# ASYMPTOTIC BEHAVIOR OF RESTRICTED EXTREMAL POLYNOMIALS AND OF THEIR ZEROS

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**Introduction.** Progress in the study of polynomials has recently been made in two directions: (i) asymptotic properties of sequences of polynomials of least norm on a given set (Leja, [7]; Davis and Pollak, [1]; Fekete, [3]; Walsh and Evans, [10]; Fekete and Walsh, [5]); (ii) geometry of the zeros of polynomials of prescribed degree minimizing a given norm on a given set, where one or more coefficients are preassigned (Zedek, [12]; Fekete, [4]; Walsh and Zedek, [11]; Fekete and Walsh, [6]). The object of the present paper is to combine these two trends, by studying the asymptotic properties of sequences of polynomials of least norm on a given set, where the polynomials are restricted by prescription of one or more coefficients.

If $S$ is a given compact point set and $N[A_n(z), S]$ any norm on $S$ of the polynomial $A_n(z)\equiv z^n+a_{1n}z^{n-1}+\cdots+a_{nn}$ we are interested in the asymptotic relations for (restricted) polynomials $A_n(z, N)$ of least $N$-norm

$$(1)\qquad \lim_{n\to\infty}\nu_n^{1/n}=\tau(S)\ ,\qquad \nu_n=N[A_n(z, N), S]\ ,$$

$$(2)\qquad \lim_{n\to\infty}|A_n(z, N)|^{1/n}=|\varphi(z)|\ ,$$

where $\tau(S)$ is the transfinite diameter of $S$, $|\varphi(z)|\equiv e^{G(z)}\tau(S)$, $G(z)$ being Green's function with pole at infinity for the maximal infinite region $K$ containing no point of $S$, and where (2) is considered uniformly on a more or less arbitrary compact set in $K$.

Part I is devoted primarily to (1); we show for instance that for the unit circle, with the first $k\equiv k(n)$ coefficients $a_{jn}$ of the extremal polynomial $A_n(z, N)$ prescribed and uniformly of the order $O\left(\binom{n}{j}\right)$ in their totality, a necessary and sufficient condition for (1) for all such choices of coefficients is $k=o(n)$, where $N$ is any classical norm. We prove similar results for other sets $S$. Part II is devoted primarily to (2); first we use as hypothesis the analogue of (1), namely

$$\{N[A_n(z), S]\}^{1/n}\to\tau(S)\ ,$$

for arbitrary polynomials $A_n(z)$; and then we use (1) as hypothesis, for extremal polynomials $A_n(z, N)$ with $k$ prescribed coefficients and $N$

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monotonic. If $A_n(z, N)$ has zeros in $K$, under suitable conditions the corresponding factors of $A_n(z, N)$ can be omitted in whole or in part, and the analogue of (2) is valid for the remaining factor, uniformly on any closed set in $K$ containing no limit point of zeros of that factor; for instance if $k$ is constant we can omit the factors of $A_n(z, N)$ corresponding to the zeros of $A_n(z, N)$ exterior to the inflated convex hull $H_k(S)$, and (2) is valid uniformly on any compact set exterior to $H_k(S)$; as another instance, if $k=1$ and if the prescribed center of gravity of the zeros of $A_n(z, N)$ is fixed and different from the conformal center of gravity of $S$, then precisely one zero of $A_n(z, N)$ becomes infinite and (2) is valid uniformly on any compact set exterior to the convex hull $H_0$ of $S$. Finally, we study (1) for extremal polynomials some of whose zeros are prescribed.

## PART I

### ASYMPTOTIC PROPERTIES OF THE LEAST $N$-NORM OF RESTRICTED POLYNOMIALS ON A GIVEN POINT SET

1. **In pursuing the objective indicated,** we start our considerations with remarks relevant to both (1) and (2). Let $A_n = A_n(\gamma_j, 1 \leqq j \leqq k)$ denote the aggregate of all polynomials $A_n(z) \equiv z^n + a_{1n}z^{n-1} + a_{2n}z^{n-2} + \cdots + a_{nn}$ satisfying

$$(3) \qquad a_{jn} = \gamma_j\ , \qquad 1 \leqq j \leqq k,\ 1 \leqq k \leqq n-1\ .$$

The reader may easily prove the existence for each $n(\geqq n^* = n^*(N))$ and for each $\gamma_j = \gamma_j(n)$ and $k = k(n)$ of a polynomial $A_n(z, N)$ in $A_n(\gamma_j, 1 \leqq j \leqq k)$ of least $N$-norm, provided $N$ belongs to the wide category of quasi-Tchebycheff (q.T.) norms continuous in $A_n$ on $S$; such norms are broad generalizations of the classical norms, including the (ordinary) Tchebycheff norm

$$M = M[A_n(z), S] = [\max |A_n(z)|,\ z \text{ on } S]\ .$$

We recall [5, p. 53] that $N[A_n(z), S]$ is a q.T. norm on $S$ provided for all

$$A_n(z) \equiv z^n + a_{n1}z^{n-1} + \cdots$$

we have

$$(4) \qquad \frac{N[A_n(z), S]}{M[A_n(z), S]} \leqq U(S, N)\ , \qquad n = 1, 2, \cdots\ ,$$

$$(5) \qquad \frac{N[A_n(z), S]}{M[A_n(z), S]} \geqq L_n(S, N, \varepsilon) > 0 \qquad \text{for } n = n_0(\varepsilon)\ ,$$

$$\lim_{n\to\infty} \{L_n(S, N, \varepsilon)\}^{1/n} = f(N, \varepsilon) , \qquad \lim_{\varepsilon\to 0} f(N, \varepsilon) = 1 .$$

A norm $N[A_n(z), S]$ is *continuous* in $A_n$ on $S$ provided to an arbitrary $A_n^*(z) \in A_n$ and $\varepsilon\,(>0)$ there corresponds a $\delta = \delta(\varepsilon, A_n^*)\ (>0)$ such that for an arbitrary polynomial $A_n^{**}(z)$ in $A_n$ the inequality $|A_n^*(z) - A_n^{**}(z)| < \delta$ on $S$ implies

$$|N[A_n^*(z), S] - N[A_n^{**}(z), S]| < \varepsilon .$$

Such continuity of $N[A_n(z), S]$ on a certain subset of $A_n$ is also necessary for the existence of a polynomial $A_n(z, N)$ in $A_n$ of least $N$-norm, since there exist instances of noncontinuous q.T.-norms $N$ for which

$$N[A_n(z), S] > [\inf N[A_n(z), S], A_n(z) \in A_n]$$

holds for all $A_n(z) \in A_n$. For the purposes of (1) if $N$ is not continuous one may replace $\min N$ in our considerations by $\inf N$. Henceforth we consider only continuous q.T. norms $N[A_n(z), S]$.

**2. The writers have already proved** [5, Theorem 2] that the relation

$$(6) \qquad \lim_{n\to\infty} \{N_0[A_n(z), S]\}^{1/n} = \tau(S)$$

for an arbitrary compact set $S$, a given q.T.-norm $N_0$, and an arbitrary sequence of polynomials $A_n(z) \equiv z^n + \cdots$, implies the relation

$$(7) \qquad \lim_{n\to\infty} \{N[A_n(z), S]\}^{1/n} = \tau(S)$$

for any other q.T.-norm on $S$. There follows

THEOREM 1. *Equation* (1) *holds for every choice of q.T. norm $N$ if and only if* (1) *holds for a particular choice of $N$, where all polynomials are restricted to $A_n$.*

From (6) with $A_n(z)$ the polynomials $A_n(z, N_0)$ we deduce (7) involving these same polynomials, and this (7) as a majorant relation proves (1); we use here the consequence of (5) that no matter what the polynomials $A_n(z)$ may be, the first member of (7) is not less than $\tau(S)$. Conversely, if (6) is not valid for a particular $N_0$ and the polynomials $A_n(z, N_0)$, then (7) is not valid for either the $A_n(z, N_0)$ or the $A_n(z, N)$, so (1) is not valid.

The importance of Theorem 1 for our investigation of (1) is that in the sequel we may instead investigate (6) with $A_n(z) \equiv A_n(z, N_0)$ for a particular $N_0$ conveniently chosen with respect to $S$.

**3. As a first such application of** Theorem 1 we prove

THEOREM 2. *Let $A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ be given, and $S$ the unit disc*

$|z| \leqq 1$. *A necessary and sufficient condition for* (1) *with* $A_n(z, N) \in A_n$ *is*

$$\lim_{n\to\infty} \{1+|\gamma_1|^2+|\gamma_2|^2+\cdots+|\gamma_k|^2\}^{1/2n}=1 , \tag{8}$$

*independently of the* q.T.*-norm* $N$.

In fact, with $A_n(z) \in A_n$ and $n \geqq k+1$, the choice

$$\begin{aligned}\{N_0[A_n(z), S]\}^2 &= \frac{1}{2\pi}\int_{|z|=1} |A_n(z)|^2 |dz| \\ &= 1+|\gamma_1|^2+\cdots+|\gamma_k|^2+|a_{k+1,n}|^2+\cdots\end{aligned} \tag{9}$$

leads to the unique minimizing polynomial

$$A_n(z, N_0) \equiv z^n+\gamma_1 z^{n-1}+\cdots+\gamma_k z^{n-k} ,$$

$$\nu_n = N_0[A_n(z, N_0), S] = \{1+|\gamma_1|^2+\cdots+|\gamma_k|^2\}^{1/2} . \tag{10}$$

Since $\tau(S)=1$, (1) with $N_0$ for $N$ is equivalent to (8). To complete the proof we recall Theorem 1.

4. **A noteworthy corollary of** Theorem 1 is

THEOREM 3. *With the notation and hypothesis of Theorem* 2, *suppose we have*

$$\gamma_j = O\left[\binom{n}{j}\right], \qquad 1 \leqq j \leqq k , \tag{11}$$

*uniformly in* $j$. *Then a necessary and sufficient condition for* (1) *with arbitrary* q.T.*-norm* $N$ *is*

$$k = o(n) . \tag{12}$$

With $N_0$ of (9) for $N$, hypothesis (11) in case (12) entails in view of (10)

$$1 \leqq \nu_n = O\left[n\binom{n}{j}\right] \qquad \text{as } n \to \infty , \tag{13}$$

since $\binom{n}{j}$ increases with $j$ provided $2j < n$. It is sufficient to prove (1) for every sequence of values $n \to \infty$, so it is sufficient to prove (1) under the alternate assumptions $k(n) \equiv O(1)$ and $k(n) \to \infty$. Relations (13) prove (1) if $k=O(1)$. If however $k=k(n) \to \infty$ still subject to (12), by Stirling's formula

$$\binom{n}{k} \sim \left[\left(\frac{n}{k}-1\right)^k \Big/ \left(1-\frac{k}{n}\right)^n\right] \sqrt{1/2\pi k} , \tag{14}$$

$$\binom{n}{k}^{1/n} \sim \left(\frac{n}{k}-1\right)^{k/n} \to 1 ,$$

implying (1) with $N=N_0$, and hence for all q.T.-norms $N$.

To prove the necessity of (12) for (1) with $N\equiv N_0$ (and hence with an arbitrary q.T.-norm), choose

$$\gamma_j = \binom{n}{j}, \qquad 1 \leqq j \leqq k .$$

Then by (10)

$$\nu_n > \binom{n}{j}, \qquad 1 \leqq j \leqq k = k(n) .$$

If (12) is false, for a suitably chosen sequence we have

$$\lim k(n)/n = \varepsilon , \qquad 0 < \varepsilon \leqq 1 ;$$

in case $0<\varepsilon<1$, by (14) follows

$$\nu_n^{1/n} > \binom{n}{k}^{1/n} \to \frac{1}{(1-\varepsilon)^{1-\varepsilon}\varepsilon^{\varepsilon}} \to 1 ,$$

while in case $\varepsilon=1$ we have

$$\nu_n^{1/n} > \binom{n}{\left[\frac{n}{2}\right]}^{1/n} \to 2 \qquad \text{as } n \to \infty .$$

This contradiction of (1) completes the proof of Theorem 2.

5. **By modifying slightly the above argument**, the reader may easily prove the following proposition, a generalization for $S$ the disc $|z| \leqq R$ of the previous two theorems.

THEOREM 4. *A necessary and sufficient condition for* (1) *with* $S: |z| \leqq R$ *and* $A_n(z, N) \in A_n(\gamma_1, \cdots, \gamma_k)$ *is*

$$\lim \{R^{2n} + |\gamma_1|^2 R^{2n-2} + \cdots + |\gamma_k|^2 R^{2n-2k}\}^{1/2n} = R . \tag{15}$$

*With the particular choice* (*uniformly in* $j$)

$$\gamma_j = O\left[R^j \binom{n}{j}\right], \qquad 1 \leqq j \leqq k , \tag{16}$$

*a necessary and sufficient condition for* (1) *is* (12).

A word is in order to justify the form of (16). Much of the present paper is devoted to the study of polynomials $A_n(z)$ in $A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$

where we have

$$\gamma_j=(-1)^j\binom{n}{j}c_j \tag{17}$$

and the numbers $c_j$ are independent of $n$. For instance the center of gravity of the zeros of $A_n(z)$ is $c_1$, and (17) may prescribe $c_1$ independent of $n$. Here it is significant (Theorem 11, below) that a necessary condition for (1) with the zeros of the $A_n(z)$ bounded is (17) with $c_1 \to c$, where of course $c_1$ is not necessarily independent of $n$, and where $c$ is the conformal center of gravity of $S$, a number depending wholly on $S$ itself.

We shall call the number $c_j$ defined by (17) *the centroid of order* $j$ of the zeros of $A_n(z)$.

Another comment on (16) is that if the $z$-plane is transformed by a simple stretching $z'=Rz$, the transfinite diameter of every set is multiplied by $R$, and the $j$th centroid of the zeros of a polynomial is multiplied by $R^j$; thus the factor $R^j$ in (16) is appropriate.

6. **We shall shortly indicate** (§§ 7, 8) that Theorems 2, 3, and 4 admit at least partial extensions to arbitrary sets whose boundaries are rectifiable. The usefulness of these extensions in the study of still more general point sets is now to be shown.

If $S$ is an arbitrary compact set, and if the maximal infinite region $K$ belonging to the complement of $S$ is regular in the sense that the classical Green's function $G(z)$ for $K$ with pole at infinity exists, we denote by

$$w=\varphi(z)\equiv \exp\left[G(z)+iH(z)+\log \tau(S)\right],$$

where $H(z)$ is conjugate to $G(z)$ in $K$, a function which maps $K$ onto $|w|>\tau(S)$ with $\varphi(\infty)=\infty$.

The locus $C_R$: $|\varphi(z)|=R\tau(S)$, $R>1$, in $K$ consists of a finite number of rectifiable Jordan curves which are mutually exterior except perhaps for a finite number of points each of which may belong to several curves; we denote the sum of the closed interiors of these curves by $S_R$. As $R\to 1$, the locus $C_R$ approaches the boundary of $K$.

THEOREM 5. *Let $S$ be a compact set, and let the infinite region $K$ belonging to the complement of $S$ be regular. Let $A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ be given and restrict $A_n(z, N)$ to $A_n$. A necessary and sufficient condition that* (1) *be valid for all* q.T.*-norms on $S$ is that* (1) *be valid for all* q.T.*-norms on all $S_R$.*

By Theorem 1, we may restrict ourselves to the consideration of the Tchebycheff norms on $S$ and $S_R$. We denote the respective extremal polynomials by $T_n(z, S)$ and $T_n(z, S_R)$. To prove the sufficiency of the condition, we write

$$M[T_n(z, S_R), S_R] \geqq M[T_n(z, S_R), S] \geqq M[T_n(z, S), S] \geqq \tau(S)^n ,$$

$$\begin{aligned} \lim \{M[T_n(z, S_R), S_R]\}^{1/n} &= \tau(S_R) \\ &= R\cdot\tau(S) \geqq \limsup \{M[T_n(z, S), S]\}^{1/n} \geqq \tau(S) , \end{aligned}$$

and $R \to 1$ establishes (1). With this reasoning as given, *it is also sufficient if* (1) *is valid on a sequence of sets* $S_m$ *each containing* $S$, *with* $K$ *regular or not, provided* $\tau(S_m) \to \tau(S)$; *the sets* $S_m$ *may be taken as the closed interiors of lemniscates.*

Conversely, by use of the generalized Bernstein Lemma [9, p. 77] we have

$$\{\tau(S_R)\}^n \leqq M[T_n(z, S_R), S_R] \leqq M[T_n(z, S), S_R] \leqq M[T_n(z, S), S]R^n ,$$

$$\begin{aligned} \tau(S_R) \leqq \liminf \{M[T_n(z, S_R), S_R]\}^{1/n} &\leqq \limsup \{M[T_n(z, S_R), S_R]\}^{1/n} \\ &\leqq \lim \{M[T_n(z, S), S]\}^{1/n}\cdot R = \tau(S_R) , \end{aligned}$$

$$\lim \{M[T_n(z, S_R), S_R]\}^{1/n} = \tau(S_R) .$$

**7. Theorem 5 emphasizes the importance** in considering (1) of sets with rectifiable boundary, both for their own sake and for the study of more general sets. For the former we have the great advantage of orthogonal polynomials as a tool. Thus we prove[2]

THEOREM 6. *Let the point set* $S$ *consist of a finite number of rectifiable Jordan arcs, and let the polynomials* $P_n(z) \equiv z^n + \cdots$ *of respective degrees* $n$ *be mutually orthogonal on* $S$, *with*

$$\int_S |P_n(z)|^2 |dz| = p_n .$$

*Let the norm* $N_0$ *be defined by*

$$\{N_0[A_n(z)]\}^2 = \int_S |A_n(z)|^2 |dz| ,$$

*and let us set*

$$B_n(z) \equiv z^n + \gamma_1 z^{n-1} + \cdots + \gamma_k z^{n-k} \equiv d_0 P_n(z) + d_1 P_{n-1}(z) + \cdots + d_n P_0(z) ,$$

$$p_h d_h = \int_S B_n(z)\overline{P_h(z)}|dz| , \qquad d_h = d_h(n) ,\ 0 \leqq h \leqq n ,$$

[2] An analogous theorem obviously exists if $S$ consists of a finite number of mutually disjoint Jordan regions, and least-square norm is measured by surface instead of line integrals.

*where the* $\gamma_j = \gamma_j(n)$ *and* $k = k(n)$ *are prescribed. For each* $n \geqq k+1$, $\min N_0[A_n(z)]$ *with* $A_n(z) \in A_n(\gamma_j)$ *is* $N_0[C_n(z)]$ *where*

$$C_n(z) \equiv d_0 P_n(z) + d_1 P_{n-1}(z) + \cdots + d_k P_{n-k}(z) \ ,$$

*and is assumed by no other polynomial. A necessary and sufficient condition for* (1) *with* $N = N_0$ *or with* $N$ *an arbitrary* q.T.*-norm on* $S$ *is*

$$\lim_{n\to\infty} \{|d_0|^2 p_n + |d_1|^2 p_{n-1} + \cdots + |d_k|^2 p_{n-k}\}^{1/2n} = \tau(S) \ .$$

An arbitrary polynomial $A_n(z) \equiv z^n + \cdots$ may obviously be expressed as a linear combination $A_n(z) \equiv b_0 P_n(z) + b_1 P_{n-1}(z) + \cdots + b_n P_0(z)$ by considering successively the coefficients of $z^n, z^{n-1}, \cdots, 1$. Then the coefficients $b_h$ can also be computed by use of the orthogonality relations,

$$\int_S A_n(z)\overline{P_h(z)}\,|dz| = p_h b_{n-h} \ ,$$

and we have

$$\{N_0[A_n(z)]\}^2 = |b_0|^2 p_n + |b_1|^2 p_{n-1} + \cdots + |b_n|^2 p_0 \ .$$

The condition $A_n(z) \in A_n(\gamma_j)$ is equivalent to specific prescription of $b_0, b_1, \cdots, b_k$, so it is clear that

$$\min N_0[A_n(z)] = N_0[C_n(z)] = \{|d_0|^2 p_n + |d_1|^2 p_{n-1} + \cdots + |d_k|^2 p_{n-k}\}^{1/2} \ ,$$

a minimum assumed by no other polynomial than $C_n(z)$ in $A_n$; the remainder of Theorem 6 follows from Theorem 1.

Both Theorem 2 and the first part of Theorem 4 are clearly generalized in Theorem 6. We proceed to a corresponding generalization of the necessity of condition (16) in the second part of Theorem 4.

8. **The number $R$ plays two roles in** Theorem 4: it is both $\tau(S)$ and a parameter restricting the order of $\gamma_j$ in (16) if $k = k(n)$ is not bounded.

In extending the second part of Theorem 4 to a compact set $S$ of connected complement $K$ whose boundary $B$ consists of a finite number of rectifiable Jordan arcs or even to a more general set $S$ with regular connected complement $K$, the second of these roles is kept for $R$. To be more explicit we shall prove the following.

THEOREM 7. *Let* $S$ *be a compact set of connected regular complement* $K$ *and* $R$ *an arbitrary positive number such that the disc* $|z| \leqq R$ *contains* $S$ *in its interior.*

*Suppose that* (16) *holds with* $k = k(n) \to \infty$, $k(n) \neq o(n)$. *Then there exist polynomials* $A_n(z, N) \in A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ *of least* q.T.*-norm* $N = N[A_n(z), S]$ *on* $S$ *for which* (1) *is not valid.*

We chose $R_1=R_1(R)>1$ so that not only $S$ but also $S_{R_1}$ of § 6 is covered by $|z|<R$ (thus $\tau(S_{R_1})<R$). We know (Cf. § 6) that with $T_n(z, S)\in A_n$, $T_n(z, S_{R_1})\in A_n$,

$$R_1^n M[T_n(z, S), S]\geqq M[T_n(z, S), S_{R_1}]\geqq M[T_n(z, S_{R_1}), S_{R_1}]\ . \tag{18}$$

On the other hand if $C_{R_1}$ denotes the boundary of $S_{R_1}$,

$$\begin{aligned}\{M[T_n(z, S_{R_1}), S_{R_1}]\}^2\int_{C_{R_1}}|dz| &\geqq \int_{C_{R_1}}|T_n(z, S_{R_1})|^2|dz| \\ &\geqq \int_{C_{R_1}}|A_n(z, N_0)|^2|dz|]\ ,\end{aligned} \tag{19}$$

where $A_n(z, N_0)\in A_n$ is of least square norm $N_0$ on $S_{R_1}$,

$$\{N_0[A_n(z), S_{R_1}]\}^2=\int_{C_{R_1}}|A_n(z)|^2|dz|\ .$$

Fix $\gamma_j=\gamma_j(n)$, $1\leqq j\leqq k=k(n)$, by

$$\begin{aligned}A_n(z)\equiv P_n(z)+R^h\binom{n}{k}P_{n-h}(z)\Big/\binom{n-h}{h}+R^k\binom{n}{k}P_{n-k}(z)\\ +\lambda_1 P_{n-k-1}(z)+\cdots+\lambda_{n-k}P_0(z)\end{aligned}$$

where $h=\left[\dfrac{k}{2}\right]$, and the $P_m(z)=z^m+\cdots$ are mutually orthogonal on $C_{R_1}$. All zeros of the $P_m(z)$ lie in $|z|\leqq R$ (Fejér, [2]) so (16) is satisfied; indeed $\binom{n}{k}\binom{n-h}{j}=\binom{n-h}{h}\binom{n}{n-h-j}$ for $1\leqq j\leqq h$. For $A_n(z)\in A_n(\gamma_1, \cdots, \gamma_k)$ so chosen we have

$$A_n(z, N_0)=P_n(z)+R^h\binom{n}{k}P_{n-h}(z)\Big/\binom{n-h}{h}+R^k\binom{n}{k}P_{n-k}(z)\ ,$$

$$\{N_0[A_n(z, N_0), S_{R_1}]\}^2=p_n+R^{2h}\binom{n}{k}^2 p_{n-k}\Big/\binom{n-h}{h}^2+R^{2k}\binom{n}{k}^2 p_{n-k}$$

where

$$p_j=\int_{C_{R_1}}|P_j(z)|^2|dz|\ ,\qquad\qquad j=0, 1, 2, \cdots\ .$$

Hence

$$N_0[A_n(z, S_{R_1}), S_{R_1}]>\max\left\{R^h\binom{n}{k}p_{n-h}^{1/2}\Big/\binom{n-h}{h},\ R^k\binom{n}{k}p_{n-k}^{1/2}\right\}\ .$$

By Fekete-Walsh [5]

$$p_n^{1/2n} \to \tau(S_{R_1}) \qquad \text{as } n \to \infty .$$

Consider any sequence of $n$ for which $\lim \dfrac{k}{n} = \varepsilon$ exists with $0 < \varepsilon \leqq 1$. In case $0 < \varepsilon < 1$, (Cf. § 4)

$$\begin{aligned}(20) \qquad \liminf \{N_0[A_n(z, S_{R_1}), S_{R_1}]\}^{1/n} &\geqq R^{\varepsilon}\tau(S_{R_1})^{1-\varepsilon}(1-\varepsilon)^{\varepsilon-1}\varepsilon^{-\varepsilon} \\ &> R_1\tau(S)(1-\varepsilon)^{\varepsilon-1}\varepsilon^{-\varepsilon} ;\end{aligned}$$

for we have $R > \tau(S_{R_1}) = R_1\tau(S)$.
In case $\varepsilon = 1$, by $\binom{n}{k}\Big/\binom{n-h}{h} \to 1$ as $n \to \infty$, we have

$$(21) \qquad \liminf \{N_0[A_n(z, N_0), S_{R_1}]\}^{1/n} \geqq R^{1/2}\tau(S_{R_1})^{1/2}\cdot 2^{1/2} \geqq R_1\tau(S)\cdot 2^{1/2} .$$

Combining (20) or (21) with (18) and (19) we obtain

$$\limsup \{M[T_n(z, S), S]\}^{1/n} > \tau(S)\cdot((1-\varepsilon)^{-1}\varepsilon^{-\varepsilon} \text{ or } 1) \geqq \tau(S) .$$

Thus the proof is complete for the classical Tchebycheff norm $M$, and the theorem follows by Theorem 1.

Theorem 7 can be extended to arbitrary compact sets $S$ of positive transfinite diameter $\tau(S)$ with connected nonregular complement $K$, the role of $R$ being taken by any positive number such that the disc $|z| < R$ contains a level locus $C_{R_1}$: $G(z) = \log R_1$ which consists of finitely many Jordan contours and separates $S$ from infinity. The proof is similar to the above one but uses the generalized Bernstein lemma in its extended form. (Walsh, [9], § 4.9). Corresponding extensions to the case of $K$ nonregular can be made for Theorem $8^{\text{bis}}$ and the second part of Theorem 9 below (concerning the respective necessary conditions for the validity of (1)).

**9. We have studied in some detail** the conditions (12) and (16) singly and in combination, especially if $S$ is a circular disc, and in particular have shown in Theorem 7 for a more general set $S$ that (12) is necessary[3] for (1) provided (16) is assumed with the choice $R > [\max |z|, z \text{ on } S]$. We are not in a position to prove that conversely

[3] Nevertheless $k = k(n) = O(n)$, more precisely $k(n) = n-1$, is compatible with the validity of (1) for $k$-fold restricted $n$th degree extremal polynomials $A_n(z, N) \equiv z^n + a_{1n}z^{n-1} + \cdots + a_{n-1,n}z + a_{nn}$ with suitably preassigned coefficients $a_{jn} = \gamma_j = \gamma_j(n)$, $1 \leqq j \leqq k$, fulfilling condition (16) with the new choice $R = [\max |z|, z \text{ on } S]$. In fact, the coefficients $\gamma_j(n)$ of the classical $T$-polynomial $t_n(z) = z^n + \gamma_1(n)z^{n-1} + \cdots + \gamma_{n-1}(n)z + \gamma_n(n)$ on $S$, by Fejér's theorem, satisfy (16) with this special choice of $R$, and the $n$th degree $(n-1)$-fold restricted $T$-polynomials $T_n(z, S) \in A_n(\gamma_1(n), \cdots, \gamma_{n-1}(n))$ minimizing the classical $T$-norm on $S$ obviously coincide with $t_n(z)$, thus satisfy $[\max |T_n(z, S), z \text{ on } S]^{1/n} \to \tau(S)$ as $n \to \infty$. Hence, by Theorem 1, the validity of (1) for all $A_n(z, S) \in A_n(\gamma_1(n), \cdots, \gamma_{n-1}(n))$ of least q. T. norm $N$ on $S$, subject to (16) with $R = [\max |z|, z \text{ on } S]$ as required, although (12) does not hold.

(12), with the assumption of (16), is sufficient for (1), but now prove for an arbitrary compact set $S$ that a slightly stronger condition on $k(n)$ than (12), namely (22), is sufficient for (1) with a weaker assumption than (16) concerning the centroids $c_j$ defined by (17).

THEOREM 8. *Let $S$ be an arbitrary compact set, and let $N$ be a continuous* q.T.*-norm on $S$. Let $A_n(z)\equiv z^n+\cdots$ minimize $N$ over $A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ with*

$$k=o\left(\frac{n}{\log n}\right), \tag{22}$$

$$|c_j|\leq\alpha_j , \qquad 1\leq j\leq k , \tag{23}$$

*where the $\alpha_j$ are independent of $n$.*

*Then* (1) *is valid provided the power series*

$$\sum_{h=1}^{\infty} \alpha_h z^h/h!$$

*has positive radius of convergences $r$.*[4]

Let $t_h(z)\equiv z^h+a_1^{(h)}z^{h-1}+\cdots+a_h^{(h)}$ be the $h$th degree classical Tchebycheff polynomial minimizing the norm $M[A_h(z), S]$ on $S$ among all polynomials $A_h(z)$ with the leading terms $z^h$. Then a majorant of the $n$th degree generalized Tchebycheff polynomial $T_n(z, S)\in A_n$ can be expressed as the product $t_{n-k}(z)\cdot(z^k+\lambda_1 z^{k-1}+\cdots+\lambda_k)$ where the coefficients $\lambda_j=\lambda_j(k, n)$ satisfy the linear equations

$$\lambda_1+a_1^{(n-k)}=\gamma_1,\ \lambda_2+\lambda_1 a_1^{(n-k)}+a_2^{(n-k)}=\gamma_2,\ \cdots$$

$$\lambda_k+\lambda_{k-1}a_1^{(n-k)}+\lambda_{k-2}a_2^{(n-k)}+\cdots+a_{n-2k}^{(n-k)}=\gamma_k .$$

Hence, by Fejér's Theorem [2] on the zeros of $t_h(z)$

$$|\lambda_1|\leq|\gamma_1|+|a_1^{(n-k)}|\leq\binom{n}{1}\alpha_1+\binom{n-k}{1}R<nR\left(1+\frac{\alpha_1}{R}\right)$$

where $R\geq[\max|z|,\ z\in S]+1$. Similarly,

[4] While the research here presented was in progress, Professor G. Szegö communicated to the first named author the following result. *Let $L$ be an analytic Jordan curve. Let the positive constants $\alpha_j$ satisfy the condition $|\alpha_j|<ab^j$, $j=0, 1, 2, \cdots$, where $a$ and $b$ are arbitrary positive constants. Let the integer $k=k(n)$ satisfy the condition $k(n)=o(n/\log n)$. There exist polynomials $A_n(z)\equiv z^n+\cdots$ satisfying condition* (23) *such that*

$$\lim_{n\to\infty}\max_{z\in L}|A_n(z)|^{1/n}=\tau(L)$$

*where $\tau(L)$ is the transfinite diameter of $L$.*

This communication induced us to study the problem dealt with in Theorem 8.

$$|\lambda_2| \leqq |\gamma_2| + |\lambda_1|\,|a_1^{(n-k)}| + |a_1^{(n-k)}| \leqq \binom{n}{2}\alpha_2 + nR\left(1+\frac{\alpha_1}{R}\right)\binom{n-k}{1}R$$

$$+\binom{n-k}{2}R^2 < 2n^2R^2\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}\right),$$

$$\lambda_3 \leqq |\gamma_3| + |\lambda_2|\,|a_1^{(n-k)}| + |\lambda_1|\,|a_2^{(n-k)}| + |a_3^{(n-k)}|$$

$$\leqq \binom{n}{3}\alpha_3 + 2n^2R^2\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}\right)\binom{n-k}{1}R$$

$$+nR\left(1+\frac{\alpha_1}{R}\right)\binom{n-k}{2}R^2 + \binom{n-k}{3}R^3 \leqq 2^2n^3R^3\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}+\frac{\alpha^3}{3!R^3}\right),$$

and so on; finally

$$|\lambda_k| \leqq 2^{k-1}n^kR^k\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}+\cdots+\frac{\alpha_k}{k!R^k}\right).$$

Hence, for $R > 1/r$ we have

$$M[T_n(z,S),S] < M[t_n(z),S]\cdot(k+1)2^{k-1}n^kR^k\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}\right.$$

$$\left.+\cdots+\frac{\alpha_k}{k!R^k}+\cdots \text{ in inf.}\right)$$

and, therefore, by (22) and

$$\lim\,\{M[t_n(z),S]\}^{1/n} = \tau(S),$$

$$\limsup\,\{M[T_n(z,S),S]\}^{1/n} \leqq \tau(S).$$

On the other hand

$$\liminf\,\{M[T_n(z,S),S]\}^{1/n} \geqq \tau(S).$$

Thus the proof is complete for the classical Tchebycheff norm $M$, and the theorem follows by Theorem 1.

As a converse of the proposition just demonstrated we prove

THEOREM 8 bis. *Let $S$ satisfy the assumptions of Theorem* 7. *Suppose $k=k(n)\to\infty$ as $n\to\infty$ but $k(n)\neq o(n/\log n)$. Then there exist polynomials $\alpha_n(z,N)\in A_n(\gamma_1,\gamma_2,\cdots,\gamma_k)$ of least* q.T.*-norm $N=N[\alpha_n(z),S]$ on $S$ with $|\gamma_j^{(n)}|\leqq\binom{n}{j}\alpha_j$ for $1\leqq j\leqq k(n)$ and* $\limsup\,[\alpha_h/h!]^{1/h}<\infty$ *for which* (1) *is not valid.*

Since Theorem 7 established the existence in question in the case $\alpha_j=R^j$ if $k(n)\neq o(n)$, we assume $\limsup\,(k\log n/n)>0$ with $k/n\to 0$. Then we fix $\gamma_j=\gamma_j(n)$, $1\leqq j\leqq k=k(n)$, by

$$\alpha_n(z) \equiv P_n(z) + \binom{n}{k}(k!)^{1/2}R^k P_{n-k}(z) + \lambda_1 P_{n-k-1}(z) + \cdots + \lambda_{n-k}P_0(z) ,$$

where $P_n(z)$, $C_{R_1}$, and $R_1$ have the same meaning as in § 8; thus $|\gamma_j(n)| < \binom{n}{j}R^j$ for $1 \leqq j \leqq k-1$ and $|\gamma_k(n)| < \binom{n}{k}$ $[1+(k!)^{1/2}]R^k$ and therefore $\alpha_h = 1+(h!)^{1/2}$ satisfies our hypothesis. The $n$th degree polynomial $\alpha_n(z, N_0)$ of least square norm

$$N_0 = N_0[\alpha_n(z), C_{R_1}] = \left\{\int_{C_{R_1}} |\alpha_n(z)|^2 |dz| \right\}^{1/2}$$

on $C_{R_1}$ among all

$$\alpha_n(z) \equiv z^n + \cdots \in A_n(\gamma_1, \cdots, \gamma_n)$$

is obviously

$$P_n(z) + \binom{n}{k}(k!)^{1/2}R^k P_{n-k}(z) ,$$

and we have

$$\int_{C_{R_1}} |\alpha_n(z, N_0)|^2 |dz| > R^{2k}k! \int_{C_{R_1}} |P_{n-k}(z)|^2 |dz| = R^{2k}k! p_{n-k} .$$

In view of $k(n) = o(n)$, $p_n^{1/2n} \to \tau(C_{R_1}) = R_1\tau(S)$, $\limsup\,[k \log n/n] = \varepsilon(>0)$, we have now

$$\limsup \left\{\int_{C_{R_1}} |\alpha_n(z, N_0)|^2 |dz|\right\}^{1/2n} \geqq R_1\tau(S) \limsup\,(k!)^{1/2n}$$
$$\geqq R_1\tau(S)e^{\varepsilon/2} > R_1\tau(S) .$$

Combination of this inequality with (18) and (19) yields

$$\limsup\,\{M[T_n(z, S), S]\}^{1/2} > \tau(S)$$

for the polynomial $T_n(z, S) \in A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ minimizing the classical norm $M[\alpha_n(z), S]$ on $S$, and the theorem is established.

**10. We conclude our investigations** concerning the validity of (1) with $A_n(z, N) \in A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$ by considering the particular case

$$k = k(n) = O(1)$$

nearest to the simplest one: $k = $const. Using arguments similar to those just applied in the proof of Theorem 8 we obtain at once, for arbitrary compact set $S$

$$\tau(S)^n \leqq M[T_n(z, S), S]$$
$$\leqq M[t_n(z), S]\cdot(k+1)\cdot 2^{k-1}n^k R^k\left(1+\frac{\alpha_1}{R}+\frac{\alpha_2}{2!R^2}+\cdots+\frac{\alpha_k}{k!R^k}\right)$$

provided $|\gamma_j| \leqq \binom{n}{j}\alpha_j$, $\alpha_j = \alpha_j(n)$. Hence by the hypothesis $k = O(1)$

$$\tau(S) \leqq \liminf \{M[T_n(z, S), S]\}^{1/n} \leqq \limsup \{M[T_n(z, S), S]\}^{1/n}$$
$$\leqq \tau(S)\cdot \max(1, \limsup \{[\max \alpha_j(n), 1 \leqq j \leqq k = k(n)]\}^{1/n}) .$$

Hence the validity of (1) for the classical Tchebycheff norm $M$ and thus also for each continuous q.T.-norm $N$ provided

$$\limsup \{\alpha_j(n)\}^{1/n} \leqq 1 , \qquad 1 \leqq j \leqq \max_{n \geqq 2} k(n) . \tag{24}$$

Conversely, the condition (24) in case $|\gamma_j| \leqq \binom{n}{j}\alpha_j(n)$, $1 \leqq j \leqq k(n) = O(1)$, is also necessary for (1) with $A_n(z, N) \in A_n(\gamma_1, \cdots, \gamma_k)$ for $N$ the classical Tchebycheff norm $M$ and hence for arbitrary q.T.-norms $N$ continuous in $A_n$ on a compact set $S$ of connected regular complement $K$.

In fact, fix $\gamma_j = \gamma_j(n)$, $1 \leqq j \leqq k = k(n)$, by

$$A_n(z) \equiv P_n(z) + \alpha_k(n)\binom{n}{k}P_{n-k}(z) + \lambda_1 P_{n-k-1}(z) + \cdots + \lambda_{n-k}P_0(z) ,$$

where the $P_m(z) = z^m + \cdots$ are mutually orthogonal on $C_{R_1}$ and $R$, $R_1(R)$, $C_{R_1}$, and $S_{R_1}$ are defined as in § 8. Then with $\alpha_j(n) \equiv R^j$ for $1 \leqq j \leqq k-1$, the condition (24) is fulfilled. We shall show that (1) cannot hold under our hypotheses unless (24) is satisfied also for $j = k$. In the proof we may restrict ourselves to the classical Tchebycheff norm $M$, minimized by the generalized Tchebycheff polynomial $T_n(z, S) \in A_n(\gamma_1, \cdots, \gamma_k)$. Applying the results (18) and (19) of § 8 we can write

$$R_1^n M[T_n(z, S), S]\left\{\int_{C_{R_1}} |dz|\right\}^{1/2} \geqq \left\{\int_{C_{R_1}} |A_n(z, N_0)|^2 |dz|\right\}^{1/2} , \tag{25}$$

where $A_n(z, N_0) \in A_n(\gamma_1, \cdots, \gamma_k)$ is the $n$th degree polynomial of least square norm $N_0(A_n(z), S_{R_1}) = \left\{\int_{C_{R_1}} |A_n(z)|^2 |dz|\right\}^{1/2}$ on $C_{R_1}$. Our above choice of $A_n(\gamma_1, \cdots, \gamma_k)$ yields

$$A_n(z, N_0) \equiv P_n(z) + \alpha_k(n)\binom{n}{k}P_{n-k}(z) ,$$

whence

$$(26) \qquad \{N_0[A_n(z, N_0), S_{R_1}]\}^2 = p_n + \left\{ \alpha_k(n)\binom{n}{k} \right\}^2 p_{n-k} .$$

Combining (25) with (26) leads in view of $\lim p_m^{1/m} = \tau(S_{R_1})$ to

$$R_1 \limsup \{M[T_n(z, S), S]\}^{1/n} \geqq \tau(S_{R_1}) \limsup \{\alpha_k(n)\}^{1/n} ,$$

which by $\tau(S_{R_1}) = R_1\tau(S)$ is equivalent to

$$\limsup \{M[T_n(z, S), S]\}^{1/n} \geqq \tau(S) \limsup \{\alpha_k(n)\}^{1/n} .$$

This establishes (24) as a necessary condition for (1) in case of the $M$-norm and hence also for all q.T.-norms $N$ on any set $S$ of the type considered. We summarize the foregoing results in

THEOREM 9. *Let $S$ be an arbitrary compact set, and let $N$ be a continuous* q.T.*-norm on $S$. Let $A_n(z) = z^n + \cdots$ minimize $N$ over $A_n(\gamma_1, \cdots, \gamma_k)$ with*

$$(27) \qquad k = k(n) = O(1) ,$$

$$(28) \qquad |\gamma_j| \leqq \binom{n}{j}\alpha_j(n) , \qquad 1 \leqq j \leqq k .$$

*Then* (1) *is valid provided* (24) *holds. Conversely, in case* (27) *is fulfilled,* (24) *is also necessary for* (1) *provided $S$ is a compact set of regular complement $K$.*

**11. In the previous sections we developed** conditions, necessary or sufficient or both, for the validity of (1). By Theorem 1 such conditions are the same for all q.T.-norms $N$ which are defined on the set $S$ considered. If (1) does not hold, there might be two possibilities:

(a) $\lim \{N[A_n(z, N), S]\}^{1/n} = \omega(S, N, \gamma_1, \gamma_2, \cdots, \gamma_k)$, with the polynomials $A_n(z, N)$ of least $N$-norm restricted to some given $A_n(\gamma_1, \cdots, \gamma_k)$, but $\omega$ is different from $\tau(S)$;

(b) the $\{N[A_n(z, N), S]\}^{1/n}$ have no limit as $n \to \infty$ if $A_n(z, N) \in A_n$, that is

$$\liminf \{N[A_n(z, N), S]\}^{1/n} = \alpha(S, N, \gamma_1, \cdots, \gamma_r)$$

is actually smaller than

$$\limsup \{N[A_n(z, N), S]\}^{1/n} = \beta(S, N, \gamma_1, \cdots, \gamma_k) .$$

**12. It is easy to show that both** possibilities (a) and (b) may eventually occur. In the light of this fact the following result has some intrinsic interest:

THEOREM 10. *Let $S$ be an arbitrary compact set and let $N=N[A_n(z), S]$ be any given* q.T.*-norm defined and continuous in $A_n=A_n(\gamma_1, \cdots, \gamma_k)$ on $S$. Let the least $N$-norm $\nu_n$ on $S$ for polynomials $A_n(z) \in A_n$ satisfy*

$$\liminf \nu_n^{1/n}=\alpha(S, N, \gamma_1, \cdots, \gamma_k) ,$$

$$\limsup \nu_n^{1/n}=\beta(S, N, \gamma_1, \cdots, \gamma_k) .$$

*Then $\alpha$ and $\beta$ are independent of the particular choice of $N$ and, consequently* $\lim \nu_n^{1/n}$ *exists or not for all* q.T.*-norms $N$, the coexisting limits having the same value $\omega(S, \gamma_1, \cdots, \gamma_n)$.*

Applying (5) with $A_n(z) \equiv A_n(z, N)$ the polynomial of least $N$-norm on $S$ for $A_n(z) \in A_n$, we obtain

$$\begin{aligned} (29) \qquad \liminf \{N[A_n(z, N), S]\}^{1/n} &\geqq \liminf \{M[A_n(z, N), S]\}^{1/n} \\ &\geqq \liminf \{M[A_n(z, M), S]\}^{1/n} , \\ \limsup \{N[A_n(z, N), S]\}^{1/n} &\geqq \limsup \{M[A_n(z, N), S]\}^{1/n} \\ &\geqq \limsup \{M[A_n(z, M), S]\}^{1/n} , \end{aligned}$$

while (4) applied to $A_n(z) \equiv A_n(z, M) \in A_n$ yields

$$\begin{aligned} (30) \qquad \liminf \{N[A_n(z, N), S]\}^{1/n} &\leqq \liminf \{M[A_n(z, M), S]\}^{1/n} , \\ \limsup \{N[A_n(z, N), S]\}^{1/n} &\leqq \limsup \{M[A_n(z, M), S]\}^{1/n} . \end{aligned}$$

Combining (29) with (30) leads to

$$\alpha(S, N, \gamma_1, \cdots, \gamma_k)=\alpha(S, M, \gamma_1, \cdots, \gamma_k) ,$$

and we similarly obtain

$$\beta(S, N, \gamma_1, \cdots, \gamma_k)=\beta(S, M, \gamma_1, \cdots, \gamma_k)$$

for all q.T.-norms $N$ defined and continuous in $A_n$ on $S$. Thus the proof for the independence of $\alpha$ and $\beta$ of the choice of $N$ is complete and hence the rest of the theorem follows if $\alpha=\beta$.

## PART II

### ASYMPTOTIC PROPERTIES OF THE MODULI, AND OF THE ZEROS OF POLYNOMIALS OF LEAST NORM

**1. In Part I we have developed primarily** sufficient conditions for the validity of (1); we propose now to consider necessary conditions for (1), namely consequences of (1) such as (2) which are significant in the study of restricted extremal polynomials. Our first three theorems are entirely general, without special reference to extremal polynomials.

THEOREM 11. *Let $S$ be a point set of positive transfinite diameter whose complement $K$ is a region containing the point at infinity, and let the zeros of the polynomials $p_n(z) \equiv z^n + a_{n1}z^{n-1} + \cdots + a_{nn}$ be uniformly bounded. Then*

$$\limsup_{n\to\infty} P_n^{1/n} \leqq \tau(S) , \qquad P_n = [\max |p_n(z)|, \ z \text{ on } S] , \tag{31}$$

*implies*

$$\lim_{n\to\infty} (a_{n1}/n) = a_1 , \tag{32}$$

*where $-a_1$ is the conformal center of gravity of $S$. That is, the center of gravity of the zeros of $p_n(z)$ approaches the conformal center of gravity of $S$.*

For any sequence of polynomials $p_n(z) \equiv z^n + \cdots$ we have

$$\liminf_{n\to\infty} P_n^{1/n} \geqq \tau(S) , \tag{33}$$

since the corresponding relation holds for the Tchebycheff polynomials of $S$; thus (31) is equivalent to

$$\lim_{n\to\infty} P_n^{1/n} = \tau(S) . \tag{34}$$

If $G(z)$ is the generalized Green's function for $K$ with pole at infinity, a suitably chosen level locus $C_R$: $G(z) = \log R (> 0)$ in $K$ consists of a single Jordan curve containing in its interior both $S$ and all the zeros of the $p_n(z)$. It then follows from (34) that we have exterior to $C_R$, uniformly on any closed bounded set exterior to $C_R$,

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\phi(z)| , \tag{35}$$

where $\phi(z) \equiv \exp[G(z) + iH(z) + \log \tau(S)]$ and $H(z)$ is conjugate to $G(z)$ in $K$; we need merely apply a previous result [Fekete and Walsh, [5], Theorem 11], where (31) is used to establish (loc. cit.)

$$\limsup_{n\to\infty} [\max |p_n(z)|, \ z \text{ on } C_R]^{1/n} \leqq R \cdot \tau(S) ;$$

the closed interior of the present $C_R$ contains all zeros of the $p_n(z)$ and has the transfinite diameter $R \cdot \tau(S)$.

We write (35) in the form

$$\lim_{n\to\infty} |p_n(z)/z^n|^{1/n} = |\phi(z)/z| , \tag{36}$$

which holds uniformly in some neighborhood of the point at infinity.

We set

$$(37) \qquad \phi(z) \equiv z + a_1 + a_2 z^{-1} + \cdots ;$$

it is of course no loss of generality to choose $\phi'(\infty)=1$, and here $-a_1$ is by definition the conformal center of gravity of $S$. If the $n$th root in (36) is suitably chosen, namely with the value unity at infinity, (36) implies

$$(38) \qquad \lim_{n\to\infty}\left[1+\frac{a_{n1}}{z^2}+\frac{a_{n2}}{z}+\cdots\right]^{1/n}=1+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots$$

uniformly in some neighborhood of the point at infinity. We use here the theory of normal families of functions. Any infinite sequence of the functions in the first member of (38) is bounded and admits a subsequence converging uniformly in the neighborhood of infinity. All limit functions are analytic in this neighborhood, have the same modulus there, and are equal at infinity; hence these limit functions are identical, and the original sequence converges uniformly in a neighborhood of infinity to this limit function. Equation (38) implies (32).

2. **Of course this same reasoning** applies to the higher coefficients in (38); for instance

$$\frac{2a_{n2}-a_{n1}^2}{2n} \to a_2 - \frac{a_1^2}{2},$$

but (32) would seem to be the most interesting of these relations.

Equation (32) has been previously established by Schiffer [8] for the case that $K$ possesses a classical Green's function, and where the $p_n(z)$ are the Fekete polynomials for $S$, whose zeros lie on $S$ and maximize the discriminant.

In the hypothesis of Theorem 11 we may replace (31) by the corresponding inequality involving an arbitrary quasi-Tchebycheff norm; compare [Fekete and Walsh, [5], Theorem 2].

3. **The significance of Theorem 11** in the theory of once-restricted and $k$-fold restricted extremal polynomials is that *if* (31) *is satisfied, then unless* (32) *is also satisfied the zeros of the* $p_n(z)$ *cannot be bounded*; thus (36) cannot be valid uniformly in the neighborhood of infinity, and may not be valid on every compact set in $K$. Of course (36) is valid uniformly in the neighborhood of infinity for all classical extremal polynomials [Fekete and Walsh, [5], Theorems 11 and 13].

An illustration here is illuminating; we choose $S$ as $|z| \leq 1$ and prescribe merely the (constant) center of gravity $c_1$ ($\neq 0$) of the zeros of each $p_n(z)$. The extremal polynomials with the least-square norm on the boundary of $S$ are

$$p_n(z) \equiv z^n - nc_1 z^{n-1} \equiv z^{n-1}(z - nc_1) ,$$

and the zeros of these polynomials are not bounded. It is striking that (36) continues to hold, but *nonuniformly*, in the neighborhood of infinity. Moreover, if we replace the prescribed $c_1$ by $c_1^{(n)}$, where $c_1^{(n)} \to 0$, then the zeros of $p_n(z)$ are bounded if and only if the numbers $nc_1^{(n)}$ are bounded.

Theorem 11 (like later results) does not require that the $p_n(z)$ be defined for every $n$; it is sufficient if these polynomials are given for an infinite sequence of values of $n$. Thus, *if the $p_n(z)$ are given and* (32) *is not satisfied, the zeros of the sequence $p_n(z)$ cannot be bounded; if* (32) *is satisfied for no subsequence of indices $n$, the zeros of no subsequence of the $p_n(z)$ can be bounded; in particular if the $p_n(z)$ are once-restricted extremal polynomials $p_n(z) \equiv z^n + na_1 z^{n-1} + \cdots$ with constant $a_1$ different from the conformal center of gravity of $S$, there exists a unique sequence for $n$ sufficiently large of zeros $\mathfrak{z}_n$ of the respective $p_n(z)$, where $\mathfrak{z}_n$ lies exterior to the extended convex hull $H_1$ of $S$, with $\mathfrak{z}_n \to \infty$, and all other zeros of $p_n(z)$ lie in $H_1$*; compare Fekete and Walsh [**6**; Theorem VII].

**4. Deeper results can be established** concerning the zeros of the $p_n(z)$ which become infinite.

THEOREM 12. *Let $S$ be a point set of positive transfinite diameter whose complement $K$ is a region containing the point at infinity. Let $G(z)$ be the generalized Green's function for $K$ with pole at infinity, and let $\Gamma$ be a Jordan curve in $K$ which separates $S$ from the point at infinity. Suppose*

$$p_n(z) \equiv z^n + a_{n1} z^{n-1} + \cdots + a_{nn} ,$$

*and suppose* (31) *is valid. Let us write $p_n(z) \equiv q_{n-\sigma}(z) \cdot r_\sigma(z)$, where $r_\sigma(z) \equiv z^\sigma + \cdots$ is a polynomial whose zeros are precisely the zeros of $p_n(z)$ exterior to $\Gamma$. Then we have*

$$\sigma = \sigma(n) = o(n) , \tag{39}$$

$$\lim_{n\to\infty} Q_{n-\sigma}^{1/(n-\sigma)} = \tau(S), \qquad Q_{n-\sigma} = \max\,[|q_{n-\sigma}(z)|,\ z \text{ on } S] . \tag{40}$$

Equation (39) follows at once [Walsh and Evans, [**10**]], for the number $n-\sigma$ of zeros of $p_n(z)$ on and interior to $\Gamma$ satisfies $(n-\sigma)/n \to 1$.

Since $S$ is closed, the distance $d$ from $S$ to $\Gamma$ is positive, so for $z$ on $S$ we have $|r_\sigma(z)| > d^\sigma$. Consideration of a point $z$ of $S$ at which $|q_{n-\sigma}(z)| = Q_{n-\sigma}$ then yields

(41) $$|q_{n-\sigma}(z)|\cdot d^{\sigma}=Q_{n-\sigma}\cdot d^{\sigma}<Q_{n-\sigma}\cdot|r_{\sigma}(z)|\leqq P_n\ .$$

Equation (39) implies $d^{\sigma/n}\to 1$, whence from (41)

$$\limsup Q_{n-\sigma}^{1/n}\leqq \limsup P_n^{1/n}\leqq \tau(S)\ .$$

But we may write also

$$\limsup Q_{n-\sigma}^{1/n}=\limsup Q_{n-\sigma}^{1/(n-\sigma)}\ ,$$

so (40) follows by the analogue of (33).

Of course it is a consequence of (40) that the center of gravity of the zeros of $q_{n-\sigma}(z)$ approaches the conformal center of gravity of $S$.

It follows from Theorem 12 [Walsh and Evans, [10], p. 335] that on any closed set exterior to $\Gamma$ we have

$$\lim_{n\to\infty}|q_{n-\sigma}(z)|^{1/(n-\sigma)}=|\phi(z)|\ ,$$

the analogue of (35).

5. **Our main interest lies in** the zeros of $p_n(z)$ which become infinite, but Theorem 12 deals also with also with other zeros. In particular, *if $p_n(z)$ is a $k$-fold restricted ($k$=const.) extremal polynomial on $S$ for a monotonic quasi-Tchebycheff norm, and if either $\Gamma$ is the boundary of $H_k(S)$ (supposed to contain $S$ in its interior) or is a curve containing $H_k(S)$ in its closed interior, then at most $k$ zeros of $p_n(z)$ lie exterior to $\Gamma$; we have $\sigma(n)\leqq k$. Moreover, equation* (35) *is valid uniformly on any closed bounded set in $K$ containing no limit point of the zeros of the $p_n(z)$; and* (35) *with $p_n(z)$ replaced by $q_{n-\sigma}(z)$ is valid uniformly on any closed bounded set in $K$ containing no limit point of the zeros of the $q_{n-\sigma}(z)$, in particular is valid on any closed bounded set exterior to $\Gamma$* [compare Walsh and Evans, [10], p. 335].

6. **Theorem** 13 **is complementary** to Theorem 12:

THEOREM 13. *Under the conditions of Theorem* 12 *we have*

(42) $$\lim_{n\to\infty}R_\sigma^{1/n}=1,\qquad R_\sigma=[\max|r_\sigma(z)|,\ z\ on\ S]\ .$$

There exists a number $D$ independent of $Z$ such that for each fixed point $Z$ on or exterior to $\Gamma$ and as $Z_1$ and $Z_2$ range over $S$, we have for the distances

$$\frac{\max\overline{ZZ_1}}{\min\overline{ZZ_2}}\leqq D\ ,$$

for the first member depends continuously on $Z$ and approaches unity as $Z\to\infty$. The zeros of $r_\sigma(z)$ lie exterior to $\Gamma$, so for $z$ on $S$ we have

$$\frac{\max|r_\sigma(z)|}{\min|r_\sigma(z)|}\leqq D^\sigma .$$

From (41) we may write for $z$ on $S$

$$Q_{n-\sigma}\cdot[\min|r_\sigma(z)|]\leqq P_n ,$$

$$Q_{n-\sigma}\cdot R_\sigma\leqq Q_{n-\sigma}\cdot[\min|r_\sigma(z)|]D^\sigma\leqq P_n\cdot D^\sigma . \tag{43}$$

In equation (40) we may replace the exponent $1/(n-\sigma)$ by $1/n$, so from (43), (31), and (39) we have

$$\limsup_{n\to\infty} R_\sigma^{1/n}\leqq 1 . \tag{44}$$

However, (33) for arbitrary polynomials here shows

$$\liminf_{n\to\infty} R_\sigma^{1/\sigma}\geqq\tau(S) ,$$

$$\liminf_{n\to\infty}\left[\frac{1}{n}\log R_\sigma\right]\geqq 0 , \tag{45}$$

and (42) follows.

**7. A result similar to Theorems** 12 **and** 13 **is the following.**

*Under the hypothesis of Theorem* 12 *let* $\Gamma_1$ *be an arbitrary Jordan curve in* $K$ *containing on or within it no point of* $S$, *and let* $r_\sigma(z)\equiv z^\sigma+\cdots$ *be the polynomial whose zeros are the zeros of* $p_n(z)$ *on and within* $\Gamma_1$, *with* $p_n(z)\equiv q_{n-\sigma}(z)\cdot r_\sigma(z)$. *Then we have* (39), (40), (42), *and the relation* $\lim|q_{n-\sigma}(z)|^{1/(n-\sigma)}=|\phi(z)|$ *uniformly on any closed set interior to* $\Gamma_1$. Here (39) follows at once [Walsh and Evans, [10]], (40) follows as in the proof of Theorem 12, (42) follows from the boundedness of the zeros of $r_\sigma(z)$ and from (45), and the remaining remark is immediate [Walsh and Evans, [10], p. 335].

**8. Theorems** 12 **and** 13, **devoted to** arbitrary sequences of polynomials $p_n(z)$ such as those studied in Walsh and Evans [10], yield a precise result for restricted extremal polynomials in the case $k=1$.

THEOREM 14. *Let* $S$ *be a closed bounded set of positive transfinite diameter, and let* $p_n(z)\equiv z^n+\cdots$ *be the sequence of once-restricted polynomials on* $S$, *with constant center of gravity of the zeros different from the conformal center of gravity of* $S$, *extremal for a monotonic quasi-Tchebycheff norm. Let* $C$ *be a (closed) circular disk containing* $S$, *let* $C'$

*be a concentric disk whose radius is three times as great, and let*

$$r_\sigma(z) \equiv z^\sigma + \cdots, \qquad \sigma = \sigma(n) = 1 \text{ or } 0,$$

*be the polynomial whose zero is the zero of* $p_n(z)$ *if any exterior to* $C'$, *otherwise unity. Set*

$$q_{n-\sigma}(z) \equiv z^{n-\sigma} + \cdots; \qquad p_n(z) \equiv q_{n-\sigma}(z) \cdot r_\sigma(z).$$

*Then* (34), (40), *and* (42) *are valid. Moreover on any closed bounded set* $S_1$ *exterior to* $H$ *we have uniformly* (*notation of* (35))

$$\lim |p_n(z)|^{1/n} = \lim |q_{n-\sigma}(z)|^{1/n} = |\phi(z)|. \tag{46}$$

The present writers have previously [6] shown that *if a zero of* $p_n(z)$ *lies exterior to* $C'$, *then all other zeros of* $p_n(z)$ *lie in* $C$. Moreover it is remarked in § 3 that under the present conditions a zero of $p_n(z)$ lies exterior to $C'$ for $n$ sufficiently large. Thus $\sigma(n)=1$ for $n$ sufficiently large. Equation (34) is known [part I, § 10], (40) follows from Theorem 12, and (42) from Theorem 13.

*The zeros of the polynomials* $p_n(z)$ *and* $q_{n-\sigma}(z)$ *have no* (*finite*) *limit point exterior to* $H$. Indeed, if $z=\alpha$ is assumed to be such a limit point, let $\Gamma$ be a circular disc containing $H$ in its interior but to which $\alpha$ is exterior. For $n$ sufficiently large a zero of $p_n(z)$ lies exterior to the disc concentric with $\Gamma$ whose radius is three times as great, and consequently [Fekete and Walsh, [6], Theorem IX] all other zeros of $p_n(z)$ lie in $\Gamma$, which contradicts the assumption of $\alpha$ as a limit point of zeros.

Equation (46) now follows [Walsh and Evans, [10], p. 335]. If $S_1$ is a closed bounded set exterior to $H$, for $n$ sufficiently large no zeros of $p_n(z)$ lie on $S_1$.

**9. Under the conditions of Theorem** 14 we can obtain some information about the asymptotic behavior of the one zero $z_1$ among the totality of zeros $(z_1, z_2, \cdots, z_n)$ of $p_n(z)$ which becomes infinite. If $\alpha$ is the prescribed center of gravity and $a$ the conformal center of gravity of $S$, we have $z_1+z_2+\cdots+z_n=n\alpha$, and by Theorem 11

$$\frac{z_2+z_3+\cdots+z_n}{n-1} \to a, \qquad \frac{z_1+z_3+\cdots+z_n}{n} \to a, \qquad \frac{z_1}{n} \to \alpha - a.$$

**10. We are not in a position to extend** Theorem 14 to the case of $k$-fold restricted extremal polynomials, $k>1$, for with $k>1$ precise conditions are as yet unknown concerning the number of zeros of $p_n(z)$ which become infinite or indeed lie exterior to $H$. For instance, if $C$ is $|z|=1$ and we use the least-square norm on $C$ with $k=2$, the twice-restricted extremal polynomial

$$p_n(z) \equiv z^n + 0 \cdot z^{n-1} + \binom{n}{2} c_2 z^{n-2}\ , \qquad c_2 \neq 0\ ,$$

has two zeros $\pm[-n(n-1)c_2/2]^{1/2}$ which become infinite, whereas the twice-restricted extremal polynomial

$$p_n(z) \equiv z^n + nc_1 z^{n-1} + 0 \cdot z^{n-2}\ , \qquad c_1 \neq 0\ ,$$

has but one zero $-nc_1$ which becomes infinite.

Nevertheless, for $k > 1$ *if we know that $k$ zeros of $p_n(z)$ become infinite as $n \to \infty$, the remaining $n-k$ zeros of $p_n(z)$ have no limit point exterior to $H$.* For let $\phi_1, \phi_2, \cdots, \phi_n$ denote the angles subtended by $S$ at the respective zeros of $p_n(z)$. Zedek's relation (Cf. Walsh and Zedek [11])

$$\phi_1 + \phi_2 + \cdots + \phi_{k+1} \geqq \pi$$

can be written in the form

$$\phi_{k+1} \geqq \pi - (\phi_1 + \phi_2 + \cdots + \phi_k)\ ,$$

and if $(\phi_1 + \phi_2 + \cdots + \phi_k) \to 0$, then $\phi_{k+1} \to \pi$ if $\phi_{k+1} < \pi$. To be more explicit, if $\phi_j \leqq \varepsilon/k$ for $j=1, 2, \cdots, k$, then $\phi_j \geqq \pi - \varepsilon$ for $j=k+1, k+2, \cdots, n$, so the $n-k$ corresponding zeros of $p_n(z)$ lie in the locus of points from which $S$ subtends an angle greater than or equal to $\pi - \varepsilon$. Under these conditions (that $k$ zeros of $p_n(z)$ become infinite), the set of zeros of the $p_n(z)$ has no finite limit point exterior to $H$, and Theorem 14 admits a precise analogue. Even though the set of zeros of the $p_n(z)$ has no finite limit point exterior to $H$, not all zeros near $S$ need lie in $H$; compare [Fekete and Walsh, [6], § 13].

**11. If we are willing to forego an** analogue of (42), we can obtain a further result on extremal polynomials.

THEOREM 15. *Let $S$ be a closed bounded set of positive transfinite diameter, and let $p_n(z) \equiv z^n + \cdots$ be the sequence of $k$-fold restricted ($k=$ const.) polynomials on $S$ extremal for a monotonic quasi-Tchebycheff norm. Let $H_k$ be the inflated convex hull of $S$ of order $k$, and let $q_{n-\sigma}(z) \equiv z^{n-\sigma} + \cdots$ be the polynomial whose zeros are the zeros of $p_n(z)$ on $H_k$. Then on any closed bounded set $S_1$ exterior to $H_k$ we have uniformly the second of equations* (46).

Let $H_k^{(n)}$ be the (closed) point set consisting of all points at a distance not greater than $1/n$ from $H_k$, let $r_v(z) \equiv z^v + \cdots$ be the polynomial whose zeros are the zeros of $p_n(z)$ exterior to $H_k^{(n)}$, and let $t_{n-v}(z) \equiv z^{n-v} + \cdots$ be the polynomial whose zeros are the zeros of $p_n(z)$ on $H_k^{(n)}$. At most $k$ zeros of $p_n(z)$ lie exterior to $H_k$, whence $0 \leqq v \leqq k$. For $z$

on $S$ we have $|r_v(z)| \geqq 1/n^v \geqq 1/n^k$. We proceed further as in the proof of (41). If $z$ is a point of $S$ at which $|t_{n-v}(z)|$ takes its maximum value $T_{n-v}$, we have

$$|t_{n-v}(z)|/n^k = T_{n-v}/n^k \leqq T_{n-v}|r_v(z)| \leqq P_n \ .$$

There follows

$$\limsup T_{n-v}^{1/n} \leqq \limsup P_n^{1/n} = \tau(S) \ ,$$

and by (33)

$$\lim T_{n-v}^{1/n} = \tau(S) \ .$$

Thus we have for $z$ on $S_1$

$$\lim |t_{n-v}(z)|^{1/n} = |\phi(z)| \ .$$

However, $t_{n-v}(z)$ has at most $k$ zeros in $H_k$ but not in $H_k^{(n)}$, so if $d_0$ ($>0$) and $d_1$ denote the shortest and longest distances between $S_1$ and $H_k^{(n)}$ for $n$ sufficiently large, we have for $z$ on $S_1$

$$\min [1, d_0, \cdots, d_0^k] \leqq \left| \frac{t_{n-v}(z)}{q_{n-\sigma}(z)} \right| \leqq \max [1, d_1, \cdots, d_1^k] \ ,$$

so the second of equations (46) follows.

Theorem 15 is not a consequence of Theorem 12, for in Theorem 15 the set $S$ may have points in common with the boundary of $H_k$.

**12. Under the conditions of Theorem** 15, if the set $S$ is real, it is known [[6], Part II, § 15] that at most $k$ zeros of a real $p_n(z)$ lie exterior to the convex hull $H$ of $S$. Precisely the method of proof of Theorem 15 (details are left to the reader) yields

THEOREM 16. *Let the real set $S$ and the real polynomials $p_n(z)$ satisfy the conditions of Theorem* 15. *Let $q_{n-\sigma}(z) \equiv z^{n-\sigma} + \cdots$ be the polynomial whose zeros are the zeros of $p_n(z)$ on $H$. Then the second of equations* (46) *is valid uniformly on any closed bounded set $S_1$ exterior to $H$.*

**13. We return to the consideration of** asymptotic relations, without specific reference to restricted polynomials. Equation (42) is derived in Theorem 13 as a necessary condition on the polynomials $r_\sigma(z)$, but is sufficient in the following sense. *If $S$ satisfies the conditions of Theorem* 12, *if $\sigma = \sigma(n) = o(n)$, and if for polynomials $q_{n-\sigma}(z) \equiv z^{n-\sigma} + \cdots$ and $r_\sigma(z) \equiv z^\sigma + \cdots$ we have* (40) *and* (42), *then we also have* (31) *with $p_n(z) \equiv q_{n-\sigma}(z) \cdot r_\sigma(z)$.* In fact we may write $p_n \leqq Q_{n-\sigma} \cdot R_\sigma$, whence (31) follows by (40), (42), and (33). In this remark there are no geometric conditions on the zeros of $q_{n-\sigma}(z)$ and $r_\sigma(z)$, but in connection with restricted extremal polynomials the most interesting situation is that the zeros of $q_{n-\sigma}(z)$ are bounded whereas the zeros of $r_\sigma(z)$ are not necessarily bounded.

**14.** The polynomials $r_\sigma(z)$ of Theorem 13 have various interesting properties:

THEOREM 17. *If $S$ is a set of positive transfinite diameter, and if for the polynomials $r_\sigma(z) \equiv z^\sigma + \cdots$ we have (42) with $\sigma = \sigma(n) = o(n)$, then on any closed bounded set $S'$ of positive transfinite diameter we have*

$$\lim_{n\to\infty} R_\sigma'^{1/n} = 1 , \qquad R_\sigma' = [\max |r_\sigma(z)|, \ z \text{ on } S'] . \tag{47}$$

We note that (33) for arbitrary polynomials yields

$$\liminf R_\sigma'^{1/n} \geqq 1 . \tag{48}$$

The generalized Bernstein Lemma is valid [Walsh, [9], § 4.9] even if we must use the generalized Green's function $G(z)$ instead of the classical Green's function for the maximal subregion $K$ containing infinity of the complement of $S$. Let $\rho$ ($>1$) be chosen so large that the level locus $\Gamma_\rho$: $G(z) = \log \rho$ in $K$ separates $S'$ from infinity. For $z$ interior to $\Gamma_\rho$ we have (loc. cit.)

$$|r_\sigma(z)| \leqq R_\sigma \cdot \rho^\sigma , \qquad R_\sigma' \leqq R_\sigma \cdot \rho^\sigma .$$

Equation (42) yields

$$\limsup_{n\to\infty} R_\sigma'^{1/n} \leqq 1 ,$$

which with (48) gives (47). We have made no hypothesis on the location of the zeros of $r_\sigma(z)$ relative to $S$ and $S'$.

Such a sequence as $r_\sigma(z)$ of Theorem 17 may in some respects be considered a "negligible sequence" with respect to $n$, provided $\sigma = \sigma(n) = o(n)$, in the sense that

(i) its presence or absence as a factor of $q_{n-\sigma}(z) \equiv z^{n-\sigma} + \cdots$ does not alter the value of

$$\lim_{n\to\infty} [\max |q_{n-\sigma}(z)|, \ z \text{ on } S]^{1/n} ,$$

or even the value of lim inf or lim sup here, and

(ii) this property of $r_\sigma(z)$ is not dependent on the particular set $S$ (of positive transfinite diameter) chosen. *Any sequence of polynomials $r_\sigma(z) \equiv z^\sigma + \cdots$ whose zeros are bounded is in this sense a negligible sequence with arbitrary $\sigma = \sigma(n) = o(n)$,* for if these zeros and $S$ lie on a point set of diameter $D$ we have

$$R_\sigma \leqq D^\sigma ,$$

which together with (45) implies (42).

**15. As an application of this** proof of (42) we state

THEOREM 18. *Let $S$ be a closed bounded set not necessarily of positive transfinite diameter, and let an arbitrary bounded set of points $z_1^{(n)}$, $z_2^{(n)}, \cdots, z_\sigma^{(n)}$ be given with $\sigma=\sigma(n)=o(n)$. Then there exists a sequence of polynomials $p_n(z)\equiv z^n+\cdots$ vanishing in the points $z_j^{(n)}$, such that* (1) *is valid.*

Indeed, there exists a sequence of polynomials $q_{n-\sigma}(z)\equiv z^{n-\sigma}+\cdots$ satisfying (40), the polynomials $r_\sigma(z)\equiv(z-z_1^{(n)})\cdots(z-z_\sigma^{(n)})$ satisfy (44), so (1) follows from (33).

**16. For any of the classical norms** and the polynomials $p_n(z)$ of Theorem 18, the analogue of (1) holds; we state: *the analogue of* (1) *holds for the extremal polynomials for any quasi-Tchebycheff norm, if the polynomials are required to vanish in the points $z_j^{(n)}$ $(j=1, 2, \cdots)$, provided $\sigma=o(n)$.* The polynomials $A_n(z, N)$ of least q.T. $N$-norm on $S$ with $A_n(z_j^{(n)}, N)=0$ by (4) fulfill

$$N[A_n(z, N), S]\leqq N[P_n(z), S]\leqq U(S, N)P_n ,$$

while by (5) they satisfy

$$N[A_n(z, N), S]\geqq L_n(S, N, \varepsilon)\tau(S)^n$$

for $n\geqq n_0(N, \varepsilon)$. Hence the validity of (1) by virtue of $\lim P_n^{1/n}=\tau(S)$.

**Addendum.** Using a device, communicated by Prof. Szegö to the first named author after the conclusion of the research above presented we can prove the following counterpart of our Theorem 7:

THEOREM 7 bis. *Let $S$ be an arbitrary compact set and $R$ an arbitrary positive number. Suppose that* (16) *holds with $k=k(n)$ subject to* (12). *Then for all polynomials $A_n(z, N)\in A_k(\gamma_1, \cdots, \gamma_k)$ of least* q.T. *norm $N=N(A_n(z), S)$ on $S$,* (1) *is valid.*

By Theorem 1 we may restrict the proof of (1) to the particular case $N=M(A_n(z), S)$; thus $A_n(z, N)\equiv T_n(z, S)$, the $k$-fold restricted Tchebycheff polynomial in $A_n(\gamma_1, \gamma_2, \cdots, \gamma_k)$; and by a remark to Theorem 5 we can reduce this proof to the special case $S$: $|p(z)|=\rho^m$ with $p(z)\equiv z^m-(p_1z^{m-1}+\cdots+p_m)$, a lemniscate of radius $\delta=\tau(S)$. Then a majorant for $T_n(z, S)$ is the product $z^tp(z)^s(z^k+\delta_1z^{k-1}+\cdots+\delta_k)$, where $t$ and $s$ are nonnegative integers satisfying

$$n-k-ms=t\leqq m-1 ,$$

while $z^k+\delta_1 z^{k-1}+\cdots+\delta_k$ is the principal part of the Laurent development

$$z^k+\delta_1 z^{k-1}+\cdots+\delta_k+\delta_{k+1}z^{-1}+\delta_{k+2}z^{-2}+\cdots$$

around $z=\infty$ of

$$(z^n+\gamma_1 z^{n-1}+\cdots+\gamma_k z^{n-k})z^{-t}p(z)^{-s} .$$

For each $z\neq 0$ with $|p_1|\,|z|^{-1}+\cdots+|p_m|\,|z|^{-m}<1$ we have

$$\begin{aligned}&(z^n+\gamma_1 z^{n-1}+\cdots+\gamma_k z^{n-k})z^{-t}[p(z)]^{-s}\\ &\quad\equiv(z^k+\gamma_1 z^{k-1}+\cdots+\gamma_k)\{1-(p_1 z^{-1}+\cdots+p_m z^{-m})\}^{-s}\\ &\quad\equiv(z^k+\gamma_1 z^{k-1}+\cdots+\gamma_k)\Big\{1+s(p_1 z^{-1}+\cdots+p_m z^{-m})\\ &\qquad+\binom{s+1}{2}(p_1 z^{-1}+\cdots+p_m z^{-m})^2\\ &\qquad+\binom{s+2}{3}(p_1 z^{-1}+\cdots+p_m z^{-m})^3+\cdots\Big\}\end{aligned}$$

whence

$$\delta_1=\gamma_1+sp_1,\ \ \delta_2=\gamma_2+\gamma_1 sp_1+\cdots,\ \ \delta_3=\gamma_3+\gamma_2 sp_1+\cdots+\binom{s+2}{3}p_1^3,\ \cdots .$$

Similarly, for the aforesaid values of $z$,

$$\begin{aligned}&(z^k+|\gamma_1|z^{k-1}+\cdots+|\gamma_k|)\{1-(|p_1|z^{-1}+\cdots+|p_m|z^{-m})\}^{-s}\\ &\quad\equiv z^k+\Delta_1 z^{k-1}+\cdots+\Delta_k+\Delta_{k+1}z^{-1}+\cdots\end{aligned}$$

with

$$\begin{aligned}&\Delta_1=|\gamma_1|+s|p_1|\geqq|\delta_1| ,\\ &\Delta_2=|\gamma_2|+|\gamma_1|s|p_1|+\cdots\geqq|\delta_2|,\ \ \Delta_3\geqq|\delta_3| ,\ \cdots .\end{aligned}$$

Hence, for every $r>0$ large enough to satisfy

$$|p_1|r^{-1}+\cdots+|p_m|r^{-m}<1 ,$$

$$\begin{aligned}&\max\nolimits_{|z|=r}|z^k+\delta_1 z^{k-1}+\cdots+\delta_k|\\ &\quad\leqq r^k+\Delta_1 r^{k-1}+\cdots+\Delta_k+\Delta_{k+1}r^{-1}+\Delta_{k+2}r^{-2}+\cdots\\ &\quad=(r^k+|\gamma_1|r^{k-1}+\cdots+|\gamma_k|)\{1-(|p_1|r^{-1}+\cdots+|p_m|r^{-m})\}^{-s} ;\end{aligned}$$

thus in case $|z|\leqq r$ covers the lemniscate $|p(z)|=\rho^m$ we have a fortiori

$$\begin{aligned}\max\nolimits_{|p(z)|=\rho^m}|z^k+\delta_1 z^{k-1}+\cdots+\delta_k|^{1/n}&\leqq r^{k/n}(1+|\gamma_1|r^{-1}+\cdots+|\gamma_k|r^{-k})^{1/n}\\ &\quad\times\{1-(|p_1|r^{-1}+\cdots+|p_m|r^{-m})\}^{-\frac{(n-k-t)}{n}\cdot\frac{1}{m}} .\end{aligned}$$

By (16) and (12), for $r > R$ we obtain

$$\limsup \max_{|p(z)|=\rho^m} |z^k + \delta_1 z^{k-1} + \cdots + \delta_k|^{1/n} \leqq \{1 - (|p_1| r^{-1} + \cdots + |p_m| r^{-m})\}^{1/m}$$

which ($r \to \infty$) yields $\limsup \max_{|p(z)|=\rho^m} |z^k + \delta_1 z^{k-1} + \cdots + \delta_k|^{1/n} \leqq 1$. We therefore have

$$\limsup [\max_{z \in S} |T_n(z, S)|]^{1/n} \leqq \rho = \tau(S).$$

Combining this with $\liminf \max_{z \in S} |T_n(z, S)|^{1/n} \geqq \tau(S)$ leads to the validity of (1) for the special norm and special set considered, whence its validity for arbitrary q.T. norms and arbitrary sets, as stated.

Using the above argument to obtain an upper bound for

$$[\max |z^k + \delta_1 z^{k-1} + \cdots + \delta_k|, \ z \in S \cdot |p(z)| = \rho^m]^{1/n}$$

if $k = k(n)$ is subject to (22), and $\gamma_j = \gamma_j(n)$, for $1 \leqq j \leqq k = k(n)$ subject to (23) with $\limsup \{\alpha_h (h!)^p\}^{1/h} < \infty$, we can prove (1) for all q.T. norms and arbitrary compact sets $S$ provided $p$ is a positive constant. This generalization of our Theorem 8 is due to Professor Szegö.

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