ON GENERALIZED EUCLIDEAN AND
NON-EUCLIDEAN SPACES

W. L. STAMEY

Introduction. The present paper develops necessary and sufficient conditions that a complete, convex, metric space with extendible segments shall be generalized euclidean, r-hyperbolic, r-spherical, or r-elliptic. Blumenthal and others have given four-point conditions which characterize these generalized spaces among certain classes of spaces, and the results of this paper follow the general plan of these earlier works.

1. Definitions, notation and previous results. Unless otherwise noted all terms used have the same meanings as those given in [1]. The distance between two points p and q of a semi-metric space is denoted by pq, a point s distinct from p and from q is *between p and q*, denoted by psq, provided $ps+sq=pq$, and a triple of points (not necessarily distinct) is a *mid-point triple*, denoted by (psq), provided $ps=sq=pq/2$. A metric space is said to be generalized {euclidean, r-hyperbolic, r-spherical, r-elliptic} provided each of its n-dimensional subspaces is congruent with $\{E_n, H_{n,r}, S_{n,r}, \mathscr{E}_{n,r}\}$, where these four symbols represent n-dimensional euclidean, hyperbolic, spherical, elliptic space respectively, the last three of space constant $r>0$. A metric space is said to have the weak {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each of its quadruples containing a triple of points congruent to three points of $\{E_1, H_{1,r}, S_{1,r}, \mathscr{E}_{1,r}\}$ is itself congruent to four points of $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. A space has the feeble {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each quadruple containing a mid-point triple is congruently imbeddable in $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. The weak property obviously implies the feeble property.

THEOREM 1 (Blumenthal [2]). *A complete, convex, externally convex metric space is generalized euclidean if and only if it has the feeble euclidean four-point property.*

Defining a conjugate space as one with finite metric diameter $\delta>0$ and having the further property that corresponding to each pair of points p, q of the space with $0<pq<\delta$ there exist points p^*, q^* of the space with pqp^*, qpq^*, and $pp^*=qq^*=\delta$ all holding, Hankins [4] has shown the following.

THEOREM 2. *If a complete, convex, conjugate, metric space M has diameter $\pi r/2$, $r > 0$, and if M possesses the feeble r-elliptic four-point property, then M is generalized r-elliptic.*

2. Metric characterizations. Throughout the remainder of the paper Σ will denote a space which satisfies:

(i) Σ is metric,

(ii) Σ is complete,

(iii) Σ is metrically convex,

(iv) if $T_{p,q}$ is a segment with end points p, q, there exists $\delta(p) > 0$ such that if $s \in T_{p,q}$ with $0 < ps < \delta$, then there exists a point $t \in \Sigma$ with (spt) holding.

LEMMA 1. *If Σ has the feeble euclidean, r-hyperbolic, r-elliptic, or r-spherical four point property, and if (pqs), (pqt), $qs = qt$, then $s = t$.*

Proof. Let R represent any one of the spaces E_{2}, $H_{2,r}$, $S_{2,r}$, $\mathscr{E}_{2,r}$. Then p, q, s, $t \approx p_1$, q_1, s_1, $t_1 \in R$ and $(p_1q_1s_1)$, $(p_1q_1t_1)$, $q_1s_1 = q_1t_1$ imply that $s_1 = t_1$, so that $s = t$.

REMARK. If in condition (iv) on Σ the quantity $\delta(p)$ is unbounded for all $p \in \Sigma$, then Σ is externally convex.

THEOREM 1. *If Σ is externally convex then each two points of Σ lie on a unique metric line if and only if pqs, pqt, and ps = pt imply s = t.*

Proof. The necessity is obvious. The sufficiency is proved by noting that each two points are joined by at least one metric line. Then if there are two distinct segments joining p and q, each may be prolonged beyond q along the same segment $T_{q,s}$ to a point s, but this implies that $T_{q,s}$ may be prolonged in two distinct ways beyond q to p, contrary to hypotheses. Thus p and q must determine a unique segment, and this segment can be prolonged to a metric line in exactly one way.

THEOREM 2. *If Σ has the feeble euclidean or feeble r-hyperbolic four-point property then Σ is externally convex.*

Proof. Let p, $q \in \Sigma$ with $p \neq q$. Then on a segment $T_{p,q}$ joining p and q choose a point s with $qs > 0$ and such that there exists a point t with (sqt). Then denoting either E_{2} or $H_{2,r}$ by R_{2}, the hypotheses guarantee that p, q, s, $t \gtreqless R_{2}$. This together with psq and (sqt) implies that pqt holds.

THEOREM 3. *If Σ has the feeble {euclidean, r-hyperbolic} four-point*

property, then Σ is generalized {euclidean, r-hyperbolic}.

Proof. By Theorem 2 Σ is externally convex and by Lemma 1 (along with the completeness and convexity of Σ) pqs, pqt, $qs=qt$ imply that $s=t$. Thus (Theorem 1) each two points of Σ lie on a unique metric line. Then the theorem in the euclidean case is identical with theorem 4.1 in [2]. The r-hyperbolic case is handled in the same manner as the euclidean case.

THEOREM 4. *If Σ has the feeble r-spherical four-point property, then Σ is a conjugate metric space with metric diameter equal to πr, and each point $p \in \Sigma$ determines a unique point p^* such that $pp^*=\pi r$.*

Proof. Since Σ has the feeble r-spherical four-point property, the metric diameter of Σ is at most πr. If p, $q \in \Sigma$ with $0 < pq < \pi r$ there exist points t, $v \in \Sigma$ such that ptq and (tqv) hold and $pt+tq+qv < \pi r$. The feeble r-spherical four-point property then implies that p, t, q, $v \approxeq S_{2,r}$, and this can be strengthened to p, t, q, $v \approxeq S_{1,r}$ because ptq and tqv hold.

The feeble r-spherical four-point property implies that each pair of points of Σ with distance less than πr have a unique mid point. This then implies that each two such points are joined by a unique segment. Let $T_{p,q}$ be the segment joining p and q, and let E be the set of points x of Σ such that pqx holds. All $x \in E$ such that $px < \pi r$ lie on a unique segment since repeated application of Lemma 1 will show that if pqx_1, pqx_2, $px_1=px_2$, then $x_1=x_2$. If for $x \in E$, $\alpha = \text{lub } px$, then there exists a point $\bar{x} \in E$ such that $p\bar{x}=\alpha$. If $p\bar{x} < \pi r$ there exists a point $y \in E$ such that $py > \alpha$, so $p\bar{x}=\pi r$.

If there exist two points p^*, p^{**} in Σ with $pp^*=pp^{**}=\pi r$, let q be a mid-point of p^* and p^{**}. Then p, p^*, p^{**}, $q \approx p_1$, p_1^*, p_1^{**}, $q_1 \in S_{2,r}$ and $p_1 p_1^*=p_1 p_1^{**}=\pi r$ gives $p_1^*=p_1^{**}$ so that $p^*=p^{**}$.

THEOREM 5. *If Σ has the feeble r-spherical four-point property, then Σ has the weak r-spherical four-point property.*

Proof. Let p, q, s, t be four points of Σ with p, q, $s \approxeq S_{1,r}$, to show that p, q, s, $t \approxeq S_{2,r}$. If two of the points p, q, s coincide, then p, q, s, $t \approxeq S_{2,r}$, so let it be assumed that p, q, s are pairwise distinct. Then because of the feeble r-spherical four-point property some pair, say p and q, have distance less than πr and determine a unique segment $T_{p,q}$. Let p, q, $s \approx p_1$, q_1, $s_1 \in S_{2,r}$ and let $S_{1,r}(p_1, q_1)$ be the unique $S_{1,r}$ determined by p_1 and q_1. If v and v_1 are the unique mid-points of p, q and p_1, q_1 respectively, the congruence p, q, v, $t \approx p_1$, q_1, v_1, t_1 can be extended

to $t + T_{p,q} \approx t_1 + T_{p_1, q_1}$. If $s \in T_{p,q}$, p, q, s, $t \gtreqqless S_{2,r}$. If not, suppose the labelling is such that $qs \leq ps$, and consider the congruence q, w, s, $q^* \approx q_1$, w_1, s_1, q_1^*, where w is the mid-point of the unique segment T_{q,q^*} joining q, q^* and containing s. This congruence follows from the feeble r-spherical four-point property and the free movability of $S_{2,r}$. Then this congruence can be extended to $t + T_{q,q^*} \approx t_1 + T_{q_1, q_1^*}$ and p, q, s, $t \gtreqqless S_{2,r}$.

THEOREM 6. *If Σ has the feeble r-elliptic four-point property, then Σ has metric diameter $\pi r/2$ and Σ is a conjugate space.*

Proof. Because of the feeble r-elliptic four-point property Σ has diameter at most $\pi r/2$. Let p, $q \in \Sigma$ with $pq < \pi r/2$. Then there exist points t, $v \in \Sigma$ with ptq, (tqv) holding and $pt + tq + qv < \pi r/2$. By the feeble r-elliptic four-point property p, t, q, $v \gtreqqless \mathscr{E}_{2,r}$ and this can be strengthened to p, t, q, $v \gtreqqless \mathscr{E}_{1,r}$ because of ptq and (tqv).

Let p, t, q, $v \approx p_1$, t_1, q_1, $v_1 \in \mathscr{E}_{2,r}$ and let $x \neq v$ and w be points of Σ with pwq and (wqx) holding and $px < \pi r$. Then p, w, q, $x \approx p_2$, w_2, q_2, $x_2 \in \mathscr{E}_{2,r}$ and p_2, w_2, q_2, x_2 lie on an $\mathscr{E}_{1,r}$. Then there exists a motion of $\mathscr{E}_{2,r}$ sending p_2, q_2 into p_1, q_1 respectively and sending w_2 and x_2 into uniquely determined points w_1 and x_1 on the $\mathscr{E}_{1,r}$ determined by p_1 and q_1. Thus if M is the set of $x \in \Sigma$ with pqx and $px < \pi r$ holding, the unique segment $T_{p,q}$ can be uniquely extended to $T_{p,q} \cup T_{q,x} = T_{p,x}$ for $x \in M$.

Let now $\alpha = \text{lub } px$ for $x \in M$ and let $\{x_i\}$ be a sequence of points of M such that $\lim px = \alpha$ and if $i < j$, $px_i x_j$ holds. Then since p and all of the x_i lie on the same metric segment, as $i, j \to \infty$, $x_i x_j \to 0$. The completeness of Σ then implies the existence of a point y such that $y = \lim x_i$ and $py = \alpha \leq \pi r/2$. Furthermore, since pqx_i holds for $i = 1, 2, \cdots$ and Σ is metric, pqy holds. If $py < \pi r/2$, then $y \in M$ and there exists $\bar{y} \in \Sigma$ such that $p\bar{y} > py = \alpha$, and this is impossible.

Finally the uniqueness of extensions of segments insures that if $pp^* = pp^{**} = \pi r/2$ and pqp^*, pqp^{**} hold, then $p^* = p^{**}$.

THEOREM 7. *If Σ has the feeble $\{r$-spherical, r-elliptic$\}$ four-point property, then Σ is generalized $\{r$-spherical, r-elliptic$\}$.*

Proof. The theorem follows in the spherical case from Theorem 5 upon application of Theorem 66.5 of [1] and in the elliptic case from Theorem 6 and Theorem 4.4 of [3].

M. M. Day[1] [3] has defined another four-point property which he calls the "queasy euclidean four-point property" and has shown that a

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complete, externally convex semimetric space possessing this property is generalized euclidean. The remainder of this paper is devoted to extending Day's work.

A semimetric space M will be said to have the *queasy* {euclidean, r-hyperbolic, r-spherical} four-point property provided that corresponding to each pair of points $p, s \in M$ there exists $q \in M$ such that pqs holds and for each $t \in M$, the quadruple $p, q, s, t \cong \{E_2, H_{2,r}, S_{2,r}\}$.

LEMMA 2. *If Σ has the {euclidean, r-hyperbolic} four-point property, then each two distinct points of Σ are joined by a unique metric segment.*

Proof. Since Σ is complete, convex and metric each two points are joined by at least one segment. It will be sufficient then to show that each pair of points of Σ have just one mid-point. Let $p, q_1, q_2, s \in \Sigma$ with (pq_1s), (pq_2s) and $p \neq s$ holding, and let R represent either of the spaces E_2 or $H_{2,r}$. If there exists a sequence of points $t_i \in \Sigma$, $i=1, 2, \cdots$, with $\lim t_i = q_1$, pt_iq_1, pt_iq_2 holding, then $\lim t_i = q_2$ and $q_1 = q_2$.

If $q_1 \neq q_2$, then there exists a positive number $\bar{p}_1$ such that if $pt + tq_1 = pq_1$ and $pt + tq_2 = pq_2$ then $tq_1 = tq_2 > \bar{p}_1$. Also there exists $\bar{p}_2 > 0$ such that if $q_1t + ts = q_1s$ and $q_2t + ts = q_2s$, then $tq_1 = tq_2 > \bar{p}_2$. Let ρ_1 be the least upper bound of the numbers $\bar{p}_1$ and ρ_2 that of the numbers $\bar{p}_2$. Let $\bar{p}$ and p^* be points of Σ with $p\bar{p} + \bar{p}q_1 = pq_1$, $pp^* + p^*q_2 = pq_2$ and $\bar{p}q_1 = p^*q_2 = \rho_1$. Then either $\bar{p} = p^*$ or there is a sequence p_i with pp_iq_1, pp_iq_2 holding and $\lim p_i = \bar{p}$, $\lim p_i = p^*$ so that $\bar{p} = p^*$. Thus there exist two points of Σ with q_1 and q_2 each metrically between these points but such that any segment joining the points and containing q_1 has only end points in common with a segment joining these points and containing q_2. There will be no loss of generality if these points are taken to be p and s and if q_1 and q_2 are assumed to be distinct middle points of p and s.

The queasy four-point property of Σ implies that there exist $x \in \Sigma$, $\bar{p}, \bar{x}, \bar{q}, \bar{s}, p^*, x^*, q^*, s^* \in R$ with pxs holding and

$$p, x, q_1, s \approx \bar{p}, \bar{x}, \bar{q}, \bar{s}$$
$$p, x, q_2, s \approx p^*, x^*, q^*, s^*.$$

Then since $p^*x^*s^*$ and $p^*q^*s^*$ hold, there is a motion sending the "starred" points into the corresponding "barred" ones, and $\bar{p}, \bar{x}, \bar{q}, \bar{s}$ all lie on one metric segment of R. Thus either $\bar{x} = \bar{q}$ and $x = q_1 = q_2$ or there is a metric segment joining p, q_1, s and one joining p, q_2, s with these two segments having interior point x in common. This contradiction completes the proof.

LEMMA 3. *If Σ has the queasy r-spherical four-point property, then each two distinct points having distance less than πr are joined by a unique segment.*

Proof. The proof is identical with that of the preceeding lemma if distance ps is restricted to be less than πr.

THEOREM 8. *If Σ has the queasy {euclidean, r-hyperbolic, r-spherical} four point property, then Σ is generalized {euclidean, r-hyperbolic, r-spherical}.*

Proof. It will be sufficient to show that if p, q, s, $t \in \Sigma$ with (pqs) holding, then p, q, s, $t \approxeq R$, where R represents any one of the spaces E_2, $H_{2,r}$, $S_{2,r}$. Assume for the present that if R is spherical, $ps < \pi r$. Then let $x \in R$ with pxs holding and

$$p, \; x, \; q, \; s \approx \bar{p}, \; \bar{x}, \; \bar{q}, \; \bar{s} \in R$$
$$p, \; x, \; s, \; t \approx p^*, \; x^*, \; s^*, \; t^* \in R$$

Then there exists a motion of R sending p^*, s^* into $\bar{p}$, $\bar{s}$ respectively and t^* into a point $\bar{t}$. If $qt = \overline{qt}$, then p, q, s, $t \approx \bar{p}$, $\bar{q}$, $\bar{s}$, $\bar{t}$.

If $qt \neq \overline{qt}$, let a congruence f between the segments $T_{p,s}$ and $T_{\bar{p},\bar{s}}$ be established so that $f(p) = \bar{p}$, $f(s) = \bar{s}$. Let Q represent the set of points $x \in T_{p,s}$ such that $tx = \bar{t}f(x)$. Then the continuity of the metric in Σ implies that in traversing $T_{p,s}$ from p to q there is a last point of Q encountered. Let this point be u, and let w be the last point of Q encountered in traversing $T_{p,s}$ from s toward q. Denote $\bar{u} = f(u)$, $\bar{w} = f(w)$.

Then there exists by the queasy property a point $y \in \Sigma$ with uyw holding and u, y, w, $t \approx u'$, y', w', $t' \in R$. A motion of R sends u', w', t' into $\bar{u}$, $\bar{w}$, $\bar{t}$ and y' into a unique point $\bar{y}$ with $\overline{uyw}$ holding and $\overline{yt} = y't' = yt$. This contradicts the property used to pick out u and w so that $qt = \overline{qt}$ and p, q, s, $t \approx R$.

Finally if R represents $S_{2,r}$ and $ps = \pi r$, there is a point $x \in \Sigma$ with pxs holding and

$$p, \; x, \; s, \; q \approx \bar{p}, \; \bar{x}, \; \bar{s}, \; \bar{q} \in S_{2,r}$$
$$p, \; x, \; s, \; t \approx p^*, \; x^*, \; s^*, \; t^* \in S_{2,r}.$$

Let a motion be performed sending p^*, s^* into $\bar{p}$, $\bar{s}$ respectively and t^* into a point $\bar{t}$. Consider the set of distances $\bar{t}x$ where x belongs to the $S_{1,r}$ at distance $\pi r/2$ from p. This set of numbers has a minimum m and a maximum M. Let the labelling be taken so that $pt \leq \pi r/2$. Then it is necessary that $m \leq tq \leq M$. For if $tq < m$, then $pt + m = \pi r/2$ and $pt + tq < \pi r/2 = pq$. Also if $tq > M$, $tq > tp + pq$.

Now on the $S_{1,r}$ at distance $\pi r/2$ from p there is a point $\bar{\bar{q}}$ so that $\bar{t}\bar{\bar{q}} = tq$. Then p, q, s, $t \approx \bar{p}$, $\bar{\bar{q}}$, $\bar{s}$, $\bar{t}$, and this completes the proof.

Of course the proof of Theorem 8 is not valid for $\mathscr{E}_{2,r}$ because of the strong use made of free movability. It should also be noted that when the queasy four-point property is assumed for a semi-metric space, it is unnecessary to assume convexity and metricity since the queasy property implies these.

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KANSAS STATE COLLEGE