TWO THEOREMS ON TOPOLOGICAL LATTICES

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A *topological lattice* is a pair of continuous functions

$$\wedge\colon L \times L \to L, \quad \wedge\colon L \times L \to L$$

(L a Hausdorff space) satisfying the usual conditions for lattice operations. A set A is *convex* if $x, y \in A$ and $x \leqq a \leqq y$ implies $a \in A$. This is equivalent to $A = (A \wedge L) \cap (A \vee L)$.

After proving a separation theorem involving a convex set we show that a compact connected topological lattice is a cyclic chain in the sense of G. T. Whyburn and that each cyclic element is a convex sublattice. In doing so we rely on some results recently obtained by L. W. Anderson.

THEOREM 1. *Let L be a connected topological lattice and let A be a convex set such that $L \setminus A$ is not connected. Then $L \setminus A$ is the union of the connected separated sets $(A \wedge L) \setminus A$ and $(A \vee L) \setminus A$ which are open (closed) if A is closed (open). If L is also compact then A is connected if it is either open or closed.*

Proof. Let $L \setminus A = U \cup V$ with $U^* \cap V = \phi = U \cap V^*$ and let $p \in U$, $q \in V$. The connected set $(p \wedge L) \cup (q \wedge L)$ meets both U and V; hence it meets A. Adjust the notation so that $(q \wedge L) \cap A \neq \phi$ and thus $q \in A \vee L$. If $(q \vee L) \cap A \neq \phi$ then $q \in A \wedge L$ and hence $q \in (A \wedge L) \cap (A \vee L) = A$. This being impossible we infer that $(q \vee L) \cap A = \phi$ and $q \in (A \vee L) \setminus A = (A \vee L) \setminus (A \wedge L)$. The connected set $(p \vee L) \cup (q \vee L)$ intersects U and V and so intersects A. But $(q \vee L) \cap A = \phi$ so that $(p \vee L) \cap A \neq \phi$ and hence $p \in A \wedge L$. Were $(p \wedge L) \cap A \neq \phi$ we would also have $p \in A \vee L$ and so $p \in A$, a contradiction. Thus $(p \wedge L) \cap A = \phi$ and $p \in (A \vee L) \setminus A = (A \vee L) \setminus (A \wedge L)$. Now take $y \in V$ and suppose that y is not in $A \vee L$ so that $(y \wedge L) \cap A = \phi$; then $(p \wedge L) \cap A \neq \phi$ since $(p \wedge L) \cup (y \wedge L)$ is a connected set meeting U and V. But this is contrary to the proven fact that $(p \wedge L) \cap A = \phi$. We conclude that $V \subset (A \vee L) \setminus A$ and, dually, that $U \subset (A \wedge L) \setminus A$. It follows that $L = (A \wedge L) \cup (A \vee L)$. Now $x \in (A \vee L) \setminus A$ and $x \in L \setminus V$ gives $x \in U \subset (A \wedge L) \setminus A$ and this contradicts the convexity of A. Hence $U = (A \wedge L) \setminus A$ and $V = (A \vee L) \setminus A$. To see that $U \wedge L = U$ we need only note that $x \in U$ gives $(x \wedge L) \cap A = \phi$ and thus $(x \wedge L) \cap V = \phi$ (since $x \wedge L$ is connected and contains x) and hence $x \wedge L \subset (A \wedge L) \setminus (A \vee L) = U$.

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Dually, $V \wedge L = V$ and these equalities imply that U and V are connected. If A is closed (open) then U and V are open (closed). This completes the proof of the first sentence of the conclusion. If L is also compact then $H^1(L)=0$ [3] so that (as is well known) L is unicoherent. But L is locally connected, $L=(A \wedge L) \cup (A \vee L)$, and the sets $A \wedge L$ and $A \vee L$ are connected, and open (closed) [1] if A is open (closed). Hence by a known result [2] we see that $A=(A \wedge L) \cap (A \vee L)$ is connected.

We assume that the reader is familiar with the cyclic element theory of locally connected continua as given in [4]. We recall that a locally compact connected topological lattice is locally connected [1].

THEOREM 2. *Let L be a compact connected metrizable topological lattice. Then L is a cyclic chain, each cyclic element of which is a convex sublattice. If L is topologically contained in the plane then each true cyclic element of L is 2-cell and L has the fixed-point property.*

Proof. Let C be a true cyclic element of L, let $x, y \in C$ with $x \leqq y$ and let $p \in L$ such that $x \leqq p \leqq y$. If T is a maximal chain containing x, p, and y then T is an arc from 0 to 1, as is well known [1]. Hence the set $[x, y] = \{t | t \in T \text{ and } x \leqq t \leqq y\}$ is an arc from x to y [1]. Since C is an A-set [4] we know that $[x, y] \subset C$ and thus $p \in C$. Hence C is convex. Let D be the cyclic chain from 0 to 1, that is, D is the smallest A-set containing 0 and 1 [4]. Then, by definition, $T \subset D$ and if $x \in L \setminus D$ then the maximal chain T' containing 0, x, 1 is an arc from 0 to 1 and thus $T' \subset D$, a contradiction. Hence $D=L$ and L is the cyclic chain from 0 to 1. Let T_0 be 0, 1 and all points which separate 0 and 1. Then L is the union of T_0 and all true cyclic elements meeting T_0 in two points [4]. Suppose that the true cyclic element C meets T_0 in the cutpoints p and q. Note that neither 0 nor 1 is a cutpoint [3]. If z is a cutpoint then, since $\{z\}$ is convex, $L= (z \wedge L) \cup (z \vee L)$ and thus z is comparable with each $x \in L$, by Theorem 1. We may assume that $p < q$. We will show that $C = \{x | p \leqq x \leqq q\}$. The convexity of C proves the containment "$\supset$". If $x \in C$ and if, say, $x \leqq q$ is false then we have $q < x$. By Theorem 1, $L \setminus q = ((q \wedge L) \setminus q) \cup ((q \vee L) \setminus q)$ is a separation and C meets both members, contrary to the fact that C is a true cyclic element [4] . Dually, $x \leqq p$ cannot be false, proving the containment "$\subset$" of the desired equality. It follows that C is a convex sublattice. The cases $p=0$ or $q=1$ are treated similarly. The remaining results follow from the fact that $H^1(L)=0$ [3] so that L is a locally connected continuum [1] which does not cut the plane [4].

References

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