FUNCTIONALS ASSOCIATED WITH A CONTINUOUS TRANSFOMATION

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1. Let T: $z=t(w)$, $w \in R_0$, be a continuous transformation from a simply connected polygonal region R_0, in the Euclidean plane π, into Euclidean three-space. The transformation T is a representation for an F-surface of the type of the 2-cell in Euclidean three-space, which will be called, in brief, a surface S. [4, II. 3.7, II. 3.44].

In connection with transformation T, T. Radó defines a non-negative (possibly infinite) functional $a(T)$, which he shows is independent of the representation T for the surface S. [4, V. 1.6]. Radó calls $a(T)$ the lower area of the surface, and it plays an important role in the study of surface area.

P. V. Reichelderfer has also defined a non-negative (possibly infinite) functional $eA(S)$, which he calls the essential area of the surface S. [5, p. 274]. It too is an important concept in surface area theory.

The question arises as to what relationship exists between the lower area $a(T)$ and the essential area $eA(S)$. In this paper, we show that $eA(S) = a(T)$. In addition, we introduce certain other functionals, which we show yield the same value as that of $eA(S)$ and $a(T)$. These functionals, as well as $eA(S)$ and $a(T)$, will be defined in § 3, after a discussion in § 2 of necessary topological concepts.

2. Let M be a metric space. If $A \subset M$, then $M-A$, c(A), i(A), and fr(A) denote respectively, the complement, closure, interior, and frontier of A. If $A \subset M$, $B \subset M$, then $A \cup B$, $A \cap B$, and $A-B$ denote the union, intersection, and difference of A and B. ϕ denotes the empty set. If $\{A_n\}$ is a sequence of subsets of M, then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ denote respectively the union and intersection of these sets.

Let F: $z=f(w)$, $w \in M$, be a continuous transformation from a metric space M into a metric space N. If $P \subset M$, the symbol $F|P$ denotes the transformation F with its domain restricted to P.

If $z \in N$, let $(F|P)^{-1}z$ denote the set of points w such that $w \in P$, $f(w)=z$. If $(F|P)^{-1}z \neq \phi$, then the components of $(F|P)^{-1}z$ are called maximal model components for z under $F|P$. If a maximal model component for z under $F|P$ is a continuum, then it is called a maximal model continuum (henceforth abbreviated m.m.c.) for z under $F|P$.

Now let F: $\bar{z}=f(w)$, $w \in R_0$, be a continuous transformation from a

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simply connected polygonal region R_0 in the Euclidean plane π into the Euclidean plane $\bar{\pi}$.

If R is a Jordan region, $R \subset R_0$, let $C_1, \cdots, C_{n-1}$ denote the interior boundary curves, if any, of R, oriented in the negative sense, and let C_n denote the exterior boundary curve of R, oriented the positive sense. If $\bar{z} \in F(\bigcup_{i=1}^{n} C_i)$, let $\mu(\bar{z}, F, R)=0$. If $\bar{z} \notin F(\bigcup_{i=1}^{n} C_i)$, let $\mu(\bar{z}, F, R)=\sum_{i=1}^{n} \lambda(\bar{z}, F, C_i)$, where $\lambda(\bar{z}, F, C_i)$ denotes the topological index of $\bar{z}$ with respect to the oriented closed curve $F(C_i)$, [4, II. 4.34, IV. 1.24]. If $\bar{z} \in \bar{\pi}$, then $\mu(\bar{z}, F, R)$ is an integer.

If P is a Jordan region or a domain, $P \subset R_0$, we shall call P an admissible set.

Suppose P is an admissible set, and consider $F|P$: $\bar{z}=f(w)$, $w \in P$. Suppose γ is a maximal model component for $\bar{z}$ under $F|P$. If, for every open set G containing γ, there is a Jordan region R such that $\gamma \subset \mathrm{i}(R)$, $R \subset G \cap \mathrm{i}(P)$, (note that this implies that γ is a continuum), and such that $\mu(\bar{z}, F, R) \neq 0$, then we say that γ is an essential maximal model continuum, (henceforth abbreviated e.m.m.c.), for $\bar{z}$ under $F|P$.

If P and Q are admissible sets, $Q \subset P$, and if $\bar{z} \in \bar{\pi}$, then $\kappa(\bar{z}, F|P, Q)$ will denote the number of e.m.m.c.'s for $\bar{z}$ under $F|P$ which are contained in $\mathrm{i}(Q)$. $\kappa(\bar{z}, F|P, Q)$ is possibly infinite, while, if finite, it is a non-negative integer. It may be shown that

$$\kappa(\bar{z}, F|Q, Q)=\kappa(\bar{z}, F|P, Q)=\kappa(\bar{z}, F, Q) .$$

Further, it is clear that if $P_1, \cdots, P_n$ is a collection of admissible sets with disjoint interiors, and if $P_j \subset Q$ for $j=1, \cdots, n$, then $\sum_{j=1}^{n} \kappa(\bar{z}, F, P_j) \leqq \kappa(\bar{z}, F, Q)$.

If P is an admissible set, then $\kappa(\bar{z}, F, P)$, $\bar{z} \in \bar{\pi}$, is a lower semi-continuous function, and hence is a Lebesgue measurable function. $\iint_{F(P)} \kappa(\bar{z}, F, P) d\bar{z}$ will denote the Lebesgue integral of $\kappa(\bar{z}, F, P)$ over the set $F(P)$.

3. **Let** R_0 be a simply connected polygonal region in the Euclidean plane π. We shall consider the following types of collections of sets (where it is to be understood that the collections consist of a finite number of sets, each of which is contained in R_0):

(1) Collections of disjoint simply connected polygonal regions.
(2) Collections of disjoint polygonal regions.
(3) Collections of simply connected Jordan regions with disjoint interiors.
(4) Collections of Jordan regions, with disjoint interiors.
(5) Collections of disjoint simply connected domains.

(6) Collections of disjoint domains.

Collections of the type described in (j) will be called collections of class j, $j=1, \cdots, 6$. If $A \subset R_0$, and if Φ is a collection of class j such that $R \in \Phi$ implies $R \subset A$, then we shall say that Φ is a collection of class j in A.

The transformation T: $z=t(w)$, $w \in R_0$, described in § 1, may be written T: $z=t(w)=(x_1(w), x_2(w), x_3(w))$, $w \in R_0$, where $x_1(w)$, $x_2(w)$, and $x_3(w)$ are the rectangular coordinates of $t(w)$. We now define three plane transformations.

$$T_1:\quad z_1=t_1(w)=(x_2(w),\ x_3(w)),\quad w \in R_0$$

$$T_2:\quad z_2=t_2(w)=(x_3(w),\ x_1(w)),\quad w \in R_0$$

$$T_3:\quad z_3=t_3(w)=(x_1(w),\ x_2(w)),\quad w \in R_0\,.$$

For $i=1, 2, 3$, T_i: $z_i=t_i(w)$, $w \in R_0$, is a continuous transformation from R_0 into the Euclidean plane π_i.

If P is an admissible set, (see § 2), let $g(T_i, P)=\iint_{T_i(P)} \kappa(z_i, T_i, P)dz_i$, for $i=1, 2, 3$, and let $G(T, P)=[\sum_{i=1}^{3} (g(T_i, P))^2]^{1/2}$. These quantities are non-negative and possibly infinite.

If Φ is a collection of admissible sets, let $g(T_i, \Phi)=\sum_{P \in \Phi} g(T_i, P)$, for $i=1, 2, 3$, and let $G(T, \Phi)=\sum_{P \in \Phi} G(T, P)$.

For $j=1, \cdots, 6$, let $a_j(T)=\text{l.u.b. } G(T, \Phi)$, where the least upper bound is taken with respect to all collections Φ of class j. These quantities are non-negative, possibly infinite. We note that $a_6(T)$ is precisely the lower area $a(T)$, and $a_3(T)$ is the essential area $eA(S)$, discussed in § 1, [4, V. 1.3], [5, p. 274].

The purpose of this paper is to show that the functionals $a_j(T)$, $j=1, \cdots, 6$, all yield the same value.

4. It is quite obvious from the definitions set forth in § 3. that $a_1(T) \leqq a_2(T) \leqq a_4(T)$, $a_1(T) \leqq a_3(T) \leqq a_4(T)$, and $a_5(T) \leqq a_6(T)$.

Further, if $R_1, \cdots, R_n$ is a collection of class 3, then $\text{i}(R_1), \cdots, \text{i}(R_n)$ is a collection of class 5, while, for $k=1, \cdots, n$, and $\text{i}=1, 2, 3$, we have, (see § 2), $\kappa(z_i, T_i, R_k)=\kappa(z_i, T_i, \text{i}(R_k))$. From this it follows that $a_3(T) \leqq a_5(T)$. The same type of reasoning shows that $a_4(T) \leqq a_6(T)$.

5. If D is a domain, $D \subset R_0$, then there exists a sequence $\{R_n\}$ of polygonal regions, such that $R_n \subset \text{i}(R_{n+1})$ for each n, and $\bigcup_{n=1}^{\infty} R_n = D$, [4, I. 2.48]. Then $\lim_{n \to \infty} \kappa(z_i, T_i, R_n)=\kappa(z_i, T_i, D)$, for $i=1, 2, 3$, [4; IV.

1.43], and this implies that $a_6(T) \leqq a_2(T)$.

In addition, if D is simply connected, then the polygonal regions R_n, $n=1, 2, \cdots$, may be chosen to be simply connected, and thus $a_5(T) \leqq a_1(T)$.

6. **The inequalities in** § 4 and in § 5 yield $a_1(T)=a_3(T)=a_5(T)$ and $a_2(T)=a_4(T)=a_6(T)$, while $a_1(T) \leqq a_2(T)$. To establish the equality of these six functionals, therefore, it is sufficient to show that $a_1(T) \geqq a_2(T)$.

Note that if $G(T, R_0)=+\infty$, then $a_1(T)=+\infty$, and so $a_1(T) \geqq a_2(T)$. Thus we shall assume henceforth, without loss of generality, that $G(T, R_0)<+\infty$. This in turn implies that if Φ is any collection of class j, $j=1, \cdots, 6$, then $G(T, \Phi) \leqq \sum_{i=1}^{3} g(T_i, \Phi) \leqq \sum_{i=1}^{3} g(T_i, R_0) \leqq 3G(T, R_0)$. Consequently, $a_j(T) \leqq 3G(T, R_0)<+\infty$, that is, $a_j(T)$ is finite, $j=1, \cdots, 6$.

7. **In this section,** we suppose that all sets considered are subsets of the Euclidean plane π.

Suppose A and B are connected sets, C is a closed set and $A \cup B \subset \pi - C$. We shall say that C separates A and B if A and B are contained in distinct components of $\pi - C$.

Suppose that C is closed, $C \subset R$, where R is a polygonal region. Let $Q_1, \cdots, Q_{q-1}$ denote the bounded components of $\pi - R$, (if any), and let Q_q be the unbounded component of $\pi - R$. We shall say that C separates in R if there exists k, $1 \leqq k \leqq q-1$, such that C separates Q_q and Q_k.

Let $\mathscr{G}$ be an upper semi-continuous collection of continua γ, such that $\bigcup_{\gamma \in \mathscr{G}} \gamma = R$, [4; II. 1.10]. Let E be the set of points belonging to continua of $\mathscr{G}$ which separate in R. Then E is closed. If $R-E \neq \phi$, let M be a component of $R-E$, and let $N=M \cap \mathrm{i}(R)$. Then there exist a finite number of sets, $\gamma_1, \cdots, \gamma_q$, such that either $\gamma_k=\phi$, or else γ_k is a continuum of $\mathscr{G}$, $k=1, \cdots. q$, and such that $\mathrm{fr}(N) \cap \mathrm{i}(R) \subset \bigcup_{k=1}^{q} \gamma_k$.

Suppose further that R' is a polygonal region, and $R' \subset N$. Let $Q_1', \cdots, Q_{t-1}'$ denote the bounded components of $\pi - R'$, if any, and let Q_t' denote the unbounded component $\pi - R'$. Suppose also that $Q_k' \not\subset N$, $k=1, \cdots, t$. Let $\mathscr{H}$ be an upper semi-continuous collection of continua γ' such that $\bigcup_{\gamma' \in \mathscr{H}} \gamma' = R'$, and such that if $\gamma' \in \mathscr{H}$, then there exists $\gamma \in \mathscr{G}$ for which $\gamma' \subset \gamma$. Then no continuum of $\mathscr{H}$ separates in R'.

Next, suppose $\mathscr{F}$ is an upper semi-continuous collection of continua γ, for which $\subset_{\gamma \in \mathscr{F}} \gamma = R'$, and such that no continuum of $\mathscr{F}$ separates in R. Suppose $\mathscr{L}$ is an upper semicontinuous collection of continua γ',

$\bigcup_{\gamma' \in \mathscr{L}} \gamma' = R'$, such that if $\gamma' \in \mathscr{L}$, there exists $\gamma \in \mathscr{F}$ for which $\gamma' \subset \gamma$. Then no continuum of $\mathscr{L}$ separates in R'.

8. We now state several lemmas concerning the transformation T defined in § 1 and § 3. It is assumed that $G(T, R_0) < +\infty$.

LEMMA 1. *If R is a polygonal region, $R \subset R_0$, then, for $i=1, 2, 3$, there exists a set K_i, $K_i \subset T_i(R) \subset \pi_i$, for which $m(K_i)=0$, (where $m(K_i)$ denotes the Lebesgue measure of K_i), and such that if $z_i \notin K_i$, then every m.m.c. γ for z_i under $T_i | R$ is also an m.m.c. for z_i under $T | R$.* [1; vol. 10, p. 287].

LEMMA 2. *If R is a polygonal region, $R \subset R_0$, then for $i=1, 2, 3$, there exists a set B_i, $B_i \subset T_i(R) \subset \pi_i$, for which $m(B_i)=0$, and such that $\bigcup T_i(\gamma) \subset B_i$, where the union is extended over every e.m.m.c. γ under $T_i | R$ such that $\pi - \gamma$ has more than one component.* [3; pp.593–6].

LEMMA 3. *Suppose R is a polygonal region, $R \subset R_0$. Suppose that, for $i=1, 2, 3$, F_i is a bounded Lebesgue measurable set, $F_i \subset \pi_i$. Then, given $\varepsilon > 0$, there exists a closed, totally disconnected set E_i, such that $E_i \subset F_i$ and*

$$\iint_{E_i} \kappa(z_i\, T_i, R) dz_i > \iint_{F_i} \kappa(z_i, T_i, R) dz_i - \varepsilon .$$

9. As stated previously, we wish to show that $a_j(T) = a_k(T)$, $j, k = 1, \cdots, 6$, and it was noted in § 6 that to do this, it is sufficient to show that $a_1(T) \geqq a_2(T)$ under the assumption that $G(T, R_0) < +\infty$. The proof that $a_1(T) \geqq a_2(T)$ when $G(T, R_0) < +\infty$ will be a consequence of Theorem 1 and Theorem 2, which we now consider.

THEOREM 1. *If R is a polygonal region, $R \subset R_0$, then, given $\varepsilon > 0$, there is a collection Φ_1 of class 2 in R, and a subcollection Ψ_1 of Φ_1 such that*

(a) $g(T_i, \Phi_1) > g(T_i, R) - \varepsilon$, $\quad i=1, 2, 3.$

(b) $g(T_1, \Psi_1) > g(T_1, R) - \varepsilon$.

(c) *If $\bar{R} \in \Psi_1$, then no m.m.c. under $T_1 | \bar{R}$ separates in $\bar{R}$.*

(d) *If $\bar{R} \in \Psi_1$, and if, for some i, $1 \leqq i \leqq 3$, no m.m.c. under $T_i | R$ separates in R, then no m.m.c. under $T_i | \bar{R}$ separates in $\bar{R}$.*

(There exist similar collections Φ_2, Ψ_2, and Φ_3, Ψ_3, having similar properties relative to the transformations T_2 and T_3 respectively.)

Proof. (1) If R is simply connected, then Φ_1 and Ψ_1 may both be chosen to consist of R alone.

(2) If R is not simply connected, let $Q_1, \cdots, Q_{q-1}$ denote the bounded components of $\pi-R$ and let Q_q be the unbounded component of $\pi-R$. Let $r_1, \cdots, r_q$ denote the disjoint simple closed polygons which constitute the frontier of R, in such a way that $r_k=\mathrm{fr}(Q_k)$, $k=1, \cdots, q$. Consider $T_1|R$: $z_1=t_1(w)$, $w \in R$. Let $\mathcal{G}$ denote the collection of all m.m.c.'s under $T_1|R$. Then $\mathcal{G}$ is an upper semi-continuous collection of continua γ, such that $\bigcup_{\gamma\in\mathcal{G}} \gamma=R$, and the statements of § 7 apply. Let E be the set of points which belong to m.m.c.'s under $T_1|R$ which separate in R. E is closed.

(3) If E is empty, then Φ_1 and Ψ_1 may both be chosen to consist of R alone.

If $\mathrm{i}(R)\subset E$, then $E=R$. In this case, every m.m.c. γ under $T_1|R$ is such that $\pi-\gamma$ has more than one component. Consequently, by Lemma 2, there is a set B_1, $B_1\subset T_1(R)\subset\pi_1$, $m(B_1)=0$, such that $\bigcup T_1(\gamma)\subset B_1$, where the union is extended over every e.m.m.c. γ under $T_1|R$. If $z_1\notin B_1$, we have $\kappa(z_1, T_1, R)=0$, so $g(T_1, R)=0$. Thus in this case we may let Φ_1 consist of R alone, and we may let Ψ_1 be the empty collection.

(4) From (3), we may assume $E\neq\phi$, $E\neq R$. Then $R-E\neq\phi$. $R-E$ is open relative to R, and the components of $R-E$ are open relative to R, and form at most a countably infinite collection. These components will be denoted by $C_1, C_2, \cdots$. Let $D_j=C_j\cap\mathrm{i}(R)$ for each j. D_j is non-empty, open, and connected for each j.

(5) Suppose γ is an e.m.m.c. under $T_1|R$. Then $\gamma\subset\mathrm{i}(R)$. Hence either $\gamma\subset E$ or else $\gamma\subset\mathrm{i}(R)\cap(R-E)=\bigcup_{j=1} D_j$.

In the first case, γ separates in R, and so $\pi-\gamma$ has more than one component. By Lemma 2, there is a set B_1, $B_1\subset T_1(R)\subset\pi_1$, $m(B_1)=0$, and $\bigcup T_1(\gamma)\subset B_1$, where the union is extended over every e.m.m.c. γ under $T_1|R$ for which $\pi-\gamma$ has more than one component.

In the second case, since D_j is a component of $\bigcup_{j=1} D_j$, there exists j such that $\gamma\subset D_j$. Hence γ is an e.m.m.c. under $T_1|D_j$. This implies that if $z_1\notin B_1$, then $\sum_{j=1}\kappa(z_1, T_1, D_j)=\kappa(z_1, T_1, R)$. Since $m(B_1)=0$, we have $\sum_{j=1} g(T_1, D_j)=g(T_1, R)$. There is an integer n such that $\sum_{j=1}^{n} g(T_1, D_j)>g(T_1, R)-\varepsilon/2$.

(6) For each j, $j=1, \cdots, n$, and for each k, $k=1, \cdots, q$, we have, from § 7, a set γ_{jk} such that either $\gamma_{jk}=\phi$, or else γ_{jk} is an e.m.m.c. under $T_1|R$, such that $\mathrm{fr}(D_j)\cap\mathrm{i}(R)\subset\bigcup_{k=1}^{q}\gamma_{jk}$, for each j, $j=1, \cdots, n$.

Therefore, $\bigcup_{j=1}^{n}\mathrm{fr}(D_j)\cap\mathrm{i}(R)\subset\bigcup_{j=1}^{n}\bigcup_{k=1}^{q}\gamma_{jk}$, and $T_1(\bigcup_{j=1}^{n}\mathrm{fr}(D_j)\cap\mathrm{i}(R))$ is a finite set. Also, $\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \subset (\bigcup_{j=1}^{n} \bigcup_{k=1}^{q} \gamma_{jk}) \cup \mathrm{fr}\,(R) = (\bigcup_{j=1}^{n} \bigcup_{k=1}^{q} \gamma_{jk}) \cup (\bigcup_{k=1}^{q} r_k)$.

(7) Let $F = \bigcup_{j=1}^{n} \mathrm{c}(D_j)$. F is closed, $F \subset R$, and $R-F$ is open relative to R. Let $C'_{n+1}, C'_{n+2}, \cdots$ denote the components of $R-F$. These components are open relative to R, and form at most a countably infinite collection. For each j, let $D'_{n+j} = C'_{n+j} \cap \mathrm{i}(R)$. D'_{n+j} is open and connected. (We are assuming $R-F \neq \phi$. If $R-F=\phi$, the proof is essentially the same and somewhat simpler.)

Also, it is easily seen that $\bigcup_{j=1} \mathrm{fr}\,(D'_{n+j}) \subset (\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j)) \cup (\bigcup_{k=1}^{q} r_k)$,

$(\bigcup_{j=1}^{n} D_j) \cup (\bigcup_{j=1} D'_{n+j}) \cup (\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j)) \cup (\bigcup_{j=1} \mathrm{fr}(D'_{n+j})) = R$, and

$(\bigcup_{j=1}^{n} D_j) \cup (\bigcup_{j=1} D'_{n+j}) \cup (\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j)) \cup (\bigcup_{k=1}^{q} r_k) = R$.

(8) Consider the transformation $T_2|R$: $z_2 = t_2(w)$, $w \in R$. Let γ be an e.m.m.c. under $T_2|R$. Then either γ intersects $(\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j)) \cup (\bigcup_{j=1} \mathrm{fr}\,(D'_{n+j}))$, or not.

In the first case, from (7), γ intersects $\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \cap \mathrm{i}(R)$. In (6), we have seen that $T_1(\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \cap \mathrm{i}(R))$ is a finite set, so $T_2(\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \cap \mathrm{i}(R))$ is a set of measure zero. Then $\bigcup T_2(\gamma) \subset T_2(\bigcup_{j=1}^{n} \mathrm{fr}(D_j) \cap \mathrm{i}(R))$, where the union is extended over every e.m.m.c. γ under $T_2|R$ such that $\gamma \cap ((\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j)) \cup (\bigcup_{j=1} \mathrm{fr}\,(D'_{n+j}))) \neq \phi$.

In the second case, $\gamma \subset (\bigcup_{j=1}^{n} D_j) \cup (\bigcup_{j=1} D'_{n+j})$, from (7). If there exists j, $1 \leqq j \leqq n$, such that $\gamma \cap D_j \neq \phi$, then, since γ is connected, and $\gamma \cap \mathrm{fr}\,(D_j) = \phi$, it follows that $\gamma \subset D_j$, γ is an e.m.m.c. under $T_2|D_j$.

If there is a j such that $\gamma \cap D'_{n+j} \neq \phi$, then the same reasoning shows that γ is an e.m.m.c. under $T_2|D'_{n+j}$.

Hence, if $z_2 \notin T_2(\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \cap \mathrm{i}(R))$, we have

$$\sum_{j=1}^{n} \kappa(z_2\, T_2,\, D_j) + \sum_{j=1} \kappa(z_2,\, T_2,\, D'_{n+j}) = \kappa(z_2,\, T_2,\, R).$$

Since

$$T_2(\bigcup_{j=1}^{n} \mathrm{fr}\,(D_j) \cap \mathrm{i}(R))$$

is a set of measure zero, we have

$$\sum_{j=1}^{n} g(T_2,\, D_j) + \sum_{j=1} g(T_2,\, D'_{n+j}) = g(T_2,\, R)\,.$$

(In similar fashion, $\sum_{j=1}^{n} g(T_3, D_j)+\sum_{j=1} g(T_3, D'_{n+j})=g(T_3, R)$.)

(9) Choose n' so that

$$\sum_{j=1}^{n} g(T_i, D_j)+\sum_{j=1}^{n'} g(T_i, D'_{n+j})>g(T_i, R)-\varepsilon/2, \quad \text{for} \quad i=1, 2, 3.$$

We can determine polygonal regions R_j, $j=1, \cdots, n+n'$, so that $R_j \subset D_j$ and no component of $\pi - R_j$ is contained in D_j for $j=1, \cdots, n$, and so that $R_{n+j} \subset D'_{n+j}$, and no component of $\pi - R_{n+j}$ is contained in D'_{n+j}, for $j=1, \cdots, n'$, and such that

$$\sum_{j=1}^{n+n'} g(T_i, R_j)>\sum_{j=1}^{n'} g(T_i, D_j)+\sum_{j=1}^{n'} g(T_i, D'_{n+j})-\varepsilon/2,$$

for $i=1, 2, 3$.

Let $m=n+n'$. Then

$$\sum_{j=1}^{m} g(T_i, R_j)>g(T_i, R)-\varepsilon, \qquad i=1, 2, 3,$$

and

$$\sum_{j=1}^{n} g(T_1, R_j)>g(T_1, R)-\varepsilon.$$

For each j, $j=1, \cdots, n$, consider the transformation $T_1|R_j$: $z_1=t_1(w)$, $w \in R_j$. Let $\mathscr{H}_j$ denete the collection of m.m.c.'s under $T_1|R_j$. Then $\mathscr{H}_j$ is an upper semi-continuous collection of continua γ', with $\bigcup_{\gamma' \in \mathscr{H}_j} \gamma'=R_j$. Further, if $\gamma' \in \mathscr{H}_j$, there exists $\gamma \in \mathscr{G}$, such that $\gamma' \subset \gamma$. In addition, no component of $\pi - R_j$ is contained in D_j. From § 7, no continuum of $\mathscr{H}_j$ separates in R_j, that is no m.m.c. under $T_1|R_j$ separates in R_j, $j=1, \cdots, n$.

In a similar fashion, we find from § 7 that if, for some i, $1 \leq i \leq 3$, no m.m.c. under $T_i|R$ separates in R, then no m.m.c. under $T_i|R_j$ separates in R_j, $j=1, \cdots, n$.

(10) Let Φ_1 be the collection consisting of the disjoint polygonal regions $R_1, \cdots, R_m$, and let Ψ_1 be the collection consisting of $R_1, \cdots, R_n$. These collections satisfy the requirements of the theorem. Assertions (a), (b), (c), and (d) of the theorem have been verified in (9).

10. We now prove the following.

THEOREM 2. *Let* R *be a polygonal region,* $R \subset R_0$, *and give* $\varepsilon>0$. *Let* $i_1, \cdots, i_h$, $1 \leq h \leq 3$, *denote those subscripts, if any, such that no* *m.m.c. under* $T_{i_j}|R$ *separates in* R, $j=1, \cdots, h$. *Then there exists a collection* Φ *of class* 1 *in* R *such that* $g(T_{i_j}, \Phi)>g(T_{i_j}, R)-\varepsilon$, $j=1, \cdots, h$.

Proof. We shall prove the theorem in the case where no m.m.c. under $T_i|R$ separates in R, for $i=1, 2, 3$. Then proofs in the remaining case are similar, and simpler.

(1) If R is simply connected, then Φ may be chosen to consist of R alone.

(2) If R is not simply connected, then let $Q_1, \cdots, Q_{q-1}$ denote the bounded components of $\pi-R$, and let Q_q denote the unbounded component of $\pi-R$. Let $r_1, \cdots, r_q$ denote the disjoint simple closed polygons which constitute the frontier of R in such a way that $r_k=\mathrm{fr}\,(Q_k)$, $k=1, \cdots, q$.

By Lemma 1, there is for $i=1, 2, 3$, a set K_i, $K_i\subset T_i(R)\subset\pi_i$, such that $m(K_i)=0$, and such that if γ is an m.m.c. under $T_i|R$, and if $T_i(\gamma)\notin K_i$, then γ is an m.m.c. under $T|R$. By Lemma 3, there is for $i=1, 2, 3$, a closed and totally disconnected set E_i, such that $E_i\subset(\pi-K_i)\cap T_i(R)$, and such that

$$\iint_{E_i}\kappa(z_i, T_i, R)dz_i>\iint_{(\pi-K_i)\cap T_i(R)}\kappa(z_i, T_i, R)dz_i-\frac{\varepsilon}{2}\,.$$

Since

$$\iint_{K_i}\kappa(z_i, T_i, R)dz_i=0\,,$$

we have

$$\iint_{E_i}\kappa(z_i, T_i R)dz_i>g(T_i, R)-\frac{\varepsilon}{2}\,.$$

Let $\overline{E}_i=(T_i|R)^{-1}E_i$, for $i=1, 2, 3$. Then $\overline{E}_i$ is closed, and also, the components of $\overline{E}_i$ are m.m.c.'s under $T_i|R$. No component of $\overline{E}_i$ separates in R, and $\overline{E}_i$ does not separate in R, for $i=1, 2, 3$, [2; p. 117].

(3) Let γ_1 be a component of $\overline{E}_1$. Suppose $\gamma_1\cap\overline{E}_2\neq\phi$. Then there is a component γ_2 of $\overline{E}_2$ such that $\gamma_1\cap\gamma_2\neq\phi$. γ_1 and γ_2 are, respectively, m.m.c.'s under $T_1|R$ and $T_2|R$, while $T_1(\gamma_1)\notin K_1$ and $T_2(\gamma_2)\notin K_2$. Consequently, γ_1 and γ_2 are both m.m.c.'s under $T|R$, so $\gamma_1=\gamma_2$.

Therefore, if γ_1 is a component of $\overline{E}_1$, then $\gamma_1\cap\overline{E}_2$ is connected. Thus $\overline{E}_1\cup\overline{E}_2$ does not separate in R, [2; p. 120].

Let γ_3 be a a component of $\overline{E}_3$. As above, either $\gamma_3\cap\overline{E}_1=\phi$ or else $\gamma_3\cap\overline{E}_1=\gamma_3$, and either $\gamma_3\cap\overline{E}_2=\phi$ or else $\gamma_3\cap\overline{E}_2=\gamma_3$. Hence, $\gamma_3\cap(\overline{E}_1\cup\overline{E}_2)$ is connected, and so $\overline{E}_1\cup\overline{E}_2\cup\overline{E}_3$ does not separate in R, [2; p. 120].

(4) Let $\bar{E}=\bar{E}_1\cup\bar{E}_2\cup\bar{E}_3$. $\bar{E}$ is closed, so $\pi-\bar{E}$ is open. Also, since $\bar{E}$ does not separate in R, the components Q_k, $k=1,\cdots,q$, of $\pi-R$ are contained in the same component of $\pi-\bar{E}$. Denote this component by D. Since D is open and connected, there exist polygonal arcs p_k, $k=1,\cdots,q-1$, so that for each k, $p_k\cap\bar{E}=\phi$, and $p_k\cup S_k\cup S_q$ is connected, where $S_k=Q_k\cup r_k$, $k=1,\cdots,q$.

Let $G=\mathrm{i}(R)-\bigcup_{k=1}^{q-1}p_k$. Then G is open, $G\subset R$. Let $D_1,\cdots,D_j,\cdots$ be the components of G. For each j, $D_j\subset R$, and $\mathrm{fr}(D_j)\subset\mathrm{fr}(G)\subset\pi-G=\bigcup_{k=1}^{q-1}(p_k\cup S_k\cup S_q)\cup S_q$. Then $\pi-G$ is connected, so $\pi-G$ is contained in a single component of $\pi-D_j$. But each component of $\pi-D_j$ contains just one component of $\mathrm{fr}(D_j)$, so $\pi-D_j$ has only one component, that is, D_j is a simply connected domain, [2; p. 118].

(5) If, for some i, $1\leq i\leq 3$, γ is an e.m.m.c. under $T_i|R$, then $\gamma\subset\mathrm{i}(R)$. Either $\gamma\cap(\bigcup_{k=1}^{q-1}p_k)\neq\phi$, or else $\gamma\subset G$.

In the first case, $T_i(\gamma)\notin E_i$, for otherwise

$$\gamma\subset(T_i|R)^{-1}T_i(\gamma)\subset(T_i|R)^{-1}E_i=\bar{E}_i\ ,$$

while

$$\bar{E}_i\cap(\bigcup_{k=1}^{q-1}p_k)=\phi\ .$$

Hence $\gamma\not\subset G$ implies $T_i(\gamma)\notin E_i$.

If $\gamma\subset G$, then since γ is connected, it follows that γ is contained in a component D_j of G, and γ is an e.m.m.c. under $T_i|D_j$.

Therefore, if $z_i\in E_i$, then each e.m.m.c. γ under $T_i|R$, for which $T_i(\gamma)=z_i$, is also an e.m.m.c. under $T_i|D_j$, for some j. Then

$$\iint_{E_i}\kappa(z_i,T_i,R)dz_i=\sum_{j=1}\iint_{E_i}\kappa(z_i,T_i,D_j)dz_i\ ,$$

and

$$\sum_{j=1}g(T_i,D_j)\geq\sum_{j=1}\iint_{E_j}\kappa(z_i,T_i,D_j)dz_i=\iint_{E_i}\kappa(z_i,T_i,R)dz_i>g(T_i,R)-\frac{\varepsilon}{2}\ ,$$

for $i=1,2,3$.

(6) There is an integer n for which

$$\sum_{j=1}^{n}g(T_i,D_j)>g(T_i,R)-\frac{\varepsilon}{2}\ ,$$

for $i=1, 2, 3$. Each domain D_j is simply connected, so there is a collection $R_1, \cdots, R_n$ of class 1, such that

$$R_j \subset D_j \subset R, \text{ and } g(T_i, R_j) > g(T_i, D_j) - \frac{\varepsilon}{2n},$$

for $j=1, \cdots, n$, $i=1, 2, 3$. Then

$$\sum_{j=1}^{n} g(T_i, R_j) > g(T_i, R) - \varepsilon,$$

and the collection $R_1, \cdots, R_n$ serves as the collection Φ in the statement of Theorem 2.

11. From Theorem 1 and Theorem 2, the following theorems are readily proved.

THEOREM 3. *If R is a polygonal region, $R \subset R_0$ and if $\varepsilon > 0$, then there is a collection Φ_1 of class 1 in R such that $g(T_1, \Phi_1) > g(T_1, R) - \varepsilon$.*

(Similar collections Φ_2 and Φ_3 exist relative to the transformations T_2 and T_3.)

THEOREM 4. *If R is a polygonal region, $R \subset R_0$, and if $\varepsilon > 0$, then there is a collection Φ_3 of class 1 in R such that $g(T_1, \Phi_3) > g(T_1, R) - \varepsilon$, and $g(T_2, \Phi_3) > g(T_2, R) - \varepsilon$.*

(Similar collections Φ_2 and Φ_1 exist relative to the transformations T_3 and T_1, and to the transformations T_2 and T_3.)

THEOREM 5. *If R is a polygonal region $R \subset R_0$, and if $\varepsilon > 0$, then there is a collection Φ of class 1 in R, such that $g(T_i, \Phi) > g(T_i, R) - \varepsilon$, for $i=1, 2, 3$.*

12. From Theorem 5, it follows that if R is a polygonal region, $R \subset R_0$, and if $\varepsilon > 0$, then there is a collection of class 1 in R, such that $G(T, \Phi) > G(T, R) - \varepsilon$.

This in turn implies, of course, that if Ψ is a collection of class 2 in R_0, and if $\varepsilon > 0$, then there is a collection Φ of class 1 in R_0, such that $G(T, \Phi) > G(T, \Psi) - \varepsilon$. Hence $a_1(T) \geqq a_2(T)$, and so each of the functionals $a_j(T)$, $j=1, \cdots, 6$, defined in § 3, yields the same value. We have shown in particular that the essential area of Reichelderfer, $a_3(T)$ is equal to the lower area of Radó, $a_6(T)$.

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