(x)]\| < (2r\|W(x)\| + 1)/s.
\]

This shows that \( TW(x) - W(x) \in Z(G) \).

3.7 Lemma. Let \( G \) be a group in \( \mathcal{S}(E) \). Let \( x \in E \) where \( (T-I)(x) \) is stable with respect to \( G \) for all \( T \in G \). Then \( (T-I)U(x) \) is also stable for all \( T \in G, U \in G \).

Proof. Observe that \( (T-I)U(x) = U(U^{-1}TU-I)(x) \). Since \( (U^{-1}TU-I)(x) \) is stable with respect to \( G \), it follows readily that so is \( (T-I)U(x) \).

3.8 Theorem. Let \( E \) be a Banach space and \( G \) a solvable group of bounded linear operators on \( E \). Let \( Q \) be the set of elements of \( E \) stable with respect to each \( G_i \). If there exists a non-void open subset \( \mathcal{S} \) of \( Q \) such that \( (T-I)\mathcal{S} \subseteq \mathcal{Q} \) for each \( T \in G \) then every element of \( B(G) \) is null-stable with respect to \( G \). If also there is at least one element of \( E \) not null-stable with respect to \( G \) then there exists a non-trivial invariant functional.

Proof. Assume the condition on the set \( \mathcal{S} \). We show by induction starting with \( n \), where \( G^{(n)} = \{I\} \), that \( B(G^{(n)}) \) consists entirely of elements null-stable with respect to \( G_i^{(n)} \), \( j = 0, \ldots, n \). This is automatic for \( j = n \); suppose that it holds for \( j = i + 1, \ldots, n \). Let \( S, T \in G^{(n)}, x \in \mathcal{S} \). In the notation of Lemma 3.3, we can write \( S[T(x) - x] = z + TS(x) - S(x) \) where \( z \) is a linear combination of elements of the form \( U_iTS_j(x) - TS_j(x), U_i \in G^{(i+1)}, S_j \in G^{(j)} \). By hypothesis and Lemma 3.7, \( U_iTS_j(x) - TS_j(x) \in \mathcal{Q} \).
For any $V \in G^{(i)}_1$, $V[U,T S_j(x) - T S_j(x)] \in B(G^{(i+1)}) \subset Z(G^{(i+1)}) \subset Z(G^{(i)})$ by Lemma 3.4 and the induction hypothesis. Hence by Lemma 3.5, $U,T S_j(x) - T S_j(x)$ is null-stable with respect to $G^{(i)}_1$ and thus, by Lemma 3.1 so is $z$.

Consider the constant $r$ connected with the null-stability of $z$ with respect to $G^{(i)}_1$. Take $\varepsilon > 0$. Since $x \in \mathcal{S}$, by Lemma 3.6 there exists $W \in G^{(i)}_1$ such that $\| W[T S_j(x) - S(x)] \| \leq \varepsilon (2r)$. Furthermore there exists $R \in G^{(i)}_1$, $R \leq r$ such that $\| R W(z) \| < \varepsilon / 2$. Therefore $\| R W[T S_j(x) - S(x)] \| < \varepsilon$ which shows that $S[T(x) - x] \in Z(G^{(i)}_1)$ for all $S \in G^{(i)}_1$. Since $T(x) - x \in \mathcal{Q}$ it follows from Lemma 3.5 that $T(x) - x$ is null-stable with respect to $G^{(i)}_1$. Let $P = \{ x \in E | T(x) - x$ is null-stable with respect to $G^{(i)}_1 \}$. By Lemma 3.1, $P$ is a linear manifold. But $\mathcal{S} \subset P$. Therefore $P = \mathcal{E}$. In view of Lemma 3.1, every element of $B(G^{(i)})$ is null-stable with respect to $G^{(i)}_1$. This completes the induction.

Suppose also that some element of $E$ is not null-stable with respect to $G_i$. Then (a) and (c) of Lemma 3.2 are ruled out. Thus there exists a sphere in $E$ given by Lemma 3.2 which by the above is disjoint with $B(G)$. Hence, by the Hahn-Banach theorem there exists a bounded linear functional $\neq 0$ which vanishes on $B(G)$. This is an invariant functional.

4. Positive invariant functionals. We point out next that the arguments used above and in [7] for $B(G) \subset Z(G_i)$ have wider applicability than is apparent on the surface and in particular contain implicitly results obtained by Krein and Rutman [5].

In the terminology of [5] by a linear semi-group $\mathfrak{S}$ in a real normed linear space $E$ is meant a (proper) subset of $E$ where $ax + \beta y \in \mathfrak{S}$ if $x, y \in \mathfrak{S}$ and $a \geq 0, \beta \geq 0$ are scalars. We say that $x \leq y \ (y \geq x)$ if $y - x \in \mathfrak{S}$, $x, y \in E$. Suppose that $\mathfrak{S}$ is given with $\text{Int}(\mathfrak{S})$ non-void.

Let $G$ be a multiplicative semi-group of linear operators on $E$. Following [6] we call $G$ left-solvable if there exists a finite sequence of sub-semi-groups $G = G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n)} = \{ I \}$ such that given $T, U \in G^{(i)}$, $i = 0, \cdots, n - 1$ there exists $V \in G^{(i+1)}$ with $T U = V U T$.

The following is an extension of [5, Theorem 3.1].

4.1 THEOREM. Let $G$ be a left solvable semi-group of linear operators on $E$ such that $A(\mathfrak{S}) \subset \mathfrak{S}$, $A \in G$. Suppose that $v \in \text{Int}(\mathfrak{S})$ and

(a) for some $a > 0$, $A(v) \geq a v$, $A \in G$, and

(b) for some $r > 0$, given $U \in G^{(i)}_1$ there exists $T \in G^{(i)}_1$ such that

(1) $T(v) \leq r v$, $TU(v) \leq r v$

$i = 0, \cdots, n - 1$. Then there exists a bounded linear functional $x^*$ on $E$, invariant with respect to $G$ and $x^*(x) > 0$, $x \in \text{Int}(\mathfrak{S})$. 

Let $v \in \text{Int}(\mathfrak{R})$. As in [5] we define for each $x \in E$, $|x|_v = \inf t$, where $t > 0$ and satisfies $-tv \leq x \leq tv$. $|x|_v$ is a semi-norm for $E$. Let $A$ be a linear operator on $E$, $A(\mathfrak{R}) \subset \mathfrak{R}$. Since $v \in \text{Int}(\mathfrak{R})$, if $\alpha > 0$ is sufficiently large, then

$$-\alpha v \leq 0 \leq A(v) \leq \alpha v$$

It is easy to see that $|A(v)|_v = \inf \alpha$, $\alpha > 0$ satisfying (1). If $-tv \leq x \leq tv$ then for $\alpha$ satisfying (1),

$$-t\alpha v \leq -tA(v) \leq A(x) \leq tA(v) \leq t\alpha v$$

from which we see that $|A(x)|_v \leq |A(v)|_v|x|_v$. Since $|v|_v = 1$ we see that $A$ is bounded with respect to the semi-norm and

$$|A|_v = |A(v)|_v.$$  

We define $Z(G^{(\mathfrak{R})})$ in terms of the semi-norm $|x|_v$. By the formulas (1), (2) and (3) it is seen that for $T \in G^{(\mathfrak{R})}$ there exists $V \in G^{(\mathfrak{R})}$, $|V|_v \leq r$ where $|VT|_v \leq r$. The arguments of [6, Theorem 3] are unaffected by the use of the semi-norm rather than a true norm. As noted by Robison [6, Theorem 6.8] in this situation we then obtain $B(G) \subset Z(G)$. 

Let $x \in \text{Int}(\mathfrak{R})$. There exists $\alpha > 0$ such $x \geq \alpha v$. For each $A \in G$, by (a), $A(x) \geq \alpha ov$. Moreover if $A(x) \leq \beta ov$, $0 < \beta < \alpha$, then $\beta ov \leq \alpha ov$ which is impossible by [5, p. 11]. Hence $|A(x)|_v \geq \alpha v$. This shows that $\text{Int}(\mathfrak{R}) \cap Z(G) = \phi$. By the above, $B(G) \cap \text{Int}(\mathfrak{R}) = \phi$. An application of [4, Corollary 1.2] gives the existence of the desired functional.

As a consequence of Theorem 4.1 we obtain the following.

4.2. **COROLLARY.** Let $G$ be a left solvable semi-group of operators on $E$ satisfying the requirements of Theorem 4.1, and let $v \in \text{Int}(\mathfrak{R})$. Then for any $w \in \text{Int}(\mathfrak{R})$, $T_j \in G$, $j=1, 2, \ldots, n$,

$$\sum_{j=1}^{n} p_j T_j(w) \in \mathfrak{R} \text{ implies that } \sum_{j=1}^{n} p_j \geq 0.$$  

When $\mathfrak{R}$ is the positive cone in a space $E$ of bounded functions on a set $S$, and $G$ is a semi-group of linear operations on $E$ induced by a semi-group $\Gamma$ of one-to-one transformations of $S$ onto $S$, Hadwiger and Nef [2] have shown that the statement (4) is fundamental in the theory of integration systems.

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1 We mean $|ax| = |a||x|$, $x \in E$, $a$ real, and $|x+y| \leq |x| + |y|$, $x, y \in E$. (See [1], p. 93). In particular $|x| \geq 0$ for all $x.$
REFERENCES


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