A GEOMETRIC PROBLEM OF SHERMAN STEIN

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1. Introduction. Recently, Sherman Stein [1] has proposed the following problem:

Let $J \subset R_2$ be a rectifiable Jordan curve, with the property that for each rotation R. there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nonzero length. Must J contain the arc of a circle?

We interpret "length" to be the measure induced on J by arc length, and in § 2 we give an example to show that J need not contain the arc of a circle. In § 3 we show that if "nonzero length" is replaced by "nondegenerate component", then J must necessarily contain an arc of a circle.

2. An example. Let C be a circle in R_2, and let L be the circumference of C. Using standard arguments, we can obtain a subset D of C which is open relative to C, which is dense in C, and which has length less than $L/3$. We define J to be the point set which is obtained if we modify C by replacing each component K of D by the line segment whose end points are the end points of K. J is obviously a rectifiable Jordan curve. If R is a rotation, we choose T in such a way that TR maps C onto C. It follows that $(TRJ) \cap J$ contains $C-(D \cup TRD)$. Since D and TRD each have length less than $L/3$, we see that $(TRJ) \cap J$ has length greater than $L/3$. The curve J which we have defined satisfies the conditions of Stein's problem, but J does not contain an arc of a circle.

3. A theorem about Jordan curves. Before stating our theorem, it is convenient to prove first a key lemma about arcs in R_2. It seems to the author that this lemma is quite interesting in itself.

LEMMA. *If A and B are topological arcs in R_2 and A contains an infinite number of subarcs, each of which is congruent to B, then B is either an arc of a circle or a segment of a straight line.*

Proof. We assign natural linear orderings to A and B, and define G to be the set of all isometries of R_2 onto R_2 which map B into A. Either an infinite number of members of G are order preserving or an infinite number of members of G are order reversing, and we may

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assume without loss of generality that an infinite number of members of G are order preserving. We define S to be the set of all subarcs of A which are the images of B under order preserving members of G. Let a be one of the end points of A. For each $s \in S$, we define s^* to be the end point of s which is between a and the other end point of s. There exists an arc $\sigma \in S$ such that σ^* is a limit point of the set of points s^* for $s \in S$.

Suppose that σ is not an arc of a circle or a segment of a straight line. Then, there exist four points Q_1, Q_2, Q_3, Q_4 on σ which do not all lie on any one circle or line. There exists $\varepsilon > 0$ such that if q_1, q_2, q_3, q_4 are points on σ and the distance from q_i to Q_i is less than 2ε for $i=1, 2, 3, 4$ then the points q_1, q_2, q_3, q_4 do not all lie on any one circle or line.

We now choose $\tau \in S$ such that the subarc of A from σ^* to τ^* is nondegenerate and has diameter less than ε. It is easy to see that $\sigma \cap \tau$ must be a nondegenerate arc and either $\tau^* \in \sigma$ or $\sigma^* \in \tau$. We may assume without loss of generality that $\tau^* \in \sigma$.

Next, we let f be an isometry of R_2 onto itself that maps σ onto τ with $f(\sigma^*)=\tau^*$. There exists a maximal finite sequence $p_1, p_2, \cdots, p_n$ of points on σ such that $p_1=\sigma^*$ and $p_k=f(p_{k-1})$ for $1<k\leqq n$. It is easy to see that the straight line segments $\overline{p_k p_{k+1}}$ in R_2 are all the same length for $k=1, \cdots, n-1$, and that the straight line segments $\overline{p_k p_{k+2}}$ are all the same length for $k=1, \cdots, n-2$. Thus, the angles formed by the segments $\overline{p_k p_{k+1}}$ and $\overline{p_{k+1} p_{k+2}}$ are all the same for $k=1, \cdots, n-2$.

If f is orientation preserving on R_2, then it follows that the points $p_1, p_2, \cdots, p_n$ all lie on some circle in R_2; if f is orientation reversing on R_2, then the points p_k, for odd k, lie on a straight line in R_2, and the points p_k, for even k, lie on a parallel line. In either case, there exists either a circle or a line which contains all of the points p_k, for odd k.

New, we choose odd integers $k(1), k(2), k(3), k(4)$ such that the distance from $p_{k(i)}$ to Q_i is less than 2ε for $i=1, 2, 3, 4$. Finally, we obtain a contradiction by letting $q_i=p_{k(i)}$ for $i=1, 2, 3, 4$. Thus σ, and hence also B, must be either an arc of a circle or a segment of a straight line.

We are now ready for our theorem.

THEOREM. *If $J \subset R_2$ is a (not necessarily rectifiable) Jordan curve, and H is an uncountable set of rotations about some one point such that for each $R \in H$ there is a translation T such that $(TRJ) \cap J$ has a nondegenerate component, then J contains an arc of a circle.*

Proof. Let E be a countable dense subset of J, and let F be the set of all subarcs of J whose end points are members of E. It is easily verified that if $R \in H$ and T is a translation for which $(TRJ) \cap J$ has a nondegenerate component, then there exist arcs U and V in F such that $TRU \subset V$. Since H is uncountable and there are only a countable number of pairs U, V of members of F, there exist arcs A, B in F and an uncountable subset H' of H such that for each $R \in H'$ there is a translation T such that $TRB \subset A$. A given subarc of A can be expressed in the form TRB for at most two rotations R in H', and hence there is an infinite number of subarcs of A which are congruent to B. By our lemma, B is either an arc of a circle or a segment of a line. Since A contains subarcs of the form TRB for an infinite number of rotations R, it is easily seen that B cannot be a line segment. It follows that A, and hence also J, contains an arc of a circle.

By making use of the example defined in § 2, it is easy to show that it is not possible to replace "uncountable" by "infinite" in our theorem.

REFERENCE

1. Bull. Amer. Math. Soc., **61** (1955), 465, research problem 25.

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