INVARIANT FUNCTIONALS

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1. Introduction. Let E be a normed linear space and G a solvable group of bounded linear operators on E. If there exists a non-trivial bounded linear functional invariant under G then there exists $x_0 \in E$ such that $\inf \|T(x_0)\| > 0$, $T \in G_1$, the convex envelope of G. Assume that such an x_0 exists. If G is bounded then there exists an invariant functional [7]. If G is unbounded, however, such a functional may or may not exist.

For simplicity we discuss here the abelian case. In a previous work [7] it was shown that the invariant functional exists if there is a constant $K > 0$ such that to each $U \in G_1$ there corresponds $V \in G_1$ where $\|V\| \leq K$ and $\|VU\| \leq K$. A consequence of this condition is that for each $x \in E$

$$(1) \qquad \inf_{\substack{\|T\| \leq K \\ T \in G_1}} \|T(x)\| \leq K \inf_{T \in G_1} \|T(x)\| .$$

Now call an element y *stable* if (1) holds for some $K = K(y)$ for all x of the form $U(y)$, $U \in G_1$. We show here that the invariant functional exists if E is complete and if there exists an open set S in E such that for all $x \in S$, $T \in G$, x and $T(x) - x$ are stable. An analogous result is shown to hold if G is solvable.

The problem of the existence and extension of functionals invariant under solvable groups of operators has been considered by Agnew and Morse and by Klee (see [3] for references). These authors use for E any real linear space while we take E to be a Banach space in order to utilize category arguments.

2. Notations. Let E be a Banach space and $\mathfrak{E}(E)$ be the set of all bounded operators on E. Let H be a (multiplicative) semi-group in $\mathfrak{E}(E)$. By H_1 we mean the convex envelope of H (the smallest convex subset of $\mathfrak{E}(E)$ which contains H). As in [7] we adopt the following notation. By $B(H)$ we mean the linear manifold generated by elements of the form $T(x) - x$, $x \in E$, $T \in H$. By $Z(H)$ we mean $\{x \in E \mid \inf \|T(x)\| = 0, T \in H\}$.

We introduce the following notation. An element $x \in E$ is *stable* with respect to H if there exist positive numbers K, L such that

$$\inf_{\substack{\|T\| \leq K \\ T \in H}} \|T(y)\| \leq L \inf_{T \in H} \|T(y)\|$$

for all y of the form $U(x)$, $U \in H$.

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We use the following symbolism,

$$\delta(y, H) = \inf_{T \in H} \|T(y)\|$$

$$\delta(y, H, r) = \inf_{\substack{\|T\| \leq r \\ T \in H}} \|T(y)\| .$$

It is readily seen that x is stable with respect to H if and only if there exists a constant $r > 0$ such that

$$(2) \qquad \delta(y, H, r) \leqq r\delta(y, H)$$

for all y of the form $U(x)$, $U \in H$. Such an r is called a constant connected with the stability of x with respect to H. If x is stable with respect to H and if the right-hand side of (2) is zero for all y of the form $U(x)$, $U \in H$, we say that x is *null-stable* with respect to H.

If G is a solvable group, then $G^{(i)}$ will represent the ith derived subgroup.

3. Invariant functionals for solvable groups.

3.1 LEMMA. *If* $y_1, \cdots, y_n$ *are null-stable with respect to* H *then so is* $y_1 + \cdots + y_n$.

Proof. It is enough to show this for $y_1 + y_2$. Let M be the maximum of the constants in the definition of the null-stability of y_1 and y_2. Take $U \in H$, $\varepsilon > 0$. There exist $V_i \in H$, $\|V_i\| \leqq M$, $i = 1, 2$ such that $\|V_1 U(y_1)\| < \varepsilon/(2M)$ and $\|V_2 V_1 U(y_2)\| < \varepsilon/2$. Then $\|V_2 V_1 U(y_1 + y_2)\| < \varepsilon$ with $\|V_2 V_1\| \leqq M^2$. Similarly we see that $y_1 + \cdots + y_n$ is null-stable with constant M^n if M is the maximum of the constants connected with the y_i.

3.2 LEMMA. *Let* E *be a Banach space and* G *a solvable group of bounded linear operators on* E. *Then either* (a) *every element of* E *is null-stable with respect to* G_1 *or* (b) *there exists a non-void open set of* E *containing only elements not null-stable with respect to* G_1 *or* (c) *the set of elements not stable with respect to every* $G_1^{(i)}$ *is dense.*

Proof. Let $Q_n = \{x \in E \mid x$ is stable with respect to each $G_1^{(i)}$ with constant $n\}$, $n = 1, 2, \cdots$. We show that Q_n is closed. Let $x_m \in Q_n$, $x_m \to y$. Then for each i and each x_m we have

$$(1) \qquad \delta(x_m, G_1^{(i)}, n) \leqq n\delta(x_m, G_1^{(i)}) .$$

We show that (1) also holds for y. If $\delta(y, G_1^{(i)}, n) = 0$ this is clear. Otherwise set $\delta = \delta(y, G_1^{(i)}, n)$ and take $0 < 2\varepsilon < \delta$. Select $T \in G_1^{(i)}$. Choose m

so large that

$$(2)\qquad \|T(y-x_m)\|<\varepsilon/n\ ,\qquad \|y-x_m\|<\varepsilon/n\ .$$

Then from (1) and (2) we obtain

$$(3)\qquad n\|T(y)\|\geqq n\|T(x_m)\|-\varepsilon\geqq\delta(x_m, G_1^{(i)}, n)-\varepsilon\geqq\delta(y, G_1^{(i)}, n)-2\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary in (3),

$$(4)\qquad n\|T(y)\|\geqq\delta(y, G_1^{(i)}, n)\ .$$

Since the T of (4) is arbitrary in $G_1^{(i)}$, (1) holds for y. Since the same argument is applicable to every $V(y)$, $V\in G_1^{(i)}$ as well as for y and for each i, $y\in Q_n$.

Suppose that some Q_n contains an open sphere S. Let Σ be the collection of elements of S which are null-stable with respect to G_1. If Σ is dense in S we show that $\Sigma=S$. For let $y_m\in\Sigma$, $m=1,\cdots, y_m\to z\in S$. For each m, $U\in G_1$, we have $\delta(U(y_m), G_1, n)=0$. This implies that $\delta(U(z), G_1, n)=0$ which in turn shows that $z\in\Sigma$. In this case by Lemma 3.1, (a) holds since the set of elements which are null-stable with respect to G_1 forms a linear manifold with interior. If Σ is not dense in S then erthe is an open subset S_1 of S on which (b) holds.

Suppose next that no Q_n contains a sphere. By a theorem of Baire, the intersection P of the sets $E-Q_n$ is dense. If $x\in P$, then x fails to be stable with respect to at least one of the semi-groups $G_1^{(i)}$, for otherwise $x\in Q_n$ for all sufficiently large n.

3.3 Lemma. *Let G be a solvable group in $\mathfrak{E}(E)$. If $S\in G_1^{(i)}$, $T\in G^{(i)}$, $x\in E$ then $S[T(x)-x]$ can be expressed in the form $z+TS(x)-S(x)$ where $z\in B(G^{(i+1)})$, $i=0,\cdots,n-1$.*

Proof. Let

$$S=\sum_{j=1}^{m}\alpha_jS_j,\quad \alpha_j\geqq 0\ ,\qquad \sum_{j=1}^{m}\alpha_j=1,\quad S_j\in G^{(i)}\ .$$

For each $j=1,\cdots,m$ there exists $U_j\in G^{(i+1)}$ such that $S_jT=U_jTS_j$. Then

$$(5)\qquad \begin{aligned} S[T(x)-x]&=\sum_{j=1}^{m}\alpha_j[U_jTS_j(x)-S_j(x)]\\ &=\sum_{j=1}^{m}[U_jTS_j(\alpha_jx)-TS_j(\alpha_jx)]+TS(x)-S(x)\end{aligned}$$

which is in the required form.

3.4 Lemma. *If $S\in G_1^{(i)}$, $T\in G^{(i+1)}$, $x\in E$ then $S[T(x)-x]\in B(G^{(i+1)})$.*

This follows from Lemma 3.3.

3.5 LEMMA. *Let H be a semi-group in $\mathfrak{C}(E)$. Suppose that x is stable with respect to H and that $U(x) \in Z(H)$ for all $U \in H$. Then x is null-stable with respect to H.*

This follows directly from the definitions.

3.6 LEMMA. *Let G be a group in $\mathfrak{C}(E)$, $x \in E$ where x is stable with respect to G_1. Then $TW(x) - W(x) \in Z(G_1)$ for all $T \in G$, $W \in G_1$.*

Proof. Set $V = (I + T + \cdots + T^{s-1})/s$. Then $V[TW(x) - W(x)] = [T^s W(x) - W(x)]/s$. Let r be the constant connected with the stability of x. Then since $T^{-s} \in G$,

$$\delta(T^s W(x), G_1, r) \leqq r\delta(T^s W(x), G_1) \leqq r\| W(x)\| .$$

Pick $U \in G_1$, $\|U\| \leqq r$ where $\|UT^s W(x)\| < r\| W(x)\| + 1$. Then

$$\|UV[TW(x) - W(x)]\| < (2r\| W(x)\| + 1)/s .$$

This shows that $TW(x) - W(x) \in Z(G_1)$.

3.7 LEMMA. *Let G be a group in $\mathfrak{C}(E)$. Let $x \in E$ where $(T-I)(x)$ is stable with respect to G_1 for all $T \in G$. Then $(T-I)U(x)$ is also stable for all $T \in G$, $U \in G$.*

Proof. Observe that $(T-I)U(x) = U(U^{-1}TU - I)(x)$. Since $(U^{-1}TU - I)(x)$ is stable with respect to G_1 it follows readily that so is $(T-I)U(x)$.

3.8 THEOREM. *Let E be a Banach space and G a solvable group of bounded linear operators on E. Let Q be the set of elements of E stable with respect to each $G_1^{(i)}$. If there exists a non-void open subset $\mathfrak{S}$ of Q such that $(T-I)\mathfrak{S} \subset Q$ for each $T \in G$ then every element of $B(G)$ is null-stable with respect to G_1. If also there is at least one element of E not null-stable with respect to G_1 then there exists a non-trivial invariant functional.*

Proof. Assume the condition on the set $\mathfrak{S}$. We show by induction starting with n, where $G^{(n)} = \{I\}$, that $B(G^{(j)})$ consists entirely of elements null-stable with respect to $G_1^{(j)}$, $j = 0, \cdots, n$. This is automatic for $j = n$; suppose that it holds for $j = i+1, \cdots, n$. Let $S, T \in G^{(i)}$, $x \in \mathfrak{S}$. In the notation of Lemma 3.3, we can write $S[T(x) - x] = z + TS(x) - S(x)$ where z is a linear combination of elements of the form $U_j TS_j(x) - TS_j(x)$, $U_j \in G^{(i+1)}$, $S_j \in G^{(i)}$. By hypothesis and Lemma 3.7, $U_j TS_j(x) - TS_j(x) \in Q$.

For any $V \in G_1^{(i)}$, $V[U_jTS_j(x)-TS_j(x)] \in B(G^{(i+1)}) \subset Z(G_1^{(i+1)}) \subset Z(G_1^{(i)})$ by Lemma 3.4 and the induction hypothesis. Hence by Lemma 3.5, $U_jTS_j(x)-TS_j(x)$ is null-stable with respect to $G_1^{(i)}$ and thus, by Lemma 3.1 so is z.

Consider the constant r connected with the null-stability of z with respect to $G_1^{(i)}$. Take $\varepsilon > 0$. Since $x \in \mathfrak{S}$, by Lemma 3.6 there exists $W \in G_1^{(i)}$ such that $\| W[TS(x)-S(x)]\| < \varepsilon/(2r)$. Furthermore there exists $R \in G_1^{(i)}$, $\|R\| \leq r$ such that $\|RW(z)\| < \varepsilon/2$. Therefore $\|RW[ST(x)-S(x)]\| < \varepsilon$ which shows that $S[T(x)-x] \in Z(G_1^{(i)})$ for all $S \in G_1^{(i)}$. Since $T(x)-x \in Q$ it follows from Lemma 3.5 that $T(x)-x$ is null-stable with respect to $G_1^{(i)}$. Let $P = \{x \in E \mid T(x)-x$ is null-stable with respect to $G_1^{(i)}\}$. By Lemma 3.1, P is a linear manifold. But $\mathfrak{S} \subset P$. Therefore $P = E$. In view of Lemma 3.1, every element of $B(G^{(i)})$ is null-stable with respect to $G_1^{(i)}$. This completes the induction.

Suppose also that some element of E is not null-stable with respect to G_1. Then (a) and (c) of Lemma 3.2 are ruled out. Thus there exists a sphere in E given by Lemma 3.2 which by the above is disjoint with $B(G)$. Hence, by the Hahn-Banach theorem there exists a bounded linear functional $\neq 0$ which vanishes on $B(G)$. This is an invariant functional.

4. Positive invariant functionals. We point out next that the arguments used above and in [7] for $B(G) \subset Z(G_1)$ have wider applicability than is apparent on the surface and in particular contain implicitly results obtained by Krein and Rutman [5].

In the terminology of [5] by a *linear semi-group* $\mathfrak{K}$ in a real normed linear space E is meant a (proper) subset of E where $\alpha x + \beta y \in \mathfrak{K}$ if x, $y \in \mathfrak{K}$ and $\alpha \geq 0$, $\beta \geq 0$ are scalars. We say that $x \leq y$ ($y \geq x$) if $y - x \in \mathfrak{K}$, $x, y \in E$. Suppose that $\mathfrak{K}$ is given with $\mathrm{Int}(\mathfrak{K})$ non-void.

Let G be a multiplicative semi-group of linear operators on E. Following [6] we call G left-solvable if there exists a finite sequence of sub-semi-groups $G = G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n)} = \{I\}$ such that given $T, U \in G^{(i)}$, $i = 0, \cdots, n-1$ there exists $V \in G^{(i+1)}$ with $TU = VUT$.

The following is an extension of [5, Theorem 3.1].

4.1 THEOREM. *Let G be a left solvable semi-group of linear operators on E such that $A(\mathfrak{K}) \subset \mathfrak{K}$, $A \in G$. Suppose that $v \in \mathrm{Int}(\mathfrak{K})$ and*

(a) *for some $\sigma > 0$, $A(v) \geq \sigma v$, $A \in G$, and*

(b) *for some $r > 0$, given $U \in G_1^{(i)}$ there exists $T \in G_1^{(i)}$ such that*

$$T(v) \leq rv, \; TU(v) \leq rv \tag{1}$$

$i = 0, \cdots, n-1$. Then there exists a bounded linear functional x^* on E, invariant with respect to G and $x^*(x) > 0$, $x \in \mathrm{Int}(\mathfrak{K})$.

Let $v \in \text{Int}(\mathfrak{K})$. As in [5] we define for each $x \in E$, $|x|_v = \inf t$, where $t > 0$ and satisfies $-tv \leqq x \leqq tv$. $|x|_v$ is a semi-norm[1] for E. Let A be a linear operator on E, $A(\mathfrak{K}) \subset \mathfrak{K}$. Since $v \in \text{Int}(\mathfrak{K})$, if $\alpha > 0$ is sufficiently large, then

$$(2) \qquad -\alpha v \leqq 0 \leqq A(v) \leqq \alpha v$$

It is easy to see that $|A(v)|_v = \inf \alpha$, $\alpha > 0$ satisfying (1). If $-tv \leqq x \leqq tv$ then for α satisfying (1),

$$-t\alpha v \leqq -tA(v) \leqq A(x) \leqq tA(v) \leqq t\alpha v$$

from which we see that $|A(x)|_v \leqq |A(v)|_v |x|_v$. Since $|v|_v = 1$ we see that A is bounded with respect to the semi-norm and

$$(3) \qquad |A|_v = |A(v)|_v \,.$$

We define $Z(G_1^{(i)})$ in terms of the semi-norm $|x|_v$. By the formulas (1), (2) and (3) it is seen that for $T \in G_1^{(i)}$ there exists $V \in G_1^{(i)}$, $|V|_v \leqq r$ where $|VT|_v \leqq r$. The arguments of [6, Theorem 3] are unaffected by the use of the semi-norm rather than a true norm. As noted by Robison [6, Theorem 6.8] in this situation we then obtain $B(G) \subset Z(G_1)$.

Let $x \in \text{Int}(\mathfrak{K})$. There exists $\alpha > 0$ such $x \geqq \alpha v$. For each $A \in G_1$, by (a), $A(x) \geqq \alpha\sigma v$. Moreover if $A(x) \leqq \beta\sigma v$, $0 < \beta < \alpha$, then $\beta\sigma v \leqq \alpha\sigma v$ which is impossible by [5, p. 11]. Hence $|A(x)|_v \geqq \alpha\sigma$. This shows that $\text{Int}(\mathfrak{K}) \cap Z(G_1) = \phi$. By the above, $B(G) \cap \text{Int}(\mathfrak{K}) = \phi$. An application of [4, Corollary 1.2] gives the existence of the desired functional.

As a consequence of Theorem 4.1 we obtain the following.

4.2. COROLLARY. *Let G be a left solvable semi-group of operators on E satisfying the requirements of Theorem* 4.1, *and let $v \in \text{Int}(\mathfrak{K})$. Then for any $w \in \text{Int}(\mathfrak{K})$, $T_j \in G$, $j = 1, 2, \cdots, n$,*

$$(4) \qquad \sum_{j=1}^{n} p_j T_j(w) \in \mathfrak{K} \text{ implies that } \sum_{j=1}^{n} p_j \geqq 0 \,.$$

When $\mathfrak{K}$ is the positive cone in a space E of bounded functions on a set S, and G is a semi-group of linear operations on E induced by a semi-group Γ of one-to-one transformations of S onto S, Hadwiger and Nef [2] have shown that the statement (4) is fundamental in the theory of integration systems.

[1] We mean $|ax| = |a||x|$, $x \in E$, a real, and $|x+y| \leqq |x| + |y|$, $x, y \in E$. (See [**1**], p. 93). In particular $|x| \geqq 0$ for all x.

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