A CHARACTERIZATION OF W^*-ALGEBRAS

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1. Introduction. Many results have been obtained for W^*-algebras of bounded operators on a Hilbert space. However one of the most unsatisfactory parts of the present theory of W^*-algebras is the dependence on the underlying Hilbert space.

I. Kaplansky [5] has given important developments for the removal of this difficulty, but it is now known that his AW^*-algebra is not necessarily a W^*-algebra [cf. 4].

On the other hand, J. Dixmier [3] showed that a W^*-algebra is an adjoint space, and Z. Takeda has given a kind of characterization of W^*-algebras in [10].

The purpose of this paper is to give a space-free characterization in the following theorem:

THEOREM. *A C^*-algebra is a W^*-algebra if and only if it is an adjoint space, when considered as a Banach space.*

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2. Preliminaries. In this paper, we shall always deal with a C^*-algebra with unit I. Let M be a C^*-algebra, φ a linear functional. If $\varphi(a^*a) \geqq 0$ for all $a \in M$, it is said to be positive. Any positive linear functional on a C^*-algebra M is bounded and satisfies Schwarz's inequality; that is,

$$|\varphi(a^*b)| \leqq \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}$$

for all $a, b \in M$; see [8].

An AW^*-algebra [5] is a C^*-algebra satisfying the following conditions: (a) In the set of projections, any collection of orthogonal projections has a least upper bound. (b) Any maximal commutative self-adjoint subalgebra is generated by its projections.

The notion of a Stonean space was introduced by M. H. Stone [9] as follows: a compact space Ω is said to be Stonean if it has the property that the closure of any open set is open and closed. Moreover, he showed that this property is equivalent to the following property: a uniformly bounded, increasing directed set of real valued continuous functions on Ω has a continuous function as a least upper bound.

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From these results, one can prove easily that in order that a C^*-algebra $C(\Omega)$ composed of all continuous functions on a compact space Ω be an AW^*-algebra, it is necessary and sufficient that Ω be a Stonean space.

We notice that the employment of the notion of AW^*-algebra in our course of proof is not essential because the AW^*-properties of the C^*-algebra in question which are employed are not difficult to derive directly.

Also, we shall deal with locally convex topological linear spaces. For this, we shall refer in particular to [1] and [2]. The following statements are fairly well known.

Let E be a locally convex space, A a non-void closed convex set in E, K a compact convex set in E which has void intersection with A. Then there exists a hyper-plane which separates A and K strictly [1, p. 73, prop. 4].

Let F be a Banach space, F^* the adjoint space of F, $\sigma(F^*, F)$ the topology of point-wise convergence on F^*, S the unit sphere of F^*, C a convex set in F^*. Then in order that C be $\sigma(F^*, F)$-closed, it is necessary and sufficient that $C \cap \lambda S$ be $\sigma(F^*, F)$-closed for all positive numbers λ [2, th. 23].

The proof of this last statement is given by J. Dieudonné for the case that C is a vector subspace. However we observe that it is easy to extend his proof to any convex set.

3. Lemmas. Let M be a C^*-algebra which is the adjoint space of a Banach space F, S the unit sphere of M, A the self-adjoint portion of M, P the positive portion of A. Henceforward we shall always use the topology $\sigma(M, F)$ on M; it is well known that S is $\sigma(M, F)$-compact.

LEMMA 1. *A and P are $\sigma(M, F)$-closed.*

Proof. First, we shall show that $A \cap S$ is closed. If it is not closed, there is a directed set $\{x_\alpha\}$ in $A \cap S$ such that it converges to an element $a+ib$ ($b \neq 0$), where a and b are self-adjoint. Suppose that there exists a positive number $\lambda > 0$ in the spectrum of b (otherwise consider $\{-x_\alpha\}$). Then,

$$\|x_\alpha + inI\| \leqq (1+n^2)^{1/2} < \lambda + n \leqq \|b+nI\| \leqq \|a+ib+inI\|$$

for a large number n, where $\|\cdot\|$ denotes the uniform norm.

Since $\{x_\alpha + inI\}$ converges to $a+ib+inI$ and belongs to $(1+n^2)^{1/2}S$, the compactness of $(1+n^2)^{1/2}S$ means that $a+ib+inI$ belongs also to $(1+n^2)^{1/2}S$. This contradicts the above inequality; hence $A \cap S$ is closed, so that A is closed [3, th. 23].

Moreover, since $P \cap S \subset (A \cap S)+I \subset P$, we have $P \cap S=(A \cap S) \cap \{(A \cap S)+I\}$; hence $P \cap S$ is closed, so that P is closed. This completes the proof of Lemma 1.

LEMMA 2. *Let T be the totality of $\sigma(M, F)$-continuous positive linear functionals on M. Then for any self-adjoint element $a \overline{\in} P$, there is an element φ of T such that $\varphi(a) < 0$; in particular, $\psi(b)=0$ for all $\psi \in T$ implies $b=0$.*

This follows immediately from Lemma 1 and the theorem in the preliminaries [1, p. 73, prop. 4].

We call a directed set $\{x_\alpha\}$ in A increasing if

$$x_\alpha \geqq x_\beta \quad \text{whenever} \quad \alpha \geqq \beta .$$

LEMMA 3. *Every uniformly bounded, increasing directed set converges to its least upper bound. If $x=\underset{\alpha}{\text{l.u.b.}} \{x_\alpha\}$ then $a^*xa=\underset{\alpha}{\text{l.u.b.}} \{a^*x_\alpha a\}$ for any $a \in M$.*

Proof. Let E be the set of all finite linear combinations of elements of T. It is clear that the topology $\sigma(M, E)$ is weaker than the topology $\sigma(M, F)$. Moreover $\sigma(M, E)$ is a Hausdorff topology by Lemma 2; since S is $\sigma(M, F)$-compact, $\sigma(M, E)$ is equivalent to $\sigma(M, F)$ on λS $(\lambda > 0)$. Therefore, to show that a uniformly bounded directed set $\{x_\alpha\}$ is a Cauchy directed set in $\sigma(M, F)$ topology, it is enough to show that for any $\varphi \in T$ and positive number ε there is an index α_0 such that $|\varphi(x_\alpha - x_\beta)| \leqq \varepsilon$ for $\alpha, \beta \geqq \alpha_0$.

Let $\{x_\alpha\}$ be uniformly bounded and increasing. Then $\{\varphi(x_\alpha)\}$ is so for every $\varphi \in T$; hence $\{x_\alpha\}$ is $\sigma(M, F)$-Cauchy, so that by the compactness of S, it converges to some element x. Moreover, it is clear by Lemma 2 that $x=\underset{\alpha}{\text{l.u.b.}} \{x_\alpha\}$.

If u is an invertible element, then clearly

$$\underset{\alpha}{\text{l.u.b.}} \{u^* x_\alpha u\} = u^* \{\underset{\alpha}{\text{l.u.b.}} (x_\alpha)\} u = u^* x u .$$

Finally, if a is an arbitrary element of M, then there is a suitable number $\lambda > 0$ such that $\lambda I + a$ is invertible. Then

$$\varphi((\lambda I+a)^* x_\alpha (\lambda I+a)) = \lambda^2 \varphi(x_\alpha) + \lambda\varphi(a^* x_\alpha) + \lambda\varphi(x_\alpha a) + \varphi(a^* x_\alpha a)$$
$$\rightarrow \varphi((\lambda I+a)^* x (\lambda I+a)) \quad \text{for any} \quad \varphi \in T .$$

On the other hand,

$$|\varphi(a^*(x_\alpha - x_\beta))| = |\varphi(a^*(x_\alpha - x_\beta)^{1/2}(x_\alpha - x_\beta)^{1/2})|$$
$$\leqq \varphi(a^*(x_\alpha - x_\beta)a)^{1/2})\varphi(x_\alpha - x_\beta)^{1/2} \quad \text{for} \quad \alpha \geqq \beta ,$$

and analogously,

$$|\varphi((x_\alpha - x_\beta)a)| \leqq \varphi(x_\alpha - x_\beta)^{1/2}\varphi(a^*(x_\alpha - x_\beta)a)^{1/2} \quad \text{for} \quad \alpha \geqq \beta;$$

hence

$$\lambda^2\varphi(x_\alpha) + \lambda\varphi(a^*x_\alpha) + \lambda\varphi(x_\alpha a) \to \lambda^2\varphi(x) + \lambda\varphi(a^*x) + \lambda\varphi(xa) ,$$

so that $\text{l.u.b.}_\alpha \{a^*x_\alpha a\} = a^*xa$. This completes the proof of Lemma 3.

LEMMA 4. *M is an AW^*-algebra.*

Proof. Let C be any maximal commutative self-adjoint subalgebra, Ω the spectrum space of C. Then C may be regarded as the algebra of all continuous functions on Ω.

Let $\{f_\alpha\}$ be a uniformly bounded, increasing directed set and $x_0 = \text{l.u.b.}_\alpha \{f_\alpha\}$. For any unitary element u of C, $u^*f_\alpha u = u^{-1}f_\alpha u = f_\alpha$ converges to $u^{-1}x_0u = x_0$; as C is maximal, x_0 belongs to C and so Ω is Stonean.

Let $\{e_\alpha | \alpha \in Q\}$ be an orthogonal family of projections, J any finite subset of Q and put $p_J = \sum_{\alpha \in J} e_\alpha$. Then the set $\{p_J\}$ is directed by set-inclusion and is uniformly bounded and increasing, and so admits a least upper bound p. Moreover, any maximal commutative self-adjoint subalgebra including $\{e_\alpha | \alpha \in Q\}$ contains p; it follows that p is a projection. This completes the proof of Lemma 4.

LEMMA 5. *Let e be any projection of M. Then the subalgebra eMe is $\sigma(M, F)$-closed, and moreover the mapping $x \to exe$ is $\sigma(M, F)$-continuous.*

Proof. $e(P \cap S)e$ consists clearly of those elements of $P \cap S$ which are $\leqq e$. If $\{x_\alpha\}$ is a directed set in $e(P \cap S)e$ converging to an element $x_0 \geqq 0$, then $e - x_\alpha \geqq 0$, so that $e - x_0 \geqq 0$; hence $e(P \cap S)e$ is closed. Since $e(A \cap S)e = e(P \cap S)e - e(P \cap S)e$, the compactness of $e(P \cap S)e$ implies that $e(A \cap S)e$ is closed; hence eMe is closed [2, th. 23].

Next, we shall show the continuity of the mapping. For this, it is enough to show that the kernel $(I-e)M + M(I-e)$ of the mapping is closed, because M is an algebraic direct sum of eMe and $(I-e)M + M(I-e)$.

Now, we shall show that if $\{ea_\alpha(I-e)\}$ $(a_\alpha \in A \cap S)$ converges to a, then $eae = (I-e)a(I-e) = 0$. For any integer n and complex number c $(|c| = 1)$,

$$\|ea_\alpha(I-e)+cne\|=\|\{ea_\alpha(I-e)+cne\}\{(I-e)a_\alpha e+\bar{c}ne\}\|^{1/2}$$
$$=\|ea_\alpha(I-e)a_\alpha e+n^2e\|^{1/2}\leqq(1+n^2)^{1/2}.$$

Now suppose that $eae\neq 0$ and there is a positive number $\lambda>0$ in the spectrum of $\frac{eae+ea^*e}{2}$ (otherwise consider $(-a_\alpha)$), then

$$\|eae+ne+ea(I-e)+(I-e)ae+(I-e)a(I-e)\|\geqq\|e(a+nI)e\|$$
$$\geqq\left\|\frac{eae+ea^*e}{2}+ne\right\|\geqq\lambda+n.$$

Therefore.

$$\|a+ne\|>(1+n^2)^{1/2}$$

for a large number n. This is a contradiction; hence

$$\frac{eae+ea^*e}{2}=0,$$

and analogously

$$\frac{iea^*e-ieae}{2}=0,$$

so that $eae=0$.

Similarly, suppose that $(I-e)a(I-e)\neq 0$. Then

$$\|ea_\alpha(I-e)+cn(I-e)\|=\|(I-e)a_\alpha e+\bar{c}n(I-e)\}\{ea_\alpha(I-e)+cn(I-e)\}\|^{1/2}$$
$$=\|(I-e)a_\alpha ea_\alpha(I-e)+n^2(I-e)\|^{1/2}\leqq(1+n^2)^{1/2};$$

hence we shall obtain an analogous contradiction, so that $a=ea(I-e)+(I-e)ae$; hence the closure of $(I-e)Se$ is contained in $eM(I-e)+(I-e)Me$.

By symmetry, the closure of $eS(I-e)$ is contained in $eM(I-e)+(I-e)Me$. From the above discussion and the compactness of S, we easily conclude that $eS(I-e)+(I-e)Se$ is closed, so that $eM(I-e)+(I-e)Me$ is closed; hence

$$(I-e)M+M(I-e)=(I-e)Me+eM(I-e)+(I-e)M(I-e)$$

is closed. This completes the proof of Lemma 5.

Now, define

$$V(\varphi,\ \varepsilon)=\{x\,|\,x\in P,\ \varphi(x)\leqq\varepsilon,\ \varepsilon>0 \text{ and } \varphi\in T\}$$

and

$$V^{1/2}(\varphi, \varepsilon)=\{x^{1/2} \mid x \in V(\varphi, \varepsilon)\} .$$

LEMMA 6. *The $\sigma(M, F)$-closure of*

$$\{V^{1/2}(\varphi, \varepsilon)-V^{1/2}(\varphi, \varepsilon)\} \cap \lambda S$$

is contained in

$$2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S-2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S$$

for any positive number λ.

Proof. For any $u \in S$ and $h \in V(\varphi, \varepsilon)$,

$$|\varphi(uh^{1/2})| \leqq \varphi(uu^*)^{1/2}\varphi(h)^{1/2} \leqq \|uu^*\|^{1/2}\varphi(I)^{1/2}\varepsilon^{1/2} \leqq \varphi(I)^{1/2}\varepsilon^{1/2} .$$

Now, if $a=b-c$, where $b, c \in V^{1/2}(\varphi, \varepsilon)$, then a is uniquely expressed as follows: $a=a_1-a_2$ ($a_i \in P$) where $a_1 \cdot a_2=0$. Since the spectrum space of any maximal commutative self-adjoint subalgebra of M is Stonean, there is a projection e such that $ea=a_1$ and $(e-I)a=a_2$. Since e and $(e-I)$ belong to S, we have, by the above inequality,

$$|\varphi(ea)|=|\varphi(eb)-\varphi(ec)| \leqq \varphi(I)^{1/2}\varepsilon^{1/2}+\varphi(I)^{1/2}\varepsilon^{1/2} ,$$

so that $ea=a_1$ belongs to $2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2})$. Analogously $(e-I)a=a_2$ belongs to $2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2})$; hence a belongs to $2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2})-2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2})$.

On the other hand, $a \in \lambda S$ means $a_1, a_2 \in \lambda S$. Therefore,

$$\{V^{1/2}(\varphi, \varepsilon)-V^{1/2}(\varphi, \varepsilon)\} \cap \lambda S \subset 2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S-2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S .$$

Since $V(\varphi, \varepsilon)$ is closed, $2V(\varphi, \varepsilon) \cap \lambda S$ is compact, so that

$$2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S-2V(\varphi, \varphi(I)^{1/2}\varepsilon^{1/2}) \cap \lambda S$$

is compact. This completes the proof of Lemma 6.

LEMMA 7. *Let $\{x_\alpha\}$ be a directed set in $A \cap S$ such that for any $\varepsilon>0$ and $\varphi \in T$ there is an index α_0 such that $\varphi((x_\alpha-x_\beta)^2) \leqq \varepsilon$ for all $\alpha, \beta \geqq \alpha_0$. Then, (a) x_α converges to a unique element x_0; (b) The difference x_0-x_α may be expressed as follows: $x_0-x_\alpha=y_\alpha-z_\alpha$, where y_α, $z_\alpha \in 2(P \cap S)$, and $\{y_\alpha\}$ and $\{z_\alpha\}$ converge to 0.*

Proof. Let us write $x_\alpha-x_\beta$ as follows: $x_\alpha-x_\beta=y_{\alpha,\beta}-z_{\alpha,\beta}$, where $y_{\alpha,\beta}, z_{\alpha,\beta} \in 2(P \cap S)$ and $y_{\alpha,\beta} \cdot z_{\alpha,\beta}=0$. Then $(x_\alpha-x_\beta)^2=y^2_{\alpha,\beta}+z^2_{\alpha,\beta}$. From the assumptions, for any $\varepsilon(>0)$ and $\varphi \in T$ there is an index α_0 such that $\varphi((x_\alpha-x_\beta)^2)\leqq\varepsilon$ for α, $\beta\geqq\alpha_0$; hence $\varphi(y^2_{\alpha,\beta})\leqq\varepsilon$ and $\varphi(z^2_{\alpha,\beta})\leqq\varepsilon$, so that we have $x_\alpha-x_\beta\in\{V^{1/2}(\varphi,\ \varepsilon)-V^{1/2}(\varphi,\ \varepsilon)\}\cap 2S$ for α, $\beta\geqq\alpha_0$. Since $|\varphi(x_\alpha-x_\beta)|\leqq\varphi(I)^{1/2}\varphi((x_\alpha-x_\beta)^2)^{1/2}$ and $\{x_\alpha\}$ is the directed set in question, $\{x_\alpha\}$ is Cauchy and lies in S, so that it has a unique adherent point $x_0\in S$. Hence by Lemma 6 and the above relation we have

$$x_0-x_\beta\in\{2V(\varphi,\ \varphi(I)^{1/2}\varepsilon^{1/2})\cap 2S-2V(\varphi,\ \varphi(I)^{1/2}\varepsilon^{1/2})\cap 2S\}$$

for $\beta\geqq\alpha_0$. This completes the proof of Lemma 7.

LEMMA 8. *Let (x_α) be a directed set in $A\cap S$ satisfying the assumption of Lemma 7, and x_0 be its limit. Then $\{\varphi(ax_\alpha b)\}$ converges to $\varphi(ax_0b)$ for any a, $b\in M$ and $\varphi\in T$.*

Proof. Because of the usual polarisation identity, it is enough to show that $\{\varphi(a^*x_\alpha a)\}$ converges to $\varphi(a^*x_0a)$. Let e be a projection, ψ an element of T such that $\varphi(a^*ea)\leqq\psi(e)$. Suppose that for all nonzero projections $f<e$, there exists a nonzero projection $g\leqq f$ such that $\varphi(a^*ga)>\psi(g)$. Then, let $\{g_\alpha\}$ be a maximal family of mutually orthogonal projections such that $\varphi(a^*g_\alpha a)>\psi(g_\alpha)$. If J is the set of indices α, K the set of all finite sets of J, then K is a directed set by the set-inclusion. For $\gamma\in K$, put $a_\gamma=\sum_{\alpha\in\gamma}g_\alpha$; then $\{a_\gamma\}$ is uniformly bounded and increasing, so by Lemma 3 and our assumption $\lim a_\gamma=e$ and $\lim_\gamma a^*a_\gamma a=a^*ea$. Hence,

$$\varphi(a^*ea)=\lim_\gamma\varphi(a^*a_\gamma a)=\lim_\gamma\sum_{\alpha\in\gamma}\varphi(a^*g_\alpha a)$$
$$>\lim_\gamma\sum_{\alpha\in\gamma}\psi(g_\alpha)=\psi(e)\,.$$

This is a contradiction; hence there is a nonzero projection $f\leqq e$ such that $\varphi(a^*ga)\leqq\psi(g)$ for all projections $g\leqq f$. By Lemma 4 and Lemma 5, fMf is an AW^*-algebra, so that any positive element is a uniform limit of finite positive linear combinations of mutually orthogonal projections. Therefore,

$$\varphi(a^*ba)\leqq\psi(b)\quad\text{for any}\quad b\in fMf\cap P\,.$$

Now, let $\{x_\alpha\}$ be a directed set satisfying the assumption of Lemma 7, and suppose that $\{x_\alpha\}$ converges to x_0. Then by Lemma 7, x_0-x_α may be expressed as $y_\alpha-z_\alpha$, where y_α, $z_\alpha\in 2(P\cap S)$, and $\{y_\alpha\}$ and $\{z_\alpha\}$ converge to 0,

Then by Schwarz's inequality,

$$|\varphi(a^*y_\alpha fa)|\leqq\varphi(a^*a)^{1/2}\varphi(a^*fy^2_\alpha fa)^{1/2}\leqq\varphi(a^*a)^{1/2}\psi(fy^2_\alpha f)^{1/2}$$

$$\leq \varphi(a^*a)^{1/2}\psi(fy_\alpha f)^{1/4}\psi(fy_\alpha^3 f)^{1/4} .$$

By Lemma 5 the mapping $x \to fxf$ is continuous, so $\psi_1(x)=\psi(fxf)$ is continuous; hence from the above inequality $\{\varphi(a^*y_\alpha fa)\}$ converges to 0. If $\{f_\beta|\beta \in J\}$ is a maximal family of mutually orthogonal nonzero projections such that $\lim_\alpha \varphi(a^*y_\alpha f_\beta a)=0$, then, from the preceding discussion, $\sum_{\beta \in J} f_\beta = I$ where the sum $\sum_{\beta \in J} f_\beta$ is the limit of the uniformly bounded, increasing directed set $\{\sum_{\beta \in J_\gamma} f_\beta\}$ (J_γ is any finite subset of J).

Put $h_{J_\alpha} = \sum_{\beta \in J_\alpha} f_\alpha$ for all subsets J_α of J. Then for all $\varepsilon > 0$ there is a finite set J_1 such that $\varphi(a^*h_{J_2}a) \leq \varepsilon$ for $J_2 = J - J_1$, because $\lim_\gamma a^*h_{J_\gamma}a = a^*Ia$ by Lemma 3. Therefore,

$$|\varphi(a^*y_\alpha h_{J_2}a)|^2 \leq \varphi(a^*y_\alpha^2 a)\varphi(a^*h_{J_2}a) \leq \|y_\alpha^2\|\,\varphi(a^*a)\varepsilon$$

$$\leq 4\varphi(a^*a)\varepsilon .$$

On the other hand, there is an index α_0 such that $\alpha > \alpha_0$ means $|\varphi(a^*y_\alpha h_{J_1}a)| \leq \varepsilon$. Hence,

$$|\varphi(a^*y_\alpha a)| \leq |\varphi(a^*y_\alpha h_{J_1}a)| + |\varphi(a^*y_\alpha h_{J_2}a)|$$

$$\leq \varepsilon + 2\varphi(a^*a)^{1/2}\varepsilon^{1/2} \quad \text{for} \quad \alpha \geq \alpha_0 .$$

This means that

$$\lim_\alpha \varphi(a^*y_\alpha a) = 0 ,$$

and analogously

$$\lim_\alpha \varphi(a^*z_\alpha a) = 0 ;$$

hence

$$\lim_\alpha \varphi(a^*x_\alpha a) = \varphi(a^*x_0 a) .$$

This completes the proof of Lemma 8.

4. Proof of Theorem. We shall prove the Theorem in this section. Since it was shown by J. Dixmier [3, prop. 1] that a W^*-algebra is an adjoint space, it is enough to show the converse.

Let T_1 be the totality of $\sigma(M, F)$-continuous positive linear functionals such that $\varphi(I)=1$. Let $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ be the C^*-representation of M on a Hilbert space $\mathfrak{H}_\varphi$, constructed via the element φ of T_1 [cf. 11]. Let $\mathfrak{H}$ be the direct sum of the $\mathfrak{H}_\varphi$: $\mathfrak{H} = \sum_{\varphi \in T_1} \oplus \mathfrak{H}_\varphi$. We shall consider a representation π of M on $\mathfrak{H}$ defined as follows: $\pi(x) = \sum_{\varphi \in T_1} \pi_\varphi(x)$. Then it is easily seen from Lemma 2 that π is a faithful C^*-representation of M. Let B be the algebra composed of all bounded operators on $\mathfrak{H}$. In order to avoid confusion, we shall call the usual weak topology (strong topology) on B the weak operator topology (strong operator topology).

Let $\{x_\alpha\}$ $(x_\alpha \in A \cap S)$ be a uniformly bounded Cauchy directed set with the strong operator topology. Then,

$$\|(\pi(x_\alpha)-\pi(x_\beta))I_\varphi\|^2=\varphi((x_\alpha-x_\beta)^2) \rightarrow 0 ,$$

where I_φ is the image of I in $\mathfrak{H}_\varphi$ and $\|\cdot\|$ is the norm of $\mathfrak{H}$; hence $\{x_\alpha\}$ satisfies the assumption of Lemma 7, therefore by Lemma 8,

$$\varphi(a^*x_\alpha b)=<\pi(x_\alpha)b_\varphi,\ a_\varphi>$$

converges to

$$\varphi(a^*x_0 b)=<\pi(x_0)b_\varphi,\ a_\varphi>$$

for any a, $b \in M$, where $<, >$ denotes the inner product in $\mathfrak{H}$ and a_φ, b_φ are the images of a, b in $\mathfrak{H}_\varphi$.

Since linear combinations of the images M_φ of M in $\mathfrak{H}_\varphi$ are dense in $\mathfrak{H}$, the uniform boundedness and the above relation mean that $\{\pi(x_\alpha)\}$ converges to $\pi(x_0)$ in the weak operator topology [cf. 10].

On the other hand, $\{\pi(x_\alpha)\}$ converges to an element a of B in the strong operator topology; hence $a=\pi(x_0)$ and so $\pi(A \cap S)$ is strongly closed. Finally by the result of J. Dixmier [3, prop. 1], $\pi(A \cap S)$ is weakly closed, so that $\pi(M)$ is weakly closed. This completes the proof of the Theorem.

REMARK 1. We notice that $\sigma(M, F)$ is equivalent to the weak operator topology on S.

REMARK 2. It would be desirable to find a proof of the Theorem which does not make use of projections. If we use projections fully, the discussions of Lemmas 6–8 can be replaced by the shorter ones outlined in the following: Suppose that $\{ea_\alpha(I-e)\}$ $(a_\alpha \in S)$ converges to a and $(I-e)ae \neq 0$. Since by Lemma 5,

$$a=ea(I-e)+(I-e)ae ,$$

$$\begin{aligned}\|a+n(I-e)ae\|&=\|ea(I-e)+(n+1)(I-e)ae\|\\&=\max\{\|ea(I-e)\|,\ (n+1)\|(I-e)ae\|\} ;\end{aligned}$$

hence

$$\|a+n(I-e)ae\|=(n+1)\|(I-e)ae\|$$

for a large number n. On the other hand,

$$\|ea_\alpha(I-e)+n(I-e)ae\|\leqq \max\{1,\ n\|(I-e)ae\|\}=n\|(I-e)ae\|$$

for a large number n, and this contradicts the above equality; hence $eM(I-e)$ is closed. Therefore, the mappings $x\to ex(I-e)$ and $(I-e)xe$, and so ex and xe are $\sigma(M, F)$-continuous; finally we can easily conclude that the mappings $x\to ax$ and xa, and so a^*xa are $\sigma(M, F)$-continuous for all $a\in M$.

On the other hand, it is easily seen that Lemma 7 can be proved without the use of projections, and moreover, by a slight modification of the proof of Lemma 8, we can immediately show that the use of projections is unnecessary in the commutative case. From these facts it seems that a suitable proof for the demand can be found, when the strong operator topology used.

APPENDIX. We proved the Theorem under the assumption that the C^*-algebra has a unit. But this assumption is unnecessary, since we can prove the following result.

THEOREM. *Let N be a C^*-algebra, S its unit sphere and suppose that S has an extreme point. Then N has a unit.*

The proof is obtained by a suitable modification of the discussion of R. Kadison [*Isometries of operator algebras*, the proof of Theorem 1, Ann. of Math 54 (1951)].

Proof. Let x be an extreme point of S, A the commutative C^*-subalgebra generated by x^*x and $C_0(\Omega)$ the function-representation of A, where Ω is a locally compact space.

Then one can easily take a sequence (y_n) of positive elements of $C_0(\Omega)$ such that $\|y_n\|\leqq 1$ for all n, $\|(x^*x)y_n-(x^*x)\|\to 0$ $(n\to\infty)$ and $\|(x^*x)y_n^2-(x^*x)\|\to 0$ $(n\to\infty)$.

Suppose that at some point p of Ω, x^*x takes a nonzero value less than one. Then we can take a positive element c of A, nonzero at p, such that if $r_n=y_n+c$, $s_n=y_n-c$, then $\|(x^*x)r_n^2\|\leqq 1$ and $\|(x^*x)s_n^2\|\leqq 1$. Hence xr_n and xs_n are in S.

On the other hand, $\|(xy_n-x)^*(xy_n-x)\|=\|x^*xy_n{}^2-x^*xy_n-x^*xy_n+x^*x\|\to 0$ $(n\to\infty)$. Hence $xy_n\to x$, so that $xr_n\to x+xc$, and $xs_n\to x-xc$. Since xr_n and xs_n are in S, so are $x+xc$ and $x-xc$. Therefore, by the discussion of Kadison, x is partially isometric.

Now let us represent an operator algebra N on a Hilbert space $\mathfrak{H}$, and let $\tilde{N}$ be the weak closure of N on $\mathfrak{H}$. By the discussion of Kadison, $N \supseteqq (I-xx^*)N(I-x^*x)=(0)$, where I is the unit of $\tilde{N}$, so that $(I-xx^*)\tilde{N}(I-x^*x)=0$. Hence $I=xx^*+x^*x-xx^*x^*x$ belongs to N. This completes the proof.

If a C^*-algebra is an adjoint space, by the theorem of Krein-Milman the unit sphere has an extreme point. Hence the algebra has a unit by the above theorem.

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