NOTE ON NORMAL NUMBERS

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Introduction. Let α be a real number with fractional part $.a_1a_2a_3\cdots$ when written to base r. Let Y_n denote the block of the first n digits in this representation and let $N(d, Y_n)$ denote the number of occurrences of the digit d in Y_n. The number α is said to be *simply normal* to base r if

$$\lim_{n\to\infty} \frac{N(d, Y_n)}{n} = \frac{1}{r}$$

for each of the r distinct choices of d. α is said to be *normal* to base r if each of the numbers $\alpha, r\alpha, r^2\alpha, \cdots$ are simply normal to each of the bases $r, r^2, r^3, \cdots$. These definitions, due to Emile Borel [1], were introduced in 1909. In 1940 S. S. Pillai [3] showed that a necessary and sufficient condition that α be normal to base r is that it be simply normal to each of the bases $r, r^2, r^3, \cdots$, thus considerably reducing the number of conditions needed to imply normality. The purpose of the present note is to show that α is normal to base r if and only if there exists a set of positive integers $m_1 < m_2 < m_3 < \cdots$ such that α is simply normal to base r^{m_i} for each $i \geqq 1$, and also to show that no finite set of m's will suffice.

Notation. We make use of the following additional conventions.

If B_k is any block of k digits to base r, $N(B_k, Y_n)$ will denote the number of occurrences of B_k in Y_n and $N_i(B_k, Y_n)$ will denote the number of occurrences of B_k starting in positions congruent to i modulo k in Y_n.

The term "relative frequency" will denote the asymptotic frequency with which an event occurs. For example, B_k occurs in (α), the fractional part of α, with relative frequency r^{-k} if $\lim_{n\to\infty} N(B_k, Y_n)/n = r^{-k}$.

Proof of the theorems. The following lemmas are easily proved.

LEMMA 1. *If* $\lim_{n\to\infty} \sum_{i=1}^{m} f_i(n) = 1$ *and if* $\liminf_{n\to\infty} f_i(n) \geqq 1/m$ *for* $i = 1, 2, \cdots, m$; *then* $\lim_{n\to\infty} f_i(n) = 1/m$ *for each* i.

LEMMA 2. *The real number* α *is simply normal to base* r^k *if and*

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only if $\lim_{n\to\infty} N_1(B_k, Y_n)/n = 1/kr^k$ *for every block* B_k *of* k *digits to base* r.

THEOREM 1. *The real number* α *is normal to base* r *if and only if there exist positive integers* $m_1 < m_2 < m_3 < \cdots$ *such that* α *is simply normal to each of the bases* $r^{m_1}, r^{m_2}, r^{m_3}, \cdots$.

Proof. The necessity of the condition follows immediately from the definition of normality.

Now suppose the condition of the theorem prevails. Let ν be an arbitrary positive integer and let B_ν be an arbitrary block of ν digits to base r. Choose k so large that $m_k > \nu$. It follows from Lemma 2 that

$$\lim_{n\to\infty} \frac{N_1(A_{m_k}, Y_n)}{n} = \frac{1}{m_k r^{m_k}}$$

for each block A_{m_k} of m_k digits to base r. If B_ν occurs exactly $t = t(A_{m_k})$ times in each A_{m_k}, then it follows that

$$\liminf_{n\to\infty} \frac{N(B_\nu, Y_n)}{n} \geqq \frac{T}{m_k r^{m_k}}$$

where $T = \sum t(A_{m_k})$ with the sum running over all blocks of m_k digits to base r. Now there are $r^{m_k-\nu}$ distinct blocks A_{m_k} which contain B_ν starting in position i for $i = 1, 2, \cdots, m_k - \nu + 1$ so that $T = (m_k - \nu + 1) r^{m_k - \nu}$. Thus it follows that

$$\liminf_{n\to\infty} \frac{N(B_\nu, Y_n)}{n} \geqq \frac{(m_k - \nu + 1) r^{m_k-\nu}}{m_k r^{m_k}} = \frac{1}{r^\nu} - \frac{\nu - 1}{m_k r^\nu} .$$

But, since this argument can be made for arbitrarily large values of k and $m_k \geqq k$, this implies that

$$\liminf_{n\to\infty} \frac{N(B_\nu, Y_n)}{n} \geqq \frac{1}{r^\nu} .$$

With Lemma 1 this implies that

$$\lim_{n\to\infty} \frac{N(B_\nu, Y_n)}{n} = \frac{1}{r^\nu}$$

so that α is normal to base r by a result of Niven and Zuckerman [2].

The next theorem implies that no finite set of m's will suffice in Theorem 1.

THEOREM 2. *If* $m_1, m_2, \cdots, m_s$ *is an arbitrary collection of distinct* *positive integers, then there exists at least one real number α simply normal to each of the bases $r^{m_1}, r^{m_2}, \cdots, r^{m_s}$ but not normal to base r.*

Proof. Writing to base r^m form the periodic decimal

$$\alpha = .\dot{0}12\ldots(\dot{r^m}-1)$$

where m is the least common multiple of $m_1, m_2, \cdots, m_s$. It is clear that α is simply normal to base r^m and that it is not normal to base r. To show that it is simply normal to base r^{m_i} for $i=1, 2, \cdots, s$ we prove that if d divides m then α is simply normal to base r^d.

Let $m=qd$ and let B_d be an arbitrary but fixed block of d digits to base r. In view of Lemma 2 it suffices to show that

$$\lim_{n\to\infty} \frac{N_1(B_d, Y_n)}{n} = \frac{1}{dr^d}.$$

A simple counting process shows that there are precisely $\binom{q}{i}(r^d-1)^{q-i}$ distinct blocks A_m of m digits to base r which contain B_d exactly i times starting in a position congruent to one modulo d. Therefore, since

$$\lim_{n\to\infty} \frac{N_1(A_m, Y_n)}{n} = \frac{1}{mr^m}$$

for each A_m, it follows that

$$\lim_{n\to\infty} \frac{N_1(B_d, Y_n)}{n} = \frac{1}{mr^m} \sum_{i=1}^{q} i\binom{q}{i}(r^d-1)^{q-i} = \frac{1}{dr^d}$$

as required.

REFERENCES

1. Émile Borel, *Les probabilités dénombrablés et leurs applications arithmétiques*, Rend. Circ. Mat. Palmero **27** (1909), 247-271.

2. Ivan Niven and H. S. Zuckerman, *On the definition of normal numbers*, Pacific J. Math., **1** (1951), 103-109.

3. S. S. Pillai, *On normal numbers*, Proc. Indian Acad. Sci., Sect. A, **12** (1940), 179-184.

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