ZERO-DIMENSIONAL COMPACT GROUPS OF HOMEOMORPHISMS

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1. Introduction. All spaces and topological groups referred to in this paper will be compact and metric. All topological groups will additionally be zero-dimensional, that is, either finite or homeomorphic to a Cantor set. As general references we cite Zippin [6] and Montgomery and Zippin [4]. Several of our definitions are similar to those in [6].

A *topological transformation group* of a topological space is an association of a topological group G and a topological space E in the sense that each element g of G and point x of E determine a unique point of E. If this point be called x', we write $gx=x'$. The association is subject to the following conditions:

(1) if e denotes the identity of G, $ex=x$ for all $x \in E$,

(2) $g(g'x)=(gg')x$, $g, g' \in G$, $x \in E$, and

(3) gx is continuous simultaneously in g and x.

Each element of G may, under the association, be regarded as a homeomorphism of E onto itself.

The topological transformation group G is said to be *effective* if for each $g \in G$ not the identity, there is an $x_g \in E$ for which $gx_g \neq x_g$ and is said to be *strongly effective* (or *fixed-point-free*) if for each $g \in G$ not the identity and for each $x \in E$, $gx \neq x$. We shall use the symbol $Tg(G, E)$ to denote a particular association of G with E such that G is an effective topological transformation group of E. Thus by $Tg(G, E)$ we mean a particular group of homeomorphisms of E onto itself, the group being isomorphic to and identified with G. If $Tg(G, E)$ is strongly effective we write $TgS(G, E)$.

For $x \in E$, $G(x)$ will denote the set of all images of x under G and will be called the orbit of x under G. Similarly for $X \subset E$, $G(X)$ will denote the set of images of X under G. The individual orbits may be regarded as the "points" of a space, the orbit space, $O[Tg(G, E)]$ of $Tg(G, E)$. $O[Tg(G, E)]$ is a continuous decomposition of E.

The main purpose of this paper is to prove the following theorems:

THEOREM 1. *Let G be any compact zero-dimensional topological group. Let M be the universal curve.*[1] *Then there exists a* $TgS(G, M)$

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[1] The universal curve is a particular one-dimensional locally connected continuum. Its description and a characterization of it are given in § 3.

such that $O[TgS(G, M)]$ is homeomorphic to M.

THEOREM 2. *Let G be any infinite compact zero-dimensional topological group. Let M be the universal curve. Then there exists a $TgS(G, M)$ such that $O[TgS(G, M)]$ is a regular curve*[2].

Theorem 1 asserts that the universal curve is also universal in the sense that every compact zero-dimensional group can operate on it in a fixed-point-free fashion. It is well known and is easy to prove—see Example 1—that the Cantor set also has this property.

The following two theorems are corollaries of some of the methods used in the proofs of theorems 1 and 2. In particular, the argument of § 5 gives the essential structure of an argument for Theorem 3. Theorem 4 is a corollary of Theorem 3.

THEOREM 3. *Let G be any finite group. Then there exists in E^3 a 3-manifold M with connected boundary such that $TgS(G, M)$ exists.*

THEOREM 4. *Let G be any finite group. Then there exists in E^3 a 2-manifold K (without boundary) such that $TgS(G, K)$ exists.*

Any zero-dimensional compact group G can be expressed as the inverse (or projective) limit (simultaneously in both a topological and a group sense) of a sequence $\{G_i\}$ of finite groups under a sequence $\{\pi_i\}$ of homomorphisms with, for each i, π_i carrying G_{i+1} onto G_i (see §§ 2.5–2.7 of [4]). The group G is said to be p-adic if, for each i, G_i can be taken as a cyclic group with, for each i, π_i not an isomorphism. If G is a p-adic group and sequences $\{G_i\}$ and $\{\pi_i\}$ exist such that, for each i, π_i is two-to-one then G is called the dyadic group.

AGREEMENT 1. *We shall assume henceforth that G is a particular compact zero-dimensional topological group.*

AGREEMENT 2. *We shall assume that sequences $\{G_i\}$ and $\{\pi_i\}$ with respect to which G is an inverse limit are given and to avoid subdivision of the ensuing arguments into cases we shall further assume that G is infinite and that, for no i, is π_i an isomorphism.*

It will be clear that the argument we give for Theorem 1 actually includes the essentials of the argument for the case of G finite.

[2] A locally connected continuum is said to be a regular curve provided every point of it has arbitrarily small neighborhoods with finite boundaries or, equivalently, provided every pair of points of it can be separated by a finite point set,

NOTATION. Let e be the identity of G and, for each i, let e_i be the identity of G_i. For each i, let $n(G_i)$ be the number of elements in G_i.

REMARKS. At the heart of the theory of topological transformation groups is the open question as to whether any infinite compact zero-dimensional group can operate effectively on a Euclidean manifold E. In studying such a question it is natural to consider the "nice" spaces on which such a group can operate and to consider the characteristics of the group operation[3]. Zippin [6] has observed that the known examples of even the dyadic group D effective on locally connected continua involve a type of "branching" about subsets on which D is not strongly effective, and, in fact, usually a type of "branching" about points or sets which have periodic orbits under G (see Example 2). Thus our theorems and arguments contribute to the knowledge of the ways zero-dimensional infinite compact groups can operate on locally connected continua. In this connection, we also note in Example 3 that any p-adic group can be strongly effective on the infinite dimensional compact torus.

We mention the following questions: For E a continuum and G infinite, is it possible for $TgS(G, E)$ to be such that the dimension of $O[TgS(G, E)]$ exceeds the dimension of E? If such is possible, can E be one-dimensional ?, locally connected ?, the universal curve ?, locally Euclidean ? What are conditions on E for which $\dim(O[TgS(G, E)])$ must be $\leqq \dim E$?

In the classic example of Kolmogoroff [3], G (not made explicit by him) operated effectively but not strongly effectively on a one-dimensional locally connected continuum E, and $O[Tg(G, E)]$ was two-dimensional. The more recent example by Keldys [2] of a light open mapping of a one-dimensional continuum onto a square also involved a "branching" type operation.

2. Examples. In this section we wish to give three examples of topological transformation groups. Of these A and B, at least, are

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[3] Smith, in [5], states "There exist, however, nearly periodic transformations which are not periodic. In all known examples the space M under transformation is of a highly irregular local structure which suggests the problem referred to above: Can there exist a non-periodic nearly periodic transformation T operating in M if M is fairly regular in its local structure, for example, locally Euclidean." If G is a p-adic group, if $TgS(G, M)$ exists, and if $g \in G$ with $g \neq e$, then g as a homeomorphism of M is a non-periodic nearly periodic transformation. As the universal curve is homogeneous, it is, in a sense, fairly regular in its local structure and thus our Theorems 1 and 2 contribute to this question of Smith.

well known.

A. The group G can operate on itself as follows: for each $g, h \in G$ with h thought of as a point of a space, $gh=h'$ where h' is the group-theoretic gh. With this definition G is transitive on itself. For each h, $h' \in G$ there is one (and only one) element $g \in G$ for which $gh=h'$.

If, contrary to our Agreement 2, G is finite then G can operate on itself in this same way and also G can operate on a Cantor set C as follows: let H be a collection of disjoint open and closed subsets of C such that[4] $H^*=C$ and H admits a one-to-one transformation φ onto G. For some $h \in H$ and any $g \in G$ let ρ_g be a homeomorphism of h onto $\varphi^{-1}(g\varphi(h))$ with ρ_e the identity on h. For any point $p \in C$, there exists a $g' \in G$ such that $\rho_{g'}^{-1}(p) \in h$. Define gp to be $\rho_{g''}(\rho_{g'}^{-1}(p))$ where $g''=gg'$. The technique which we use here is similar to one we shall use for Lemma 2 later in the argument for Theorems 1 and 2.

B. In this example we show that G can operate on a locally connected continuum in the plane, in fact, on a tree, the particular tree, however, depending on G. Let I be the unit interval $0 \leqq x \leqq 1$, $y=0$. Let K_1 be a collection on $n(G_1)$ disjoint subintervals of I formed by choosing every other element of a subdivision of I into $2n(G_1)-1$ equal subintervals. Inductively, for each $i>1$, let K_i be a collection of $n(G_i)$ disjoint subintervals of I formed by choosing every other one of a subdivision of each interval of K_{i-1} into $2\left(\frac{n(G_i)}{n(G_{i-1})}\right)-1$ equal subintervals. Then $\bigcap_i K_i^*$ is a Cantor set C which may, in the obvious way, be identified with G.

For each i, let Q_i be a set of $n(G_i)$ points on $y=2^{-i}$ such that for each element k of K_i, Q_i contains a point $q(k)$ whose x-coordinate is the x-coordinate of the midpoint of k. Let Q_0 be the point $(\frac{1}{2}, 1)$. Let t be $\bigcup_{i\geqq 0} Q_i + \bigcap_{i \geqq 1} K_i^* +$ for each $i \geqq 0$, the sum of all intervals with endpoints one in Q_i and the other in Q_{i+1} which project parallel to the y-axis into K_i^*. Then G may be considered as operating effectively but not strongly effectively on t such that the "branchings" of the operation of G on t occur at the points of $\bigcup_{i \geqq 0} Q_i$ and such that each point p of $t-C$ has a finite orbit under G consisting of those points of t on the horizontal line through p. In developing G we may consider that, for each i, G_i permutes the elements of K_i consistent with π_{i-1} and G_{i-1} permuting the elements of K_{i-1}.

C. Let G be a p-adic group and hence let, for each i, G_i be cyclic.

[4] If H is a collection of point sets, H^* denotes the sum of the elements of H.

Let E be the infinite dimensional compact torus $J_1 \times J_2 \times \cdots$ where, for each i, J_i may be thought of as the circle of radius 2^{-i} and center at (0, 0). Then $TgS(G, E)$ exists. For each i, let φ_i be the group of order n_i of rotations of J_i and let $Tg(G_i, E)$ be the cyclic group of order n_i on E defined coordinatewise as φ_j for $j \leqq i$ and as the identity for $j > i$. Then $TgS(G, E)$ may be defined coordinatewise as φ_i on J_i, for each i.

3. Definitions and the universal curve. Let N be the set of points in E^3 for which $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq z \leqq 1$. For $w=x$, y, z and $i=1, 2, \cdots$ let $D_i(w)$ be the set of all open intervals on the w-axis of length 3^{-i} whose endpoints have w-coordinates which are positive rational numbers less than 1, the expression for each such rational number having 3^i as a denominator when in lowest terms. The length of $D_i^*(w)$, for any i, is $\frac{1}{3}$. Let M be the set of all points (x, y, z) of N for which, for no i, do two or more of the points $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$ belong to the set $D_i^*(x)+D_i^*(y)+D_i^*(z)$. The set M is called the *universal curve*.

It is not hard to verify that M is a locally connected one-dimensional continuum with no local separating points. M is called "the universal curve" as every one-dimensional continuum can be imbedded in it.

We need several further definitions before characterizing the universal curve. We use a special case of the characterization given in [1] with resultant simpler definitions than those of [1].

If H and H' are collections of point sets, H is said to be a *refinement* of H′ if each element of H is a subset of an element of H' and each element of H' contains an element of H. A collection H of point sets is said to be *one-dimensional* provided no three elements of H intersect.

A collection H of point sets is said to be *simple* provided that (1) H is finite, and H^* is connected, (2) each element of H is a (closed) 3-cell, and (3) if two elements of H intersect their intersection is a 2-cell on the bounding 2-sphere of each such element.

Let H and H' be simple collections with H a refinement of H'. Let h be an element of H' and let Z be the collection of those elements of H in h which intersect elements of H not in h. Then H is said to *interlace* h provided that for any subdivision of Z into disjoint sets Z_1 and Z_2 with $Z_1+Z_2=Z$ there exist non-null connected sums of elements of H in h, namely X_1 and X_2 with $X_1 \supset Z_1^*$, $X_2 \supset Z_2^*$, and X_1 and X_2 having no element of H in common. H is said to *interlace* H' if H interlaces each element of H'.

A sequence $\{F_i\}$ is said to be a *λ-defining sequence* of a continuum

M provided

(1) for each i, F_i is a simple one-dimensional collection covering M,

(2) for each i, F_{i+1} is a refinement of F_i,

(3) $M = \bigcap_i F_i^*$

(4)[5] for any $\varepsilon > 0$ there exists a number n such that $m(F_n) < \varepsilon$,

(5) for each i, F_{i+1} is interlaced in F_i, and

(6) if two elements of F_i intersect then each contains two elements of F_{i+1} intersecting elements of F_{i+1} in the other but neither contains any element of F_{i+1} intersecting two elements of F_i distinct from the one containing it.

A non-degenerate continuum for which there exists a λ-defining sequence is called a *C-set.*

The following theorem is proved in [1]:

THEOREM. *Each C-set is homeomorphic to the universal curve.*

NOTATION. If E_i is a finite collection of closed point sets and $Tg(G_i, E_i^*)$ or $TgS(G_i, E_i^*)$ is such that for $h \in E_i$, and any $g \in G_i$, gh is an element of E_i then we will write $Tg(G, E_i^*, E_i)$ or $TgS(G_i, E_i^*, E_i)$ respectively. If $\{E_i\}$ is a λ-defining sequence and $TgS(G_i, E_i^*, E_i)$ and $TgS(G_{i+1}, E_{i+1}^*, E_{i+1})$ exist, then $TgS(G_{i+1}, E_{i+1}^*, E_{i+1})$ is said to *refine* $TgS(G_i, E_i^*, E_i)$ provided that for any $g \in G_{i+1}$ and any $x \in E_{i+1}$, if x' denotes the element of E_i containing x, $\pi_i(g)x'$ contains gx.

AGREEMENT 3. *In what follows we shall make many constructions in E^3 using 3-cells and homeomorphisms. Every 3-cell used is to be polyhedral and every homeomorphism defined over finite sums of 3-cells is to be piecewise-linear, that is, is to carry polyhedra into polyhedra. We interpret this understanding to apply also to appropriate subsets (2-cells) and homeomorphism over these subsets, such being used in the constructions and lemmas. All constructions are to be in E^3.*

4. Statements of lemmas and proof that the lemmas imply Theorems 1 and 2.

LEMMA 1. *Let n be any positive integer. Let K and K' be elements of a simple one-dimensional collection of 3-cells in E^3. Let D and D' be collections of n disjoint 2-cells on the boundaries of K and K' respectively. Let φ be a homeomorphism of D^* onto D'^* preserving orientation on the elements of D and D' relative respectively to K and K' as embed-*

[5] If H is a finite collection of point sets, $m(H)$ denotes the mesh of H, that is, the l.u.b. of the diameters of H.

ded in E^3. *Then there exists an orientation-preserving*[6] *homeomorphism* ψ *of* K *onto* K' *such that for each point* $p \in D^*$, $\psi(p)=\varphi(p)$.

Proof. This lemma is geometrically obvious and is well known.

LEMMA 2. *Let, for any* i, $X_1, \cdots, X_{n(G_i)}$ *be a set* X *of disjoint continua all homeomorphic to each other. For each* j, $1 \leq j \leq n(G_i)$, *let* η_j *be a homeomorphism of* X_1 *onto* X_j *with* η_1 *the identity on* X_1. *Let* ρ_i *be a one-to-one transformation of* X *onto* G_i *with* $\rho_i X_j = g_j$ *for each* j. *Then* $TgS(G_g, X^*, X)$ *exists with* gx *defined as follows:*

$$\textit{for} \quad x \in X_j,\ g_j x = \eta_{k'} \eta_k^{-1} x \quad \textit{where} \quad g_{k'} = g_j g_k .$$

Proof of Lemma 2. This lemma is almost obvious and is well known. We state it separately to simplify the argument for Lemmas 3, 3′ and 3′′. To prove the lemma it is sufficient to note that

$$g_{j_1}(g_{j_2}x) = (g_{j_1}g_{j_2})x \quad \text{for} \quad x \in x_k \quad \text{and} \quad g_{j_1},\ g_{j_2} \in G_i,$$

$$g_{j_1}(g_{j_2}x) = g_{j_1}(\eta_{k'}\eta_k^{-1}x) = \eta_{k''}\eta_{k'}^{-1}\eta_{k'}\eta_k^{-1}x = \eta_{k''}\eta_k^{-1}x$$

where $g_{k'}$ is $g_{j_2}g_k$ and $g_{k''}$ is $g_{j_1}g_{k'}$. Therefore $g_{k''}$ is $(g_{j_1}g_{j_2})g_k$ as was to be shown.

LEMMA 3. *There exists a continuum* M *and a* λ*-defining sequence* $\{F_i\}$ *of* M *such that for each* i, $TgS(G_i, F_i^*, F_i)$ *exists with* $TgS(G_{i+1}, F_{i+1}^*, F_{i+1})$ *refining* $TgS(G, F_i^*, F_i)$ *and for each element* f *of* F_i, $G_i(f)$ *consists of* $n(G_i)$ *disjoint elements of* F_i.

LEMMA 3′. *The same as Lemma* 3 *with the added condition that there exist a* λ*-defining sequence* $\{H_i\}$ *and a sequence* $\{\mu_i\}$ *such that*

(1) *for each* i, μ_i *is a mapping of* F_i^* *onto* H_i^* *with for* $f \in F_i$, $\mu_i(f) \in H_i$ *and* μ_i *a homeomorphism over* f,

(2) *for any* $g \in G_i$ *and* $x \in F_i^*$, $\mu_i(x) = \mu_i(gx)$, *and*

(3) *for each* i, $f \in F_i$ *and* $\tilde{f} \in F_{i+1}$, $\mu_i(f) \supset \mu_{i+1}(\tilde{f})$ *if and only if* $f \supset \tilde{f}$.

LEMMA 3′′. *The same as Lemma* 3 *with the added condition that there exists a sequence* $\{H_i\}$ *of simple collections and a sequence* $\{\mu_i\}$ *such that*

[6] Orientation-preserving with respect to embedding in E^3.

(1) *for each i, μ_i is an $n(G_i)$-to-one mapping of F_i^* onto H_i^* with for $f \in F_i$, $\mu_i(f) \in H_i$ and μ_i a homeomorphism over f,*

(2) *for any $g \in G_i$ and $x \in F_i^*$, $\mu_i(x) = \mu_i(gx)$*

(3) *for each i, $f \in F_i$ and $\tilde{f} \in F_{i+1}$, $\mu_i(f) \supset \mu_{i+1}(\tilde{f})$ if and only if $f \supset \tilde{f}$*

(4) *for each $h, h' \in H_i$ for which $h \cdot h'$ exists, H_{i+1} contains exactly one element in h intersecting an element of H_{i+1} in h, and*

(5) *for any $\varepsilon > 0$ there exists an n such that $m(H_n) < \varepsilon$.*

Before proving Lemmas 3, 3′ and 3″ in §§ 5 and 6 we wish to note that Lemma 3 implies a weaker form of Theorem 1 to the effect that $TgS(G, M)$ exists, that Lemma 3′ implies the full strength of Theorem 1, and that Lemma 3″ implies Theorem 2.

Clearly, from the characterization of the universal curve cited in § 3, $\bigcap_i F_i^* = M$ is a universal curve. Let $g \in G$. Then g is defined by a unique sequence $\{g_i\}$ with, for each i, $g_i \in G_i$ and $\pi_i g_{i+1} = g_i$. For any point $p \in M$, gp is defined as $\bigcap_i g_i f_i$ where $\{f_i\}$ is a sequence such that for each i, $f_i \in F_i$, $f_i \supset f_{i+1}$, and $p \in f_i$. But gp must be unique for $m(F_i) \to 0$ and if $\{f_i'\}$ is another such sequence then, for each i, $g_i f_i'$ intersects $g_i f_i$.

That such definition of the association of G and M satisfies the conditions of the definition of topological transformation group is straigtforward. First, $ex = x$ for all $x \in M$ as, for each i, e_i leaves all elements of F_i fixed. Second, as for each i, g, $g' \in G_i$ and $f \in F_i$, $g(g'f) = (gg')f$, it follows that $g(g'x) = (gg')x$ for g, $g' \in G$ and $x \in E$. Third gx is continuous simultaneously in g and x. Let $g^j \to g$ in G and let $x^j \to x$ in M. We wish to show that $g^j x^j \to vx$ in M. Let $\varepsilon > 0$. Let k be an integer such that (1) $m(F_k) < \varepsilon$, (2) for all $i > k$, x^i is in an element of F_k containing x, and (3) for all $j > k$, $\pi_k g_{k+1}^j = \pi_k g_{k+1}$ where g_{k+1}^j and g_{k+1} are the elements of G_{k+1} of the sequences $\{g_\lambda^j\}$ and $\{g_\lambda\}$ defining g^j and g respectively. Then for all $j > k$, $g^j x^j$ is at a distance of less than ε from gx as was to be shown.

We have now established that Lemma 3 imphes the weak form of Theorem 1 and it remains to show that Lemmas 3′ and 3″ establish additionally that $O[TgS(G, M)]$ is, in the first case, a universal curve and, in the second, a regular curve.

We wish to show next that $H = \bigcap_i H_i^*$ is homeomorphic to $O[TgS(G, M)]$ with $\{H_i\}$ and $TgS(G, M)$ as in either Lemma 3′ or Lemma 3″. For any $x \in H$, let $\{h_i\}$ be a sequence such that, for each i, $h_i \supset h_{i+1}$, $h_i \in H_i$, and $x \in h_i$. But then there exists a sequence $\{f_i\}$ such that, for each i, $f_i \supset f_{i+1}$, $f_i \in F_i$, and $\mu_i(f_i) = h_i$. For $x \in H$, let $\nu(x) = G(\bigcap_i f_i)$ for such a sequences $\{f_i\}$. For any other such sequence $\{f_i'\}$, $G(\bigcap_i f_i')$ is $G(\bigcap_i f_i)$. As $m(H_i) \to 0$, $m(F_i) \to 0$, and for h_i, $h_i' \in H_i$ h_i intersects

h'_i if and only if and only if for any $f_i \in F_i$ with $\mu_i(f_i)=h_i$ there exists an f'_i with $\mu_i(f'_i)=h'_i$ and f_i intersecting f'_i, then it follows that ν is one-to-one onto. A standard argument shows the continuity of ν. Hence ν is the desired homeomorphism of H onto $O[TgS(G, M)]$.

Finally for Theorem 1 we note that by the condition that $\{H_i\}$ is a λ-defining sequence in Lemma 3′ it follows that H is a universal curve.

For Theorem 2 by Condition (4) of Lemma 3″ we note that if $p \in H$ and k_i denotes the sum of all elements of H_i containing p then for any i, $H \cdot k_i$ has only a finite number of points on its boundary with respect to H. Hence H is a regular curve.

5. The first step of the proof of Lemmas 3, 3′ and 3″. The demonstration of the existence of suitable F_1 and $TgS(G, F_1^*, F_1)$ is applicable to each of the Lemmas 3, 3′ and 3″ and thus only one argument need be given.

DEFINITION. Let S denote a set of k disjoint 3-cells. A collection R is said to be an *n-developed collection about* S provided (1) R is a simple one-dimensional collection, (2) R contains S as a sub-collection, (3) $R-S$ contains $3n\binom{k}{2}$ elements, (4) for each pair of elements s_1 and s_2 of S there exist exactly n simple chains of elements of $R-S$ each consisting of 3 links and each having one end link intersecting s_1 and the other intersecting s_2, and (5) no link of any such 3-link chain intersects more than two elements of R distinct from itself.

Let S_1 be a set of $n(G_1)$ disjoint 3-cells and let R_1 be an $n(G_1)$-developed collection about S_1. Let R_1 be the desired set F_1.

For $s, s' \in S_1$ let $B(s, s')$ be the set of chains of R_1-S_1 which join s and s'. Let λ be a one-to-one transformation of S_1 onto G_1 and for $s, s' \in S_1$ let $\mu_{s,s'}$ be a one-to-one transformation of $B(s, s')$ onto G_1.

In defining $Tg(G_1, F_1^*, F_1)$ which we shall show to be strongly effective and hence $TgS(G_1, F_1^*, F_1)$ we impose consecutively the following conditions:

(A) For any $s \in S_1$ and $g \in G_1$, $gs=\lambda^{-1}g\lambda s$.

(B) For any $g \in G_1$, s, $s' \in S_1$, and f a link of an element b_f of $B(s, s')$, gf is that link of $\mu^{-1}_{gs,gs'}(g[g_{s,s'}(b_f)])$ which intersects gs, intersects gs' or intersects neither gs nor gs' according as f intersects, s, intersects s' or intersects neither s nor s'.

With these conditions being satisfied, G_1 acts in a strongly effective way on the finite set F_1 as we show. (A) implies that G_1 thus acts on S_1 by permuting the elements of S_1 among themselves, for $s \in S_1$, and $s' \in S_1$ there is a unique $g \in G_1$ for which $gs=s'$ and if $s=s'$, $g=e_1$. For

$f \in F_1 - S_1$, $e_1 f = f$ by (B). For $g, g' \in G_1$ and $f \in b_f \in B(s, s')$, $g(g'f)$ must be $(gg')f$ for

$$g[\mu^{-1}_{g's,g's'}(g'[\mu_{s,s'}(b_f)] = gb'_f = \mu^{-1}_{g(g',s),g(g's')} g[\mu_{g's,g's'}(b'_f)]$$

$$= \mu^{-1}_{(gg')s,(gg')s'}([gg'][\mu_{s,s'}(b_f)]) = b''_f$$

and consistent with this, $g(g'f)$ and $(gg')f$ are each determined solely by the orders on b_f and b'_f and on b''_f respectively relative to s and s' and $g's$ and $g's'$ on the one hand and $(gg')s$ and $(gg')s'$ on the other. It is easy to see that such operation is not only strongly effective but if $f, f' \in F_1$ with for some $g \in G_1$, $gf = f'$ then f and f' do not intersect the same element of F_1.

Furthermore, it follows directly from the construction that if f, $f' \in F_1$ intersect then for any $g \in G_i$, gf and gf' intersect.

With this information in mind we proceed to define $Tg(G_1, F_1^*, F_1)$. Let C_1 be the set of all 2-cells which are the intersections of elements of F_1. Then we may think of G_1 acting on C_1 consistent with G_1 acting on F_1, that is, for $c \in C_1$, c is $f \cdot f'$ for some f, $f' \in F_1$, and for $g \in G_1$, gc is $gf \cdot gf'$. But G_1 structures C_1 into orbits. From Lemma 2 by considering these orbits one at a time we may define $TgS(G_1, C_1^*, C_1)$ such that gc is gc as defined above and such that g is a homeomorphism of c onto gc which is oriented to be consistent with some orientation preserving homeomorphism of $f + f'$ onto $gf + gf'$ carrying f onto gf and f' onto gf'. That the orientation property of this latter statement is true follows from a consideration like that of the proof of Lemma 2. The orientation property may be made valid directly for the homeomorphisms from an element c to the elements in its orbit but any other homeomorphism between elements of such orbit is composed from these and for any f, f', $f'' \in F_1$ with f' intersecting f'' there is at most one $g \in G_1$ for which $gf = f'$ or f''.

But now Lemma 1 and Lemma 2 applied to the various orbits of the elements of F_1 under G_1 assert the existence of $TgS(G_1, F_1^*, F_1)$ as we set out to show. Clearly there exists an H_1 as in Lemmas 3′ and 3′′ such that we may map F_1^* onto H_1^* as in the Lemma.

6. The inductive step of the proofs of Lemmas 3, 3′, and 3′′. To complete the proofs of Lemmas 3, 3′, and 3′′ it now suffices to define and establish the existence of F_i and $TgS(G_i, F_i^*, F_i)$, $i > 1$, given F_1 and $TgS(G, F_1^*, F_1)$ defined as above and F_j and $TgS(G_j, F_j^*, F_j)$, $1 < j < i$, defined by the inductive procedure to be given. We seek to do this so that applicable parts of Lemma 3 are satisfied. Then we shall note variations on the argument to yield Lemmas 3′ and 3′′.

The construction we give will be similar in many ways to that of the preceding section. We shall require that $m(F_i) < 2^{-i}$.

Let C_{i-1} denote the collection of intersections of the various elements of F_{i-1} with each other. Each element of C_{i-1} is a 2-cell. Let $c \in C_{i-1}$ and let $f(c)$ and $f'(c)$ be the two elements of F_{i-1} for which $c=f(c)\cdot f'(c)$. Let $S_i(c, f(c))$ and $S_i(c, f'(c))$ be collections of exactly $\frac{n(G_i)}{n(G_{i-1})}$ disjoint 3-cells in $f(c)$ and $f'(c)$ respectively such that

(1) each element of $S_i(c, f(c))$ intersects exactly one element of $S_i(c, f'(c))$ and that in a 2-cell in c,

(2) each element of $S_i(c, f(c))$ or $S_i(c, f'(c))$ intersects $B(f(c))$[7] or $B(f'(c))$ respectively in a 2-cell and such 2-cell is in $S_i^*(c, f'(c))$ or $S_i^*(c, f(c))$ respectively, and

(3) there exist $R_i(c, f(c))$ and $R_i(c, f'(c))$ which are $n(G_i)$-developed collections about $S_i(c, f(c))$ and $S_i(c, f'(c))$ respectively such that (a) $[R_i(c, f(c))-S_i(c, f(c))]^* \subset f(c)-B(f(c))$ and $[R_i(c, f'(c))-S_i(c, f'(c))]^* \subset f'(c)-B(f'(c))$ and (b) $m[R_i(c, f(c))]<\varepsilon$ and $m[R_i(c, f'(c))]<\varepsilon$.

As it is possible to define such sets $S_i(c, f(c))$, $R_i(c, f(c))$, $S_i(c, f'(c))$ and $R_i(c, f'(c))$ for all $c \in C_{i-1}$ such that for $c' \neq c$, $R_i^*(c, f(c))+R_i^*(c, f'(c))$ does not intersect $R_i^*(c', f(c'))+R_i^*(c', f'(c'))$, we consider such a collection of sets to exist, each $c \in C_{i-1}$ being identified with just two elements $R_i(c, f(c))$ and $R_i(c, f'(c))$.

For $f \in F_{i-1}$ let $R_i(f)$ and $S_i(f)$ be the union of all such sets $R_i(c, f)$ and $S_i(c, f)$ respectively for $c \in C_{i-1}$ and $c \subset f$. Thus $S_i(f)$, for example, is a particular collection of disjoint 3-cells in f.

Definition. Let S denote a set of n disjoint 3-cells. A collection R is said to be an *(n, m)-weakly developed collection about* S provided (1) R is a simple one-dimensional collection, (2) R contains S as a subcollection, (3) $R-S$ contains $m\cdot\binom{n}{2}$ elements, and (4) for each pair of elements s_1 and s_2 there is a simple chain of m elements of $R-S$ having one end link intersecting s_1 and the other intersecting s_2 such that no link of any such chain intersects more than two elements of R distinct from itself.

Let $n(S_i(f))$ be the number of elements of $S_i(f)$. For some fixed integer m and any $f \in F_{i-1}$ let $Q_i(f)$ be an $(n(S_i(f)), m)$-weakly developed collection about $S_i(f)$ such that (1) each element of $Q_1(f)-S_i(f) \subset f-B(f)$, (2) no element of $Q_i(f)-S_i(f)$ intersects any element of $R_i(f)-S_i(f)$, and (3) $m(Q_i(f))<2^{-i}$.

Let $L_i(f)$ be that subset of $Q_i(f)$ consisting of $S_i(f)$ and all links of all chains of the development of $Q_i(f)$ between elements of $S_i(f)$ not both in any one set $S_i(c, f)$ for $c \in C_{i-1}$ and $c \subset f$.

Let $\quad S_i=\bigcup_{f\in F_{i-1}} S_i(f),\ R_i=\bigcup_{f\in F_{i-1}} R_i(f)$ and $L_i=\bigcup_{f\in F_{i-1}} L_i(f).$

[7] By $B(f)$ is meant the boundary of f.

The set F_i is defined as the set of all elements in one or more of S_i, R_i, and L_i.

Next we shall define G_i acting on F_i in a strongly effective manner such that

(a) for $f \in F_i$, and $g \in G_i$, f and gf do not intersect the same element of F_i,

(b) if f, $f' \in F_i$ for which $f \cdot f'$ exists then for each $g \in G_i$, $gf \cdot gf'$ exists, and

(c) for $f \in F_i$, $\tilde{f} \in F_{i-1}$ with $\tilde{f} \supset f$ and for any $g \in G_i$, $gf \subset \pi_{i-1}(g)\tilde{f}$.

Let D_{i-1} be the collection of all sets $G_{i-1}(c)$ for $c \in C_{i-1}$. Each element of D_{i-1} consists of $n(G_{i-1})$ 2-cells. For $d \in D_{i-1}$, let $f(d)$ and $f'(d)$ be the two sets each of which is an element of F_{i-1} containing an element of d plus the sum of its images under G_{i-1}. Let $S(f(d))$ and $S(f'(d))$ be the collection of those elements of S_i which (1) intersect d^* and (2) lie in $f(d)$ and $f'(d)$ respectively. Then $S(f(d))$ and $S(f'(d))$ each consist of $n(G_i)$ disjoint 3-cells.

For $d \in D_{i-1}$ let $\lambda_{f(d)}$ and $\lambda_{f'(d)}$ be one-to-one transformations of $S(f(d))$ and $S(f'(d))$ respectively onto G_i such that

(1) for $s \in S(f(d))$ and $s' \in S(f'(d))$, s intersects s' if and only if $\lambda_{f(d)}(s)$ is $\lambda_{f'(d)}(s')$ and

(2) for $g \in G_i$, $s \in S(f(d))$ and $f \in F_{i-1}$ for which $s \subset f$, $\pi_{i-1}(g)f \supset \lambda^{-1}_{f(d)} g \lambda_{f(d)}(s)$.

Each element of S_i belongs to exactly one set $S(f(d))$ or $S(f'(d))$ and thus S_i is structured by these sets. We may now define $TgS(G_i, S_i)$ as follows: for $g \in G_i$ and $s \in S(f(d))$, gs is $\lambda^{-1}_{f(d)} g \lambda_{f(d)}(s)$.

Next, for any s, $s' \in S_i$ for which s, $s' \in S(f)d))$ for some $d \in D_{i-1}$ and for which for some $f \in F_{i-1}$, $s + s' \subset f$, let $B(s, s')$ denote the set of 3-element chains from s to s' of the definition of R_i and let $\mu_{s,s'}$ be a one-to-one transformation of $B(s, s')$ onto G_i.

Then we may define $TgS(G_1, R_i)$. For $s \in R_i$ and $s \in S_i$ and for any $g \in G_i$, gs is gs as defined in $TgS(G_i, S_i)$. For any $g \in G_i$ and s, $s' \in S_i$ for which $B(s, s')$ is defined as above and for any x a link of an element b of $B(s, s')$, gx is that link of $\mu^{-1}_{gs,gs'}(g[\mu_{s,s'}(b)])$ which intersects gs, intersects gs' or intersects neither gs nor gs' according as f intersects s, intersects s' or intersects neither s nor s'.

Next we define $TgS(G_i, L_i)$. For $s \in L_i$ and $s \in S_i$ and for any $g \in G_i$, gs is gs as defined in $TgS(G_i, S_i)$. For s, $s' \in S_i$, $f \in F_{i-1}$, with $s + s' \supset f$ and s and s' not both elements of any set $S(c, f(c))$, there is a simple chain $\beta(s, s')$ of exactly m elements of $L_i(f) - S_i(f)$ with $\beta(s, s')$ having one end element intersecting s and the other s'. For each link x of $\beta(s, s')$ let, for $g \in G_i$, gx be that link of $\beta(gs, gs')$ which is the same number of links removed from gs as is x from s.

The definition of $Tg(G_i, F_i)$ is now complete and it may easily be verified that conditions (a)-(c) above are satisfied.

Let C_i be the set of all intersections of pairs of elements of F_i. Let $TgS(G_i, C_i)$ be defined as follows: for $c \in C_i$, c is a 2-cell which is the intersection of some two elements $f, f' \in F_i$; for $g \in G_i$, gc is $gf \cdot gf'$. Then as in § 5 employing Lemma 2, we may define $TgS(G_i, C_i^*, C_i)$ so that gc is gc as defined immediately above and g preserves orientation on c and gc relative to the orientations on (f, f') and (gf, gf') respectively.

Finally employing Lemmas 1 and 2 we may define $TgS(G_i, F_i^*, F_i)$ consistent with $TgS(G_i, F_i)$ and $TgS(G_i, C_i^*, C_i)$ so that with this inductive definition, Lemma 3 is satisfied. In this connection we note that under $TgS(G_i, F_i)$, for $f \in F_i$, $G_i(f)$ consists of $n(G_i)$ disjoint 3-cells so that Lemma 2 is applicable.

To modify the argument given so as to prove Lemma 3″ we must introduce some extra conditions. The sets H_j, $1 \leqq j \leqq i-1$ exist as in the Lemma. Then when we define S_i we also define a set $S_i(H)$ where for $h, h' \in H_{i-1}$ with h intersecting h' exactly one 3-cell is introduced in $S_i(H)$ in each of h and h' intersecting the other. In defining R_i we also define a set $R_i(H)$ where $R_i(H) - S_i(H)$ consists of exactly $3 \cdot n(G_i) \cdot N$ elements with N the number of elements in $S_i(H)$ and with for each element s of $S_i(H)$ there being $n(G_i)$ 3-link simple chains in $R_i(H) - S_i(H)$, both end links of each such chain intersecting s. We may additionally require that $m(R_i(H)) < 2^{-i}$. Then for each pair of elements of $S_i(H)$ in the same element of H_{i-1} we introduce a simple chain of 3-cells joining them, the simple chain having m links with m being so chosen that $m(H_i) < 2^{-i}$. This imposes an extra condition on the "m" of the preceding argument. It is now straightforward to see that the sequences of Lemma 3″ can be asserted to exist.

Finally to prove Lemma 3′ we need one extra device. For each $c \in C_{i-1}$, we choose not one but two pairs of sets $[S_i(c, f(c)), S_i(c, f(c))]$ and $[S_i'(c, f(c)), S_i'(c, f(c))]$ such that we may introduce two pairs of sets $[R_i(c, f(c)), R_i(c, f'(c))]$ and $[R_i'(c, f(c)), R_i'(c, f(c))]$ similar to the one pair we introduced before with additionally $R_i^*(c, f(c)) + R_i^*(c, f'(c))$ and $R_i'^*(c, f(c)) + R_i'^*(c, f'(c))$ not intersecting each other. Finally for any $f \in F_{i-1}$ we may define $S_i(f)$ in the similar fashion to that used before but with $S_i(f)$ here containing twice as many elements as the corresponding set in the preceding argument. Then we may form the set $Q_i(f)$ as an $(n(S_i(f)), m)$-weakly developed collection about $S_i(f)$ and proceed as before using extra conditions analogous to those of the argument sketched for Lemma 3″.

It is clear that under such conditions $\{H_i\}$ and $\{\mu_i\}$ can be defined so that $\{H_i\}$ will be a λ-defining sequence.

Thus Lemma 3′ is proved and our argument for Theorems 1 and 2 is completed.

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