A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL NUMBERS

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We will say that a space E is of class L if E is a separable metric space which satisfies the following conditions:

(1) *Each component of E is a point or an arc (closed, open, or half-open), and no interior point of an arc-component A is a limit point of $E-A$.*

(2) *Each point of E has arbitrarily small neighborhoods whose boundaries are finite sets.*

The purpose of this note is to show that a necessary and sufficient condition that a space be homeomorphic to a set of real numbers is that it be of class L.

This gives an affirmative answer to a question raised by de Groot in [1].

In [2] L. W. Cohen proved that a separable metric space is homeomorphic to a set of real numbers if and only if it satisfies (1) above and (3) and (4) below:

(3) *E is zero-dimensional at each of its point-components.*

(4) *If p is an end point of an arc-component A, then the space $(E-A)\cup\{p\}$ is zero-dimensional at p.*

Any set of real numbers is clearly of class L. To prove the converse it is sufficient to show that every space of class L satisfies conditions (3) and (4). To this end it is clearly enough to show the following:

If X is a component of the space E of class L and ε is a positive number. there is a set $U(X, \varepsilon)$ which is both open and closed, contains X, and is contained in the union of X with the ε-neighborhoods of its endpoints (if any).

Suppose X is a component of a space E of class L and ε is a positive number. There exists an open set V which contains X but contains no point whose distance from X exceeds ε, such that the boundary B of V is finite; if X is a point, we can apply (2) directly to obtain V; if X is an arc, let V consist of X plus type (2) neighborhoods of the end points of X (if any).

Let G denote the sets of all points p of E such that E is the union of two mutually separated sets S_p and T_p, where S_p contains X and T_p contains p.

Received May 22, 1956. Research supported by the National Science Foundation.

Case I. $E-G=X$. Then G contains B. Let R be the union of all sets T_p for p in B. Since B is finite, R is both open and closed and $V-R$ is suitable for $U(X, \varepsilon)$.

Case II. $E-G\neq X$. Since X is a component, $E-G$ is the union of two mutually separated sets Y and Z, where Y contains X and Z is not empty. It will be shown that there is a set K which is both open and closed and contains Z but does not intersect X, thus contradicting the fact that Z is not in G.

The definition of G, together with the fact that E has a countable base, implies that $G=\bigcup_{n=1}^{\infty} G_n$, where each G_n is both open and closed.

Let p be a point of Z. If q is a point of G, then T_q contains q and not p. The reasoning used in Case I shows that there is a neighborhood N_p of p which has no boundary point in G and whose diameter is less than half the distance from p to Y.

Let $\{H_n\}$ $(n=1, 2, 3, \cdots)$ be a countable base for E. If H_n is not a subset of N_p for any p in Z, put $K_n=0$. If, for some p in Z, H_n is a subset of N_p, let N be one such N_p and put $K_n=N-G_n$. Let $K=\bigcup_{n=1}^{\infty} K_n$. By the choice of N_p, K has no limit point in Y. No K_n has a boundary point in G and only finitely many sets K_n intersect any G_i. Consequently K has no boundary points in G and K is both open and closed. Since Z is a subset of K and X does not intersect K, the proof is complete.

References

1. J. de Groot, *On Cohen's topological characterization of sets of real numbers*, Nederl. Akad. Wetensch. Proc. Ser. A, **58** (1955), 33–35.

2. L. W. Cohen, *A characterization of those subsets of metric separable space which are homeomorphic with subsets of the linear continuum*, Fund. Math. **14** (1929), 281–303.

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