MULTIPLICATIVE NORMS FOR METRIC RINGS

SILVIO AURORA

1. Introduction. In his paper [19], S. Mazur stated two results concerning real normed algebras. The first of these, which asserted that the only normed division algebras over the real field were the real field, the complex field, and the division ring of real quaternions, was essentially proved by Gelfand in [10] and by Lorch in [17]. Elementary proofs of that result have also been given by Kametani [13] and Tornheim [26], while generalizations in various directions have been given by Kaplansky [16], Arens [4] and Ramaswami [23].

The second of the results given by Mazur was that a real normed algebra such that $||xy||=||x||\,||y||$ for all x and y mnst again be isomorphic to the real field, the complex field, or the division ring of real quaternions. This result was generalized in [8] by R.E. Edwards, who showed that the same conclusion holds for a Banach algebra under the weaker hypothesis that $||x||\,||x^{-1}||=1$ for all elements x which have inverses x^{-1}. A. A. Albert has also obtained results in [1], [2] and [3] similar to the second of Mazur's results.

In this paper, the second result of Mazur is generalized for certain types of metric rings. It is shown in section 6 that such rings must be division rings if the condition $||xy||=||x||\,||y||$ for all x and y holds. Similar results hold under the weaker assumption that $||x||\,||x^{-1}||=1$ for every element x which has an inverse x^{-1}. Under suitable additional conditions on the metric rings under discussion, it is shown in § 7 that the results just mentioned may be strengthened to assert that the ring is not only a division ring but is isomorphic to the real field, the complex field, or the division ring of real quaternions. Finally, the results on metric rings are applied to real normed algebras to obtain the results of Mazur and Edwards under weaker assumptions.

The author is deeply indebted to Professor E. R. Lorch of Columbia University for his invaluable guidance in the preparation of this paper; heartfelt thanks are also due to Professor Lorch and to Professor Leo Zippin of Queens College for their kindness in reading and criticizing the manuscript.

2. Topological rings, metric rings, regular and singular elements. We shall first introduce some pertinent definitions and recall some elementary results concerning topological rings and metric rings. By a topological ring is meant a structure R which is at once a Hausdorff

Received July 27, 1955 and in revised form October 1, 1956. Portions of this paper were written while the author held an Atomic Energy Commission Fellowship.

space and a ring[1] such that the applications $(a, b) \to a+b$ and $(a, b) \to ab$ of $R \times R$ into R are continuous.

If R is any ring, then a real-valued function $||x||$ defined on R is called a *norm* for R if it satisfies the following conditions:

(i) $||0||=0$ and $||a||>0$ for $a \neq 0$,

(ii) $||a+b|| \leq ||a||+||b||$ for all $a, b \in R$,

(iii) $||-a||=||a||$ for all $a \in R$,

(iv) $||ab|| \leq ||a|| \cdot ||b||$ for all $a, b \in R$. A norm for R is called an *absolute value* for R if it satisfies the following condition, which is clearly stronger than (iv):

(iv′) $||ab||=||a|| \cdot ||b||$ for all $a, b \in R$.

By a *metric ring* (*ring with absolute value*) is meant a ring R together with a norm (absolute value) for R. In any metric ring R the function $d(x, y)=||x-y||$ is a metric for R and induces in the usual way a topology for R relative to which R becomes a topological ring. Every ring admits as a norm the trivial function which takes the value 0 for the zero element of the ring and the value 1 for all other elements; in this case the induced topology is of course the discrete topology. The trivial norm is easily seen to be an absolute value for a ring if and only if the ring contains no proper zero-divisors.

For a finite ring which contains at least two elements it may be observed that the existence of an absolute value is possible only if the ring is a field and the absolute value is the trivial norm. In general, one might expect that the existence of an absolute value for a ring will require rather special properties of that ring. In the case of real normed algebras, for instance, S. Mazur stated in [**19**, second theorem] that when the norm is an absolute value the algebra must be isomorphic to the field of real numbers, the field of complex numbers, or the division ring of real quaternions. We shall consider below metric rings which satisfy various multiplicative restrictions on the norm such as (iv′), and we shall show that the class of such rings is strongly limited.

By an *isometry* of a metric ring R into a metric ring R_1 is meant a ring homorphism σ of R into R_1 such that $||\sigma x||=||x||$ for all $x \in R$; clearly, σ is necessarily an isomorphism of R into R_1. A metric ring R_1 is said to be an *extension* of the metric ring R provided that there exist an isometry of R into R_1. The notions of *limit*, *convergent sequences*, *fundamental sequences*, *complete metric ring*, and the *completion* of a metric ring are introduced in the standard way and the usual properties of these notions are easily verified.

We now exhibit some metric rings, in each case taking the obvious definitions for the operations of addition and multiplication when these are not specified, and with the ordinary absolute value as the norm in

[1] The rings in this paper are assumed to be associative and to possess a unit element, e.

examples (1)-(5):

(1) The ring of rational integers.
(2) The field of rational numbers.
(3) The field $\mathfrak{R}$ of real numbers.
(4) The field $\mathfrak{C}$ of complex numbers.
(5) The division ring $\mathfrak{Q}$ of real quaternions.
(6) The ring $C(X)$ of all continuous complex-valued functions defined on the compact Hausdorff space X, with the norm given by $||f|| = \sup |f(x)|$, where the supremum is extended over all $x \in X$.
(7) The ring $\mathscr{D}$ of all complex-valued functions which are defined and continuous on the unit disc $\{\zeta \,|\, |\zeta| \leq 1\}$ of the complex plane and analytic over the interior, $\{\zeta \,|\, |\zeta| < 1\}$, of that disc. The norm is given by $||f|| = \sup |f(\zeta)|$, where the supremum is taken for all ζ such that $|\zeta| = 1$.
(8) The field Q_p of rational numbers (where p is a fixed prime number) with the norm defined by $||q|| = p^{-r}$, where r is the uniquely determined integer such that q has a representation $q = p^r(m/n)$ with m and n integers prime to p.
(9) The field P_p of p-adic numbers, which is obtained as the completion of Q_p of example (8).
(10) The ring $C^{(n)}$ of all real-valued functions which are defined on the closed unit interval and for which the first n derivatives exist and are continuous. In this case the norm is defined to be

$$||f|| = \sum_{r=0}^{n} (r\,!)^{-1} \sup |f^{(r)}(x)| \;,$$

where each supremum is extended over all x in the closed unit interval.

All of these rings except those of examples (2) and (8) are complete metric rings; the norm is also an absolute value in all of these rings except those of examples (6) when X contains at least two points, (7) and (10).

The notions of (*left*, *right*) *inverse* of an element, (*left*, *right*) *regular* elements, (*left*, *right*) *singular* elements, and the sets $S^{\bar{l}}$, S^r, S, $G^{\bar{l}}$, G^r and G are introduced as in [24]. Clearly, $G(G^{\bar{l}}, G^r)$ is the complement of $S(S^{\bar{l}}, S^r)$. It is easily verified that $S = S^{\bar{l}} \cup S^r$ and $G = G^{\bar{l}} \cap G^r$. Also, $G^{\bar{l}}$ and G^r are multiplicative semigroups[2] and G is a multiplicative group with e as its identity element.

In many examples the distinction between left regular elements and right regular elements disappears. For example, for a ring R which has no proper idempotents it is true that $G^{\bar{l}} = G^r$. For, if $a \in G^{\bar{l}}$

[2] A *semi-group* is understood to be a non-empty system which is closed relative to an associative binary operation.

and $a'a=e$, then aa' is an idempotent distinct from 0, so $aa'=e$ and this means that $a \in G^r$. Similarly, in a ring without proper nilpotents, $G^{\bar{l}}=G^r$. For, in such a ring all idempotents are central by Lemma 1 of [9], so if $a \in G^{\bar{l}}$ where $a'a=e$, then aa' is an idempotent and therefore central. Thus,

$$aa'=a'a(aa')=a'(aa')a=(a'a)(a'a)=e ,$$

so $a \in G^r$, whence $G^{\bar{l}}=G^r$.

If R is a topological ring, its group G of regular elements is the union of a family of disjoint, maximal connected subsets,—the *components* of G. The *principal component*, G_1, is the component which contains the unit element e. It may be shown that G_1 is an invariant subgroup of G such that the cosets modulo G_1 are the components of G.

Following Kaplansky [14] we call a topological ring a *Q-ring* if the set G of its regular elements is an open set[3]. For a complete metric ring it is well known that $G^{\bar{l}}$, G^r and G are open sets so that $S^{\bar{l}}$, S^r and S are closed sets. This is shown in [18], [20] or [24] for the case of Banach algebras, and the present result, which utilizes essentially the same proof, may be found in [14]. Thus, every complete metric ring is a Q-ring.

3. Generalized divisors of zero.

In [25], G. Šilov introduced the concept of a generalized divisor of zero in a Banach algebra. A more detailed study of this concept was presented by Rickart in [24]; the present development of a theory of generalized divisors of zero in a metric ring follows closely the development presented in the latter paper, although the possibility of multiplication by complex scalars permits stronger results in the case of a Banach algebra. Šilov's results demonstrated the existence of generalized divisors of zero in any non-trivial Banach algebra; as a corollary he obtained the result of Mazur mentioned above on Banach algebras with a norm which is an absolute value. Our study of generalized divisors of zero leads in a similar way to a generalization of Mazur's result to the case of certain types of metric rings.

DEFINITION. If a is any element of a metric ring we define $\bar{l}(a)=\inf(||ax||/||x||)$ and $r(a)=\inf(||xa||/||x||)$, where in each case the infimum is taken as x ranges over the non-zero elements of the ring.

The results which follow are easily proven and in many cases follow as in Rickart's paper.

---

[3] Kaplansky,s definition is in terms of quasiregular elements, but is easily seen to be equivalent to the present one in rings with unit element.

LEMMA 1. (i) $0 \leq \bar{l}(a) \leq \|a\|$ *for any* a; (ii) $\bar{l}(a)\bar{l}(b) \leq \bar{l}(ab) \leq \|a\| \cdot \bar{l}(b)$ *for any* a *and* b; (iii) $|\bar{l}(a) - \bar{l}(b)| \leq \|a-b\|$ *for any* a *and* b.

COROLLARY. $\bar{l}(x)$ *is a continuous function of* x.

DEFINITION. $Z^{\bar{l}} = \{a | \bar{l}(a) = 0\}$, $Z^r = \{a | r(a) = 0\}$; $Z = Z^{\bar{l}} \cup Z^r$;
$H^{\bar{l}} = \{a | l(a) > 0\}$, $H^r = \{a | r(a) > 0\}$; $H = H^{\bar{l}} \cap H^r$.

It is easily observed that $Z^{\bar{l}}(Z^r, Z)$ is the complement of $H^{\bar{l}}(H^r, H)$. Since the corollary implies that $Z^{\bar{l}}$ (and also Z^r) is closed it follows that $Z = Z^{\bar{l}} \cup Z^r$ is closed. Consequently, $H^{\bar{l}}$, H^r and H are open.

An element of $Z^{\bar{l}}(Z^r, Z)$ is called a *generalized left-divisor* (*right-divisor, divisor*) *of zero*. Clearly, a (left, right) zero-divisor is always a generalized (left, right) divisor of zero. The converse, however, is not always true. For example, let R_1 be the metric ring consisting of the same elements as the ring of example (7), but where the norm of an element distinct from zero is taken as the maximum of 1 and the norm as given in example (7). The topology of R_1 is then the discrete topology. There are no proper zero-divisors in R_1, but the function $f(\zeta) \equiv \zeta - 1$ is a generalized left-divisor of zero in R_1, for if $f_n(\zeta) \equiv \zeta^n + \zeta^{n-1} + \cdots + 1$, then $\|f_n\| = n+1$, while $\|ff_n\| = 2$ since

$$(\zeta - 1)(\zeta^n + \zeta^{n-1} + \cdots + 1) \equiv \zeta^{n+1} - 1;$$

thus, $\|ff_n\| / \|f_n\| = 2/(n+1)$ for $n = 1, 2, \cdots$, so that $\bar{l}(f) = 0$ and f is a generalized left-divisor of zero.

In [24] Rickart defines a left generalized null divisor to be an element s such that there exists a sequence $\{z_n\}$ such that $\|z_n\| = 1$ for all n, and such that $sz_n \to 0$. However, he notes that s is a left generalized null divisor if and only if $\bar{l}(s) = 0$. In a metric ring, it is clear that a left generalized null divisor in the sense of Rickart satisfies the condition $\bar{l}(s) = 0$ and is thus a generalized left-divisor of zero in the sense of this paper. However, a generalized left-divisor of zero in the sense of this paper need not be a left generalized null divisor in the sense of Rickart; for example, the element f in the preceding paragraph is a generalized left-divisor of zero in R_1, but if there were a sequence $\{g_n\}$ of R_1 with $\|g_n\| = 1$ for all n and with $fg_n \to 0$, then for n large fg_n would be zero since R_1 is discrete, and, since R_1 has no proper zero-divisors, either f or g_n would be zero, and this is clearly impossible, so f can not be a left generalized null divisor in the sense of Rickart's definition.

It is nevertheless true that for many metric rings the concepts of

[4] For brevity, right-sided results are often omitted.

left generalized null divisor as defined by Rickart and that of generalized left-divisor of zero as employed in this paper coincide. One can easily show, for instance, that this is the case in a metric ring R such that for any element a distinct from zero there is an element b of $\mathscr{R}(R)$ (this set is introduced later in § 5) such that $\|a\|\cdot\|b\|=1$. It follows also that the concepts coincide in a metric ring R such that for every positive real number r there is an element b of $\mathscr{R}(R)$ such that $\|b\|=r$. In particular, this condition holds in any Banach algebra, so that the two concepts coincide in any Banach algebra, as Rickart showed.

If R_2 is the ring of elements of R_1 but with the norm taken such that $\|g\|=1$ for any g distinct from zero, then R_2 is also discrete, so that the topological rings which underlie R_1 and R_2 are identical. However, the element f defined above is a generalized left-divisor of zero in R_1, but not in R_2, for the norm of R_2 is an absolute value, whence $\|ax\|/\|x\|=\|a\|$ for all non-zero x in R_2, so $\bar{l}(a)=\|a\|$ for any a in R_2, and consequently R_2 can not contain any generalized left-divisors of zero different from zero. This shows that the notion of generalized left-divisors of zero is not a purely topological notion. In particular, this concept differs from that of a *topological zero-divisor* as defined, for example, in [15]. For, while it is easily shown that a topological left zero-divisor in a metric ring is necessarily a generalized left-divisor of zero, the converse is not true since otherwise the element f of R_1 would be a topological left zero-divisor in R_1 and hence in R_2 and hence a generalized left-divisor of zero in R_1.

LEMMA 2. (i) *If* $b \in Z^{\bar{l}}$, *then* $ab \in Z^{\bar{l}}$ *for any* a. (ii) *If* $ab \in Z^{\bar{l}}$, *then* $a \in Z^{\bar{l}}$ *or* $b \in Z^{\bar{l}}$.

LEMMA 3. $Z^{\bar{l}} \subset S^{\bar{l}}$, $Z^r \subset S^r$, $Z \subset S$, $G^{\bar{l}} \subset H^{\bar{l}}$, $G^r \subset H^r$ *and* $G \subset H$.

Lemma 3 shows that the sets $H^{\bar{l}}$, H^r and H are not empty and contain in fact all regular elements. It is also clear that the zero element belongs to the sets $Z^{\bar{l}}$, Z^r and Z; but in many instances these sets contain no element other than zero. For example, the metric rings of examples (1)–(5) possess no generalized divisors of zero other than the zero-element. However, for a complex Banach algebra distinct from $\mathfrak{C}$, G. Šilov showed in [25, lemma] that there always exist generalized divisors of zero distinct from the zero-element.[5] The results which follow give conditions under which certain types of metric rings contain nonzero generalized divisors of zero.

LEMMA 4. *For any metric Q-ring,* H *is the union of the disjoint open sets* G *and* $S \cap H$.

[5] See also the remark by Lorch in [17].

LEMMA 5. *Let R be a metric Q-ring. Let $\{a_n\}$ be a sequence of regular elements of R which converges to an element a in R. If the sequence $\{a_n^{-1}\}$ is bounded,*[6] *then a is a regular element.*

THEOREM 1. *If R is a metric Q-ring, then*[7] $[G] \cap S \subset Z^{l} \cap Z^{r}$. *If, in addition, $R^{(0)}$ is connected,*[8] *then either R is a division ring or Z contains an element distinct from zero.*

Proof. The first statement follows as in Rickart's paper.

If R is not a division ring, then the closed set S meets $R^{(0)}$. Also, the closed set $[G]$ meets $R^{(0)}$, and $R^{(0)} \subset [G] \cup S$. If $R^{(0)}$ is connected, then $R^{(0)} \cap [G] \cap S$ is not empty, so $[G] \cap S$ contains an element distinct from zero. It follows that $Z^{l} \cap Z^{r}$ contains a nonzero element, so Z also contains a nonzero element.

4. **Proper rings.** Lemma 3 asserts that the inclusion $Z \subset S$ always holds. Thus, every generalized divisor of zero is a singular element, although, as we see below, a singular element need not be a generalized divisor of zero. Indeed, the generalized divisors of zero possess the special property of permanent singularity; that is, a generalized divisor of zero does not acquire an inverse in any extension of the given ring since it is still a generalized divisor of zero and hence singular in that extension. In the ring $\mathscr{D}$ of example (7), the function $f(\zeta) \equiv \zeta$ is a singular element, but the ring $C(X)$, where X is the unit circle of the complex plane, is readily seen to be an extension of $\mathscr{D}$ in which f is a regular element.[8] Thus, f is not a permanently singular element of $\mathscr{D}$ and so f is not a generalized divisor of zero in $\mathscr{D}$, even though f is a singular element of $\mathscr{D}$. Thus, the inclusion $Z \subset S$ may be a proper inclusion.

DEFINITION. A metric ring R is said to be *proper* provided that $Z=S$, or, equivalently, that $H=G$.

The preceding discussion shows that even a complete metric ring which is connected and locally connected need not be proper; for example, $\mathscr{D}$ is not proper. However, many metric rings are proper, including any ring $C(X)$ of example (6). We see that a proper ring is a division ring if and only if there are no generalized divisors of zero other than zero. In particular, a proper ring with absolute value can have no generalized divisors of zero except zero and is therefore a divi-

[6] A set A is said to be *bounded* if there is a number M such that $\|a\| \leq M$ for all a in A.

[7] If A is any set, the symbols $[A]$ and $A^{(0)}$ denote the topological closure of A and the set of non zero elements of A, respectively.

[8] Compare [**24**].

sion ring. We shall give below some sufficient conditions for a metric ring to be proper; these conditions, in combination with the existence of an absolute value or with some other multiplicative restriction on the norm, will imply that the ring must be a division ring.

THEOREM 2. *If R is a metric Q-ring such that H is connected, then R is proper.*

THEOREM 3. *If R is a metric Q-ring (complete metric ring) such that S is nowhere dense (of first category), then R is proper.*

Proof. Either hypothesis of Theorem 3 insures that S is a closed set. Also, either hypothesis implies that S is nowhere dense, for if S is assumed to be of first category in a complete metric ring, then S is nowhere dense since a closed set of a complete metric space is of first category if and only if it is nowhere dense. The proofs of these two theorems then follow as in [24].

It must be noted that the hypothesis of completeness is needed where it occurs in Theorem 3. For, let R be the set of all functions f which belong to the ring $\mathscr{D}$ of example (7) and for which $f(0)$ is a rational number. It is easily seen that R is a metric Q-ring but is not complete. Also, R is of first category, so the set S for R is also of first category. However, R is not proper, for it contains the singular element $f(\zeta)\equiv\zeta$, which is not a generalized divisor of zero, as was noted at the beginning of this section.

DEFINITION. If R is any ring, then by an *involution* of R is meant a mapping $a \rightarrow a^*$ of R into itself such that:

(i) $(a+b)^*=a^*+b^*$ for all $a, b \in R$,

(ii) $(ab)^*=b^*a^*$ for all $a, b \in R$,

(iii) $(a^*)^*=a$ for all $a \in R$.

That is, an involution of R is an anti-automorphism of period two. For a given involution of R, an element a is said to be *self-adjoint* provided that $a^*=a$.

For $\mathfrak{C}$, for instance, the mapping which associates with each complex number its complex conjugate is an involution. Similarly, the mapping which associates with each quaternion its conjugate is an involution of $\mathfrak{Q}$. In both cases the self-adjoint elements are simply the real numbers. In the case of the ring of all bounded linear operators on a Hilbert space, the mapping which associates with each operator its adjoint is an involution, and the self-adjoint elements are of course the self-adjoint operators. Thus, many rings admit at least one involution. For metric rings, one is naturally interested in the involutions which are closely related to the metric or topological structure of the ring.

DEFINITION. An involution $a \to a^*$ of a metric ring R is said to be *bounded* provided that there is a positive constant β such that $\|a^*\| \leq \beta\|a\|$ for all a. An involution $a \to a^*$ of a metric ring R is said to be *real* if no self-adjoint element is an interior point of the set of singular elements.

The involutions described above are all bounded and real, if in the case of the ring of all bounded linear operators on a Hilbert space we take as the norm of an element its bound as an operator.

These definitions differ from the corresponding definitions of Rickart in [24] by the omission of the mention of scalars in the present definitions. Thus, the identity mapping of the field $\mathfrak{C}$ onto itself is a real and bounded involution in the present sense but is not even an involution in the sense of Rickart, since the image of $i \cdot 1$, where i is a scalar, should be $(-i)1$ but is $i \cdot 1$.

For a complex Banach algebra, an involution which is real in the sense of Rickart is also real in the present sense. For, let an involution be real in the sense of Rickart. Then, for any self-adjoint element a the spectrum of a is real. If $\{\lambda_n\}$ is a sequence of non-real complex numbers which converges to zero, then $\{a-\lambda_n \cdot e\}$ is a sequence which converges to a. Since the λ_n are not in the spectrum of a, it follows that $a-\lambda_n \cdot e$ is regular for all n. This shows that a is the limit of a sequence of regular elements and hence is not in the interior of S. Thus, the involution is real in the present sense.

The identity mapping of the ring $\mathscr{D}$ of example (7) is clearly a bounded involution relative to which all elements are self-adjoint. But the function $f(\zeta) \equiv \zeta$ is a singular element of $\mathscr{D}$, and Rouché's Theorem implies that any element of $\mathscr{D}$ whose distance from f is less than 1 is also a singular element; thus, the set of singular elements of $\mathscr{D}$ has a nonempty interior and contains the self-adjoint element f. The involution in question is consequently not real even though it is bounded.

There are also real involutions which are not bounded. For example, let R be the field obtained by adjoining x and y to a given field F, so that R consists of rational expressions in x and y with coefficients in F. If $P(x, y)$ is any irreducible polynomial belonging to $F[x, y]$, then each element of R may be represented in the form $\varphi = P^{\mu} \cdot M/N$, where M and N are elements of $F[x, y]$ which are not divisible by P, and where μ is a uniquely determined integer which depends only upon φ and P. If $\|\varphi\| = 2^{-\mu}$ where μ is the integer which corresponds to φ, then R becomes a metric ring relative to this norm. The involution of R which maps an expression $f(x, y)$ onto $f(y, x)$ is clearly real since R is not discrete and the only singular element of R is 0. In case $P(x, y)$ does not divide $P(y, x)$, let $Q(x, y) = P(y, x)$, so for any natural number n we have $\|P^n\| = 2^{-n}$, while $\|Q^n\| = 1$. But Q^n is the image of P^n relative to the involution, and $||Q^n||/||P^n||=2^n$, so that the involution is not bounded, although it is real.

For an involution to be both real and bounded, the metric ring in question must be proper, as the theorem which follows shows.

LEMMA 6. *If $a \to a^*$ is a bounded involution of the metric ring R, then $(Z^{\bar{l}})^*=Z^r$ and $(Z^r)^*=Z^{\bar{l}}$.*[9]

Proof. If $||a^*||\leqq\beta||a||$ for all $a\in R$, then $||a||=||a^{**}||\leqq\beta||a^*||$ for all a, so that

$$\bar{l}(a)\leqq ||ax^*||/||x^*||=||(xa^*)^*||/||x^*||\leqq\beta^2||xa^*||/||x||$$

for all nonzero x. Thus, $\bar{l}(a)\leqq\beta^2 r(a^*)$, so $a^*\in Z^r$ implies $a\in Z^{\bar{l}}$. That is, $(Z^r)^*\subset Z^{\bar{l}}$, while, similarly, $(Z^{\bar{l}})^*\subset Z^r$. Taking images relative to the involution, we obtain $Z^r\subset(Z^{\bar{l}})^*$ and $Z^{\bar{l}}\subset(Z^r)^*$. By combining the four inclusions, we obtain the desired results.

THEOREM 4. *If R is a metric Q-ring which admits a real, bounded involution, then R is proper.*

Proof. Let $a\to a^*$ be a real, bounded involution of R. If $a\in S^{\bar{l}}$, then $a^*\cdot a\in S^{\bar{l}}$ and $a^*\cdot a$ is self-adjoint. Since the involution is real, $a^*\cdot a\in[G]$. Thus, $a^*\cdot a\in[G]\cap S$. Theorem 1 implies that $a^*\cdot a\in Z^{\bar{l}}\cap Z^r$. Since $a^*\cdot a\in Z^{\bar{l}}$, we may conclude from Lemma 2 that $a^*\in Z^{\bar{l}}$ or $a\in Z^{\bar{l}}$. That is, $a\in(Z^{\bar{l}})^*=Z^r$ or $a\in Z^{\bar{l}}$, so $a\in Z=Z^{\bar{l}}\cup Z^r$. This shows that $S^{\bar{l}}\subset Z$. Similarly, $S^r\subset Z$, whence $S=S^{\bar{l}}\cup S^r\subset Z$. But $Z\subset S$ by Lemma 3, so $Z=S$, and R is thus proper.

5. The sets $\mathscr{L}$(R), $\mathscr{R}$(R), $\mathscr{G}$ and $\mathscr{G}'$. We shall now introduce some sets which measure to some extent how closely the norm of a given metric ring resembles an absolute value.

DEFINITION. The norm of a metric ring is said to be *multiplicative* on a set A if $||ab||=||a||\,||b||$ for all $a, b\in A$. (Thus, an absolute value is simply a norm which is multiplicative on the entire ring.) By a *μ-group* is meant a multiplicative group contained in a metric ring and on which the norm of the ring is multiplicative.

DEFINITION. If R is a metric ring, $\mathscr{L}(R)=\{a|a\in R,\ ||ax||=||a||\,||x||$

[9] If A is a set in a ring with involution $a\to a^*$, then the set of all a^*, where a is in A, is denoted by A^*. Note that the statement of the corresponding lemma in [**24**] assumes, but does not use, a *real* involution.

for all $x \in R\}$ and $\mathscr{R}(R) = \{a | a \in R,\ ||xa|| = ||x||\ ||a||$ for all $x \in R\}$.

LEMMA 7. *If R is a metric ring, then:*

(i) $a \in \mathscr{L}(R)$ *if and only if* $\bar{l}(a) = ||a||$;
(ii) $0 \in \mathscr{L}(R)$;
(iii) *if* $\mathscr{L}(R) \neq \{0\}$ *then* $||e|| = 1$;
(iv) $e \in \mathscr{L}(R)$ *if and only if* $||e|| = 1$;
(v) $\mathscr{L}(R)$ *is a closed set and a multiplicative semigroup*;
(vi) *if* $a,\ ab \in \mathscr{L}(R)$ *where* $a \neq 0$, *then* $b \in \mathscr{L}(R)$.

Proof. If $a \in \mathscr{L}(R)$, then $||ax||/||x|| = ||a||$ for all $x \neq 0$, so $\bar{l}(a) = ||a||$. Conversely, if $\bar{l}(a) = ||a||$ then $||a|| = \bar{l}(a) \leq ||ax||/||x||$ for any $x \neq 0$, so $||a||\ ||x|| \leq ||ax||$ for any x, whence $a \in \mathscr{L}(R)$.

Clearly, $0 \in \mathscr{L}(R)$. Also, $\bar{l}(e) = 1$, and since $e \in \mathscr{L}(R)$ if and only if $\bar{l}(e) = ||e||$, it follows that $e \in \mathscr{L}(R)$ if and only if $||e|| = 1$. If $\mathscr{L}(R)$ contains an element $a \neq 0$, then $||a|| = ||ae|| = ||a|| \cdot ||e||$, whence $||e|| = 1$.

$\mathscr{L}(R)$ is the set where the continuous function $||x|| - \bar{l}(x)$ vanishes, so $\mathscr{L}(R)$ is a closed set. If $a,\ b \in \mathscr{L}(R)$ then

$$||ab|| \leq ||a|| \cdot ||b|| = \bar{l}(a)\bar{l}(b) \leq \bar{l}(ab) \leq ||ab||$$

by Lemma 1 (i) and (ii), so $||ab|| = \bar{l}(ab)$, whence $ab \in \mathscr{L}(R)$. This shows that $\mathscr{L}(R)$ is a multiplicative semigroup.

Finally, if a and ab belong to $\mathscr{L}(R)$ and $a \neq 0$, then

$$||a|| \cdot ||b|| \cdot ||x|| = ||ab|| \cdot ||x|| = ||abx|| = ||a|| \cdot ||bx||\ ,$$

so $||b|| \cdot ||x|| = ||bx||$ for any x, whence $b \in \mathscr{L}(R)$.

The sets $\mathscr{L}(R)$ and $\mathscr{R}(R)$ measure the extent to which the norm resembles an absolute value. Indeed, it is easily seen that the norm of R is an absolute value if and only if $\mathscr{L}(R) = R$. For the ring $C(X)$ of example (6) the sets $\mathscr{L}(R)$ and $\mathscr{R}(R)$ coincide and consist of all functions whose absolute value is a constant function. The elements of $\mathscr{L}(R)$ in this case are then regular or equal to zero. In general, it will be useful to consider the set of regular elements of $\mathscr{L}(R)$.

DEFINITION. In a metric ring for which $||e|| = 1$, let

$$\mathscr{G} = \{a | a \in G,\ ||a|| \cdot ||a^{-1}|| = 1\}$$

and

$$\mathscr{G}' = \{a | a \in G,\ ||a|| = ||a^{-1}|| = 1\}.$$

DEFINITION. If A is any subset of a metric ring R, let $\mathscr{N}(A) = \{||a||\ |\ a \in A\}$, $\nu(a) = ||a||$ for any $a \in R$.

THEOREM 5. *Let R be a metric ring such that $||e||=1$. Then $\mathscr{L}(R)\cap G=\mathscr{G}=\mathscr{R}(R)\cap G$. Also, $\mathscr{G}$ is closed in G and is a subgroup of G. Furthermore, $\mathscr{G}$ is a maximal μ-group.*

Proof. If $a\in\mathscr{L}(R)\cap G$, then $||a||\cdot||a^{-1}||=||aa^{-1}||=1$, so $a\in\mathscr{G}$. Conversely, if $a\in\mathscr{G}$, then

$$||a||\cdot||x||=||a||\cdot||a^{-1}ax||\leqq||a||\cdot||a^{-1}||\cdot||ax||=||ax||$$

for any x, so $a\in\mathscr{L}(R)$, whence $a\in\mathscr{L}(R)\cap G$. Since $\mathscr{L}(R)$ is closed, $\mathscr{G}=\mathscr{L}(R)\cap G$ is closed in G. The proof that $\mathscr{R}(R)\cap G=\mathscr{G}$ is similar to the above.

Since $\mathscr{L}(R)$ and G are semigroups, $\mathscr{G}$ is also a semigroup. Also, e is in $\mathscr{G}$, and $\mathscr{G}$ contains the inverses of all of its elements, so $\mathscr{G}$ is a group. The norm is multiplicative on $\mathscr{L}(R)$ and hence on $\mathscr{G}\subset\mathscr{L}(R)$, so $\mathscr{G}$ is a μ-group. The definition of $\mathscr{G}$ clearly implies that $\mathscr{G}$ is the largest μ-group which is contained in G. But any μ-group which contains $\mathscr{G}$ must be contained in G since G is a maximal multiplicative group, so $\mathscr{G}$ coincides with such a μ-group and is hence a maximal μ-group.

THEOREM 6. *Let R be a metric ring with $||e||=1$. Then the restriction of ν to $\mathscr{G}$ is a homomorphism of $\mathscr{G}$ onto the multiplicative group $\mathscr{N}(\mathscr{G})$ and has $\mathscr{G}'$ as its kernel. $\mathscr{G}'$ is the largest multiplicative group on the unit sphere $U=\{x|\ ||x||=1\}$. If R is also a Q-ring, then $\mathscr{G}'$ and $\mathscr{G}\cup\{0\}$ are closed sets and $\mathscr{G}$ is closed if and only if $\mathscr{G}=\mathscr{G}'$.*

Proof. The restriction of ν to $\mathscr{G}$ is clearly a homomorphism of $\mathscr{G}$ onto $\mathscr{N}(\mathscr{G})$, and the kernel of this homomorphism is $\mathscr{G}\cap U=\mathscr{G}'$. It is also clear that $\mathscr{G}'$ is the largest multiplicative group on U.

Since $\mathscr{G}=\mathscr{L}(R)\cap G$ by the preceding theorem, we have $[\mathscr{G}]\subset\mathscr{L}(R)\cap[G]$ because $\mathscr{L}(R)$ is closed according to Lemma 7(v). If R is a Q-ring, then

$$[\mathscr{G}]\cap S\subset\mathscr{L}(R)\cap[G]\cap S\subset\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}$$

by Theorem 1. But, if $a\in\mathscr{L}(R)$ then $\bar{l}(a)=||a||$ by Lemma 7(i), while if $a\in Z^{\bar{l}}$, $\bar{l}(a)=0$. Thus, if $a\in\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}$ we have $\bar{l}(a)=||a||$ and $\bar{l}(a)=0$, so $a=0$. It follows that $\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}=\{0\}$, so $[\mathscr{G}]\cap S\subset\{0\}$. But $[\mathscr{G}]\cap G\subset\mathscr{L}(R)\cap G=\mathscr{G}$ by Theorem 5, so $[\mathscr{G}]\subset\mathscr{G}\cup\{0\}$. Then

$$[\mathscr{G}\cup\{0\}]\subset[\mathscr{G}]\cup\{0\}\subset\mathscr{G}\cup\{0\}\ ,$$

so $\mathscr{G}\cup\{0\}$ is closed. $\mathscr{G}'$ is the intersection of the closed sets U and $\mathscr{G}\cup\{0\}$ and is consequently closed.

Finally, if $\mathscr{G}=\mathscr{G}'$ then $\mathscr{G}$ is closed since $\mathscr{G}'$ is closed in a Q-ring by the preceding paragraph. Conversely, suppose $\mathscr{G}$ is closed and contains an element a not in $\mathscr{G}'$. Then theelement s a^n for $n=\pm 1, \pm 2, \cdots$ belong to $\mathscr{G}$, and since $||a||\neq 1$ and $||a^n||=||a||^n$, it follows that there are elements a^n in every neighborhood of 0. Since $\mathscr{G}$ is closed, $0\in\mathscr{G}$. This is a contradiction. Thus, $\mathscr{G}=\mathscr{G}'$ if $\mathscr{G}$ is closed.

6. Multiplicative conditions on the norm. We shall now consider several related conditions on the norm of a metric ring. In the sequel it will be assumed that $||e||=1$ in the metric rings under discussion.

M1. The norm of R is an absolute value. (Equivalently, $\mathscr{L}(R)=R$.)

M2. $\mathscr{G}=G$; that is, the norm is multiplicative on G.

M3. $\mathscr{G}$ is open.

M4. $\mathscr{G}$ fails to be nowhere dense in R.

M5. $\mathscr{L}(R)$ fails to be nowhere dense in R.

In the case of M5, Lemma 7(iii) indicates that, for a non-discrete ring, this condition can hold only if $||e||=1$. However, $\mathscr{G}$ has been defined only for metric rings for which $||e||=1$, so that M2, M3 and M4 are meaningless unless $||e||=1$; for that reason we have assumed that $||e||=1$.

It is easily seen that for any metric ring M1 implies M2, M3 implies M4, and M4 implies M5. For a metric Q-ring it is also true that M2 implies M3. Thus, for any metric Q-ring if one of the conditions M1-M5 holds then all of the later ones also hold. Under certain circumstances, two or more of the conditions M1–M5 may be equivalent.

LEMMA 8. *If R is a metric Q-ring, then conditions* M3, M4 *and* M5 *are equivalent in R.*

Proof. By the previous remarks it will suffice to show that when M5 holds then M3 holds. We may assume that R is not discrete, for if R is discrete then M3, M4 and M5 all hold. Now, if M5 holds in R, the closed set $\mathscr{L}(R)$ contains an open sphere Σ which has center $a\neq 0$ and radius $r>0$, so

$$\Sigma=\{x|\ ||x-a||<r\}\ .$$

If

$$\Sigma'=\{x|\ ||x-e||<r/||a||\}$$

is the open sphere with center e and radius $r/||a||$, then $y\in\Sigma'$ implies $||y-e||<r/||a||$, so $||ay-a||=||a||\cdot||y-e||<r$, whence $ay\in\Sigma\subset\mathscr{L}(R)$. Lemma 7(vi) implies that $y\in\mathscr{L}(R)$; this shows that $\Sigma'\subset\mathscr{L}(R)$, so e is an interior point of $\mathscr{L}(R)$. Since R is a Q-ring, e is an interior point of G, so e is an interior point of $\mathscr{L}(R)\cap G=\mathscr{G}$. Since $\mathscr{G}$ is a topolog-

ical group and is therefore homogeneous, $\mathscr{G}$ must be open,[10] so M3 holds for R. This proves the lemma.

LEMMA 9. *If R is a metric Q-ring such that $\mathscr{G}$ meets every component of G, then* M2 *and* M3 *are equivalent in R.*

Proof. If M3 holds, $\mathscr{G}$ is open. By Theorem 5, $\mathscr{G}$ is closed in G, so $\mathscr{G}$ is open and closed in G. Thus, $\mathscr{G}$ contains every component of G which it meets, so if $\mathscr{G}$ meets every component it follows that $\mathscr{G}=G$; that is, M2 holds. Conversely, it has already been pointed out that if M2 holds for a metric Q-ring then M3 also holds.

COROLLARY. *If R is a metric Q-ring such that G is connected, then* M2 *and* M3 *are equivalent in R.*

LEMMA 10. *If R is a metric ring such that G is dense in R, then* M1 *is equivalent to* M2 *in R. In particular, if R is a metric Q-ring (complete metric ring) such that S is nowhere dense (of first category), then* M1 *is equivalent to* M2 *for R.*

Proof. If R is a metric ring in which G is dense, then if M2 holds we have $R=[G]=[\mathscr{G}]\subset\mathscr{L}(R)$, so M1 holds. Thus, M1 is equivalent to M2.

If R is a metric Q-ring and the closed set S is nowhere dense, then G is dense, so that M1 is equivalent to M2 for R. For R a complete metric ring and S of first category, it follows that R is a metric Q-ring and S is nowhere dense since it is a closed first category set of a complete metric space. By the preceding result, M1 is equivalent to M2.

Note. In the presence of condition M1, a metric ring R can have no zero-divisors other than 0, for if $ab=0$, then $||a||\cdot||b||=||ab||=0$, whence $a=0$ or $b=0$. Thus, the ring contains no proper nilpotents or idempotents, and the remarks of § 2 imply that $G^{l}=G^{r}=G$, so inverses are always two-sided and unique for such a ring.

The conditions M1–M5 are strong restrictions on the algebraic structure of a metric ring, as this remark on G^{l} and G^{r} indicates. Indeed, under suitable conditions they will insure that the given ring is a division ring. Some results in this direction follow.

LEMMA 11. *Let R be a metric ring for which* M1 *holds. Then R is proper if and only if it is a division ring.*

[10] See [6].

Proof. If M1 holds for R, then $\bar{l}(a)=\|a\|=r(a)$ for all a in R, by Lemma 7 (i). Thus, $Z=\{0\}$ for this ring. Then $Z=S$ is equivalent to $S=\{0\}$; that is, R is proper if and only if it is a division ring.

THEOREM 7. *Let R be a metric Q-ring such that S is nowhere dense. If* M1 *or* M2 *holds for R, then R is a division ring and its norm is an absolute value.*

Proof. S is nowhere dense, so G is dense, whence M1 is equivalent to M2 by Lemma 10. Also, Theorem 3 implies that R is proper, and it follows from the preceding lemma that R is a division ring. Since M1 must hold in R if M1 or M2 is assumed to hold, it follows that the norm of R is an absolute value.

COROLLARY. *Let R be a complete metric ring such that S is of first category. If* M1 *or* M2 *holds for R, then R is a division ring and its norm is an absolute value.*

THEOREM 8. *Let R be a metric Q-ring such that H is connected. If* M1 *holds for R, then R is a division ring.*

Proof. By Theorem 2, R is proper, so Lemma 11 implies that R is a division ring.

If H is connected and also dense, then R is proper and G, which therefore coincides with H, is connected and dense. Lemmas 8 and 10 and the corollary to Lemma 9 imply that M1–M5 are equivalent, so that if one of the conditions M1–M5 is assumed, then M1 holds, and the theorem just established shows that R is a division ring. This establishes the following corollary.

COROLLARY. *Let R be a metric Q-ring such that H is connected and dense. If one of the conditions* M1–M5 *holds for R, then R is a division ring and its norm is an absolute value.*

THEOREM 9. *Let R be a metric Q-ring which admits a real, bounded involution and for which* M1 *holds. Then R is a division ring.*

Proof. By Theorem 4, R is proper, so Lemma 11 implies that R is a division ring.

LEMMA 12. *Let R be a metric Q-ring which satisfies one of the conditions* M1–M5. *If A is a connected subset of R which does not contain* 0, *then either $A\subset\mathscr{S}$ or A is disjoint from $\mathscr{S}$.*

Proof. Because of the relations among M1–M5, M5 holds, so Lemma 8 implies that M3 holds, whence $\mathscr{G}$ is open. But $\mathscr{G}$ is closed in $R^{(0)}$ since $\mathscr{G}\cup\{0\}$ is closed by Theorem 6. Then $\mathscr{G}$ is open and closed in $R^{(0)}$, so any connected subset A of $R^{(0)}$ must be contained in $\mathscr{G}$ or disjoint from $\mathscr{G}$.

COROLLARY. *If R is a metric Q-ring which satisfies one of the conditions* M1–M5, *then each connected component of $\mathscr{G}$ coincides with a component of G, and, in particular, $\mathscr{G}\supset G_1$.*

THEOREM 10. *Let R be a metric Q-ring such that $R^{(0)}$ is a connected set. If one of the conditions* M1–M5 *holds for R, then R is a division ring with absolute value.*

Proof. Lemma 12 implies that $\mathscr{G}$ contains the connected set $R^{(0)}$. Thus, $R=\mathscr{G}\cup\{0\}$, so R is a division ring with absolute value.

If it is assumed that S is nowhere dense and G is connected in a metric Q-ring in which one of the conditions M1–M5 holds, then Lemma 12 implies that $\mathscr{G}=G$, while G is dense since S is nowhere dense. Thus, $R=[G]$, and $R^{(0)}$ is connected since G is connected. The theorem implies that R is a division ring with absolute value in this case. The assumption of completeness again permits the requirement that S be nowhere dense to be replaced by the requirement that S be of first category.

COROLLARY 1. *Let R be a metric Q-ring* (*complete metric ring*) *for which S is nowhere dense* (*of first category*) *and G is connected. If one of the conditions* M1–M5 *holds for R, then R is a division ring with absolute value.*

COROLLARY 2. *If R is a metric Q-ring such that $R^{(0)}$ is connected, then precisely one of the following statements is valid:*

(α) *$\mathscr{L}(R)$ is nowhere dense in R.*

(β) *R is a division ring with absolute value.*

COROLLARY 3. *If R is a metric Q-ring* (*complete metric ring*) *for which G is a connected set and S is nowhere dense* (*of first category*), *then precisely one of the following statements is valid:*

(α) *$\mathscr{L}(R)$ is nowhere dense in R.*

(β) *R is a division ring with absolute value.*

Corollaries 2 and 3 follow immediately from the theorem and Corollary 1, respectively, since if (α) does not hold then M5 holds and therefore (β), which is the conclusion of the theorem and of Corollary 1, must hold.

Corollaries 2 and 3 clearly continue to hold if (α) is replaced by: (α') $\mathscr{S}$ is nowhere dense in R. In Corollaries 1 and 3 the hypothesis that G be connected may be replaced by the hypothesis that $\mathscr{S}$ meet every component of G. Another alternative for these two corollaries is to replace all conditions on G and S by the hypothesis that $\mathscr{S}$ meet every component of $R^{(0)}$.

7. Division rings with absolute value. In [21] A. Ostrowski classified the fields which admit an absolute value. However, the property of commutativity played only a minor role in Ostrowski's discussion. We outline below the classification of division rings with absolute value. By combining these results with the results of the preceding section we obtain stronger statements of those results.

DEFINITION. If R is a metric ring such that $||a+b||\leq\max(||a||, ||b||)$ for all a and b in R, then R is called a *non-archimedean ring*, and the norm for R is said to be *non-archimedean*. In the contrary case, R is called an *archimedean ring* and the norm of R is said to be *archimedean*.

For any division ring K there is a unique field P, the prime field of K, which is the smallest field contained in K. Then P is either isomorphic to the field of rational numbers, and K is said to have characteristic zero, or P is isomorphic to the field of integers modulo p, where p is a prime number, in which case K is said to have characteristic p. If K is a division ring with absolute value, then the restriction to P of the absolute value of K is an absolute value for P. The classification of the absolute value of K as non-archimedean or archimedean depends only upon its behavior on the prime field of K and, indeed, only upon its behavior on the set of elements of the form ne, where n is a natural number. (If n is a natural number, na denotes the n-fold sum $a+\cdots+a$ (n summands). If n is a negative integer, na is defined as $-[(-n)a]$, while $0a$ denotes 0.) This result, given by Ostrowski in [21], appears in Lemma 13, while a stronger result occurs in Lemma 14.

LEMMA 13. *A division ring K with absolute value is non-archimedean if and only if $||ne||\leq 1$ for every natural number n.*

LEMMA 14. *A division ring K with absolute value is non-archimedean if and only if $||2e||\leq 1$.*

Note. Lemmas 13 and 14 remain valid if we replace the hypothesis that K is a division ring with absolute value by the hypothesis that K is a commutative metric ring such that $||a^2||=||a||^2$ for all a in K. Although many metric rings have the property that $||a^2||=||a||^2$ for all a, the rings of example (10), with n positive, do not have this property. Lemma 14 also holds if 2 is replaced by any other integer greater than 1.

THEOREM 11. *If K is an archimedean division ring with absolute value, and K is complete, then K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. Furthermore, the norm in K corresponds to the ρth power of the ordinary absolute value, for some ρ such that $0<\rho\leqq 1$.*

This theorem and Lemmas 13 and 14 are easily proved. The theorem appears in essence in [21].

COROLLARY. *If K is a complete division ring with absolute value such that $\|2e\|>1$, then K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. The norm of K corresponds to the ρth power of the ordinary absolute value, for some ρ such that $0<\rho\leqq 1$.*

If we note that the completion of an archimedean division ring with absolute value is again an archimedean division ring with absolute value, the theorem implies that any archimedean division ring with absolute value is algebraically and topologically isomorphic to a dense subring of $\mathfrak{R}$, of $\mathfrak{C}$, or of $\mathfrak{Q}$.

The non-archimedean division rings with absolute value constitute a far more varied and extensive class, however. For example, even the locally compact examples are fairly numerous, as may be seen by the list given by Otobe in [22]; all of those examples are of course complete since they are locally compact. We therefore combine the results of § 6 with the preceding results on archimedean division rings for the sake of simplicity.

LEMMA 15. *If K is a non-archimedean division ring with absolute value, then K is totally disconnected.*

COROLLARY 1. *If K is a complete division ring with absolute value, then K is non-archimedean if and only if it is totally disconnected.*

COROLLARY 2. *If K is a complete division ring with absolute value, then K contains a connected set having more than one point if and only if K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

The field of rational numbers with the ordinary absolute value is archimedean and totally disconnected; this shows the necessity of assuming completeness in Corollary 1. In [7], Dieudonné constructed a connected and locally connected subfield of $\mathfrak{C}$ which is a pure transcendental extension of the field of rational numbers. The field of Dieudonné, with the ordinary absolute value, is then an example of a field which is not complete and which is connected although it is not isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. This shows that Corollary 2 requires the assumption of completeness.

By combining the results just outlined with those of the preceding section, we obtain the results which follow.

THEOREM 12. *Let R be a complete archimedean metric ring such that S is a first category set. If* M1 *or* M2 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

COROLLARY. *Let R be an archimedean metric Q-ring such that S is nowhere dense. If* M1 *or* M2 *holds for R, then R is algebraically and topologically isomorphic to a dense division subring of $\mathfrak{R}$, of $\mathfrak{C}$, or of $\mathfrak{Q}$.*

THEOREM 13. *Let R be a complete metric ring such that H is connected. If* M1 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$, to $\mathfrak{Q}$, or to the field $\mathfrak{F}$ of order* 2 *with the trivial absolute value.*

TEOREM 14. *Let R be a complete metric ring in which H is connected and dense. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$ or to $\mathfrak{Q}$.*

THEOREM 15. *Let R be a complete archimedean metric ring which admits a real, bounded involution. If* M1 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

THEOREM 16. *Let R be a complete metric ring such that $R^{(0)}$ is connected. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$, to $\mathfrak{Q}$, or to $\mathfrak{F}$.*

THEOREM 17. *Let R be a complete metric ring for which S is of first category and G is connected. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$ or to $\mathfrak{Q}$.*

COROLLARY. *Let R be a metric Q-ring for which G is connected and S is nowhere dense. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to a dense division subring of $\mathfrak{C}$ or of $\mathfrak{Q}$.*

If the requirement of completeness for R in Theorems 13–16 is replaced by the weaker requirement that R be a metric Q-ring, then the conclusion becomes that R is algebraically and topologically isomorphic to a dense division subring of one of the division rings mentioned in the conclusion of that particular theorem. In Theorem 12 and its corollary, and in Theorem 15, the assumption that R is archimedean may be replaced by the assumption that $||2e||>1$ or the assumption that R contains a connected set with more than one point.

It is easily seen that completeness is required in these theorems. For, let K be the subfield of $\mathfrak{C}$ constructed in [7] by Dieudonné. Then K is connected and locally connected, K is a dense, proper subfield of $\mathfrak{C}$, and K is a pure transcendental extension of the field of rational numbers. Clearly, K is not isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, to $\mathfrak{Q}$, or to $\mathfrak{F}$. But the set $S=\{0\}$ is nowhere dense in K, while G, H and $K^{(0)}$ coincide and are easily seen to be connected. The identity mapping of K into itself is a real, bounded involution, and M1 holds for K, so that K satisfies all of the hypotheses of these theorems except for completeness Since K does not satisfy the conclusions, completeness is needed.

8. Homogeneous metric rings and rings of quotients. In this section we consider certain types of metric rings which may be embedded in various algebras.

DEFINITION. A metric ring R is said to be *homogeneous* if $||na||=|n|\cdot||a||$ whenever n is an integer and a is in R. A metric ring R is said to be *weakly homogeneous* if $||na||=||ne||\cdot||a||$ whenever n is an integer and a is in R.

For a homogeneous ring we have $||ne||=|n|$, so every homogeneous ring is also weakly homogeneous. However, a weakly homogeneous ring need not be homogeneous; for example, the rings of examples (8) and (9) are weakly homogeneous but are not homogeneous. The rings given in the other examples are all homogeneous. It is clear that a metric ring in which M1 holds must be weakly homogeneous. We can also obtain a sufficient condition for a metric ring to be homogeneous.

LEMMA 16. *If R is a metric ring such that $||2a||=2||a||$ for every a in R, then R is homogeneous.*

Proof. For any natural number r and for $a\in R$ we have $||2^r a||=2^r||a||$. Thus, for n a natural number, we have

$$n||a||+(2^n-n)||a||=2^n||a||=||2^n a||\leq||na||+||(2^n-n)a||\leq n||a||+(2^n-n)||a||\,,$$

so that $n||a||=||na||$ for any natural number n and any a in R. It follows easily that $||na||=|n|\cdot||a||$ for any integer n and any a in R.

If R is any metric ring, and D is a nonempty multiplicative semigroup in R which does not contain 0, which lies in the center of R, and such that $D\subset\mathscr{L}(R)$, then the relation $(a, d)\sim(a', d')$ (if and only if $ad'=a'd$) is an equivalence relation in the set $R\times D$ of ordered pairs

(a, d), where a is in R and d is in D. Let R_D be the set of equivalence classes $[a/d]$ modulo this equivalence relation, with

$$[a/d]+[b/f]=[(af+bd)/df] ,$$

$[a/d]\cdot[b/f]=[ab/df]$, and $\|[a/d]\|=\|a\|/\|d\|$ as the definitions for addition, multiplication and the norm. It is clear that these definitions depend only on the equivalence classes involved and not on the representatives chosen from the classes. It is also easily verified that R_D is a metric ring, and the mapping $x \to [xd/d]$ is an isometry of R into R_D if d is in D. An element d in D may be identified with the element $[d^2/d]$ of R_D which has the inverse $[d/d^2]$ in R_D. We thus obtain the following lemma.

LEMMA 17. *Let R be a metric ring, and D a nonempty multiplicative semigroup in R which does not contain* 0. *Suppose $D \subset \mathscr{L}(R)$ and D is contained in the center of R. Then R_D is a metric ring which is an extension of R such that every element of D has an inverse in R_D.*[11]

COROLLARY. *Let R_1 be a commutative metric ring. Then there is an extension, R, of R_1 such that $\mathscr{L}(R)=\mathscr{G}\cup\{0\}$. In particular, all of the nonzero elements of $\mathscr{L}(R_1)$ have inverses in R.*

Proof. If D is the set of nonzero elements of $\mathscr{L}(R_1)$, then $R=(R_1)_D$ is the required extension of R.

COROLLARY 2. *Let R be a commutative metric ring in which* M1 *holds. Then there is a field, K, with absolute value, such that K is complete and K is an extension of R.*[12]

Proof. If D is the set of nonzero elements of R, then R_D is a field with absolute value. The completion, K, of R_D is the required field.

If K is a field with absolute value, and R is a metric ring which is also an associative linear algebra over K such that $\|ka\|=\|k\|\cdot\|a\|$ for all k in K and a in R, then R is called a *normed algebra* over K. For example, the metric rings of examples (3)–(7) and (10) are normed algebras over $\mathfrak{R}$, while the rings in examples (4), (6) and (7) are normed algebras over $\mathfrak{C}$. It will now be shown that any weakly homogeneous metric ring has an extension which is a normed algebra. Also, for homogeneous metric rings, there is an extension which is a normed algebra over $\mathfrak{R}$.

THEOREM 18. *Let R be a weakly homogeneous metric ring. Then*

[11] Compare the results on algebras of quotients in [**24**].

[12] Compare the proof of Theorem 2, Corollary 2 in [**4**], where the technique of embedding in a quotient field is also employed.

there exists an extension of R which is a complete normed algebra over some field K, where either K has the trivial norm, or K is the real field with some power of the ordinary absolute value as its norm, or K is a p-adic field, with some power of the norm given in example (9) *as the norm of K.*

Proof. Let D be the set of nonzero elements of R which have the form ne, with n an integer. Then R_D is an extension of R and contains a subset which is isomorphic to the quotient field, F, for $D \cup \{0\}$. Then R_D is a normed algebra over F, so that the completion of R_D is a normed algebra over the field K, where K is the completion of F; see, for instance, [6]. Thus, R has an extension which is a complete normed algebra over K. If the norm of F is the trivial one, then K coincides with F. In the contrary case, there is a natural number n such that $\|ne\|$ is distinct from 0 and 1. Also, F is a prime field and is therefore isomorphic to the field of rational numbers since the other prime fields are finite and would only admit the trivial absolute value. If $\|ne\|<1$, we have $\|pe\|<1$ for some rational prime p. As in Ostrowski's proof, p is unique in that case and the norm is a power of the norm described in example (8), so K is isomorphic to the field of p-adic numbers with the norm taken as some power of the p-adic norm. In case $\|ne\|>1$ for every natural number n greater than 1, we have the archimedean case, so F is the field of rational numbers with the norm taken as the ρth power of the absolute value, with $0<\rho\leq 1$. Thus, K consists of the real numbers with the norm given as the ρth power of the absolute value.

COROLLARY. *Let R be a homogeneous metric ring. Then there is an extension of R which is a complete normed algebra over $\mathfrak{R}$.*

Proof. In this case, $\|ne\|=|n|\cdot\|e\|=|n|$ for any integer n, so in the proof of Theorem 18 the norm of an element of F is the usual absolute value. Thus, K is the real field with its usual absolute value.

If K is a complete division ring with absolute value such that $\|2e\|=2$, then the corollary of Theorem 11 implies that K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$, with the norm corresponding to the ρth power of the ordinary absolute value. K is homogeneous since the condition $\|2e\|=2$ implies that $\|2a\|=2\|a\|$ for all a in K. The prime field of K is then the field of rational numbers with the ordinary absolute value as the norm, as the preceding proofs imply. But $\|a\|=|a|^\rho$ for all a in K, while for a rational, $\|a\|=|a|$. Thus, $\rho=1$, and the following theorem results.

THEOREM 19. *If K is a complete division ring with absolute value* *such that* $||2e||=2$, *then* K *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

This result implies that if the hypothesis that $||2e||=2$ is added to Theorems 11–17 and their corollaries the algebraic isomorphism of the conclusions must be an isometry. In a similar vein, Theorem 18 asserts that a weakly homogeneous metric ring R may always be embedded in a complete normed algebra, so a metric ring with absolute value may be embedded in a complete normed algebra; the addition of the strong hypothesis $||2e||=2$ yields a stronger result.

THEOREM 20. *Let* R *be a metric ring with absolute value such that* $||2e||=2$. *Then* R *is algebraically isomorphic and isometric to a subring of* $\mathfrak{Q}$.

Proof. Lemma 16 shows that R is homogeneous, so the corollary of Theorem 18 implies that there is an extension of R which is a complete normed algebra over $\mathfrak{R}$. The construction of this extension R_1 is such that R_1 also has an absolute value. If the real dimension of R_1 as a vector space is greater than one, then $R_1^{(0)}$ is connected, so, by Theorem 16, in the strengthened form just mentioned, R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. If the real dimension of R_1 is one, then R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$. In any event, R is algebraically isomorphic and isometric to a subring of R_1, R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$, each of which is algebraically isomorphic and isometric to a subset of $\mathfrak{Q}$, and the theorem follows.

Note. If r is a fixed integer greater than 1, then the condition $||re||=r$ is equivalent to the condition $||2e||=2$ and may be used as a hypothesis instead of the latter in any of the preceding results.

9. Real and complex normed algebras. The results of the last two sections may now be specialized to the case of normed algebras over $\mathfrak{R}$ or $\mathfrak{C}$. Any normed algebra over $\mathfrak{C}$ may of course be regarded as a normed algebra over $\mathfrak{R}$. A complete normed algebra over $\mathfrak{R}(\mathfrak{C})$ is called a *Banach algebra* (*complex Banach algebra*).

THEOREM 21. *Let* $\mathfrak{A}$ *be a Banach algebra for which one of the conditions* M1–M5 *holds. Then* $\mathfrak{A}$ *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

Proof. If $\mathfrak{A}$ has dimension one as a vector space over $\mathfrak{R}$, then $\mathfrak{A}$ is certainly algebraically isomorphic and isometric to $\mathfrak{R}$. If the dimension of $\mathfrak{A}$ is greater than one, then $\mathfrak{A}^{(0)}$ is clearly connected, and the result follows from the strengthened from of Theorem 16 mentioned in the previous section.

COROLLARY 1. *Let* $\mathfrak{A}$ *be a normed algebra over* $\mathfrak{R}$ *(Q-ring which is also a normed algebra over* $\mathfrak{R}$*) such that one of the conditions* M1, M3, M4 *or* M5 (M1-M5) *holds. Then* $\mathfrak{A}$ *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

Proof. The completion, $\mathfrak{A}_1$, of $\mathfrak{A}$ is a Banach algebra. Because of the relations among M1-M5, we may assume that M5 holds, and it follows that M5 holds for $\mathfrak{A}_1$. The theorem shows that $\mathfrak{A}_1$ is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. But $\mathfrak{A}$ is a dense, connected linear subspace of the finitedimensional real vector space $\mathfrak{A}_1$ and therefore coincides with $\mathfrak{A}_1$.

The theorem, with M2 assumed, is essentially the result of Edwards [8; Theorem 1] combined with the first of Mazur's theorems. The corollary, with M1 assumed is the same as Mazur's second theorem in [19].

It may be noted that the corollary does not hold when M2 is assumed and $\mathfrak{A}$ is not a Q-ring. For example, the algebra of all real polynomials $f(x)$ with the norm $||f||=\sup|f(x)|$, where the supremum is taken for all x such that $0\leq x\leq 1$, is a normed algebra over $\mathfrak{R}$ for which G consists only of the constant polynomials distinct from zero; clearly, $\mathscr{G}=G$ for this algebra, so M2 holds, even though this algebra is not even a division ring.

COROLLARY 2. *If* $\mathfrak{A}$ *is a normed algebra over* $\mathfrak{R}$ *which is not isomorphic to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$, *then* $\mathscr{L}(\mathfrak{A})$, $\mathscr{R}(\mathfrak{A})$, $\mathscr{G}$ *and all* μ*-groups of* $\mathfrak{A}$ *are nowhere dense.*

Proof. The hypothesis implies that M4 and M5 can not hold, so $\mathscr{L}(\mathfrak{A})$, $\mathscr{R}(\mathfrak{A})$ and $\mathscr{G}$ are nowhere dense in $\mathfrak{A}$.

It remains to show that all μ-groups of $\mathfrak{A}$ are nowhere dense. Suppose that A is a μ-group which fails to be nowhere dense. The unit element, j, of the group A is an idempotent, and we have the inclusion $A\subset jA\subset j\mathfrak{A}$, so that $j\mathfrak{A}$ also fails to be nowhere dense. If $\{jx_n\}$ is a sequence of elements of $j\mathfrak{A}$ which converges to an element a in $\mathfrak{A}$, then $\{j^2\cdot x_n\}$ converges to ja. But since j is an idempotent the sequences $\{jx_n\}$ and $\{j^2\cdot x_n\}$ coincide, so their limits coincide, whence $a=ja$ is in $j\mathfrak{A}$. This shows that $j\mathfrak{A}$ contains the limit of any convergent sequence of elements of $j\mathfrak{A}$, so $j\mathfrak{A}$ is closed. Because $j\mathfrak{A}$ fails to be nowhere dense it must contain a nonempty open set. But $j\mathfrak{A}$ is a right ideal and therefore, in particular, a topological group relative to addition; the homogeneity of a topological group then implies that $j\mathfrak{A}$ is open. Since

$j\mathfrak{A}$ is open and closed and nonempty in the connected space $\mathfrak{A}$, we see that $j\mathfrak{A}=\mathfrak{A}$. This shows that j has a right inverse, so $j=e$. Now, A is a μ-group which has e as its unit element, so if $a \in A$ then a has an inverse a^{-1} relative to e in A, and $||a||\cdot||a^{-1}||=||aa^{-1}||=||e||=1$, so $a \in \mathscr{G}$. This shows that $A \subset \mathscr{G}$. But it has already been observed that $\mathscr{G}$ is nowhere dense, so the assumption that A fails to be nowhere dense leads to a contradiction. This proves the corollary.

The same proof can be used to show that all μ-groups are nowhere dense in a connected metric ring for which $\mathscr{G}$ is nowhere dense.

In the case of normed algebras over $\mathfrak{C}$, one can also show that the set $\mathscr{G}'$ is generally not too extensive.

THEOREM 22. *If $\mathfrak{A}$ is a normed algebra over $\mathfrak{C}$, then $\mathscr{G}'$ consists exclusively of extreme points of the unit sphere of $\mathfrak{A}$.*

Proof. Suppose that a is an element of $\mathscr{G}'$ which is not an extreme point of the unit sphere. The mapping $x \to xa^{-1}$ is a linear automorphism of the linear space over $\mathfrak{C}$ which underlies $\mathfrak{A}$, and this mapping also preserves distances since a^{-1} belongs to $\mathscr{G}$ and has norm one. Thus, the property of failing to be an extreme point of the unit sphere is preserved, so e, the image of a relative to this mapping, is not an extreme point of the unit sphere.

If $\mathfrak{A}$ were completed, e would also fail to be an extreme point of the unit sphere of the completion, and we therefore assume, without loss of generality, that $\mathfrak{A}$ is complete. Now, e is the midpoint of a segment which lies wholly in the unit sphere of $\mathfrak{A}$, so $e=(b+c)/2$, where $||b||=||c||=1$ and $b\neq c$. Clearly, b and c commute since $c=2e-b$ is in the algebra generated by b and c, so the closed complex normed algebra which is generated by e, b and c is a commutative complex Banach algebra. If $y=(b-c)/2$, then $e-y=c$ and $e+y=b$, whence $||e-y||=||e+y||=1$ in this algebra. But the remark which follows Theorem 1 of [11] asserts that if y is an element of a commutative complex Banach algebra such that $||e-y||=||e+y||=1$, then $y=0$. It follows that $y=0$, so $b=c$. This contradiction shows that a was an extreme point of the unit sphere of $\mathfrak{A}$.

In conclusion, while the results of this paper show that the sets $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{L}(R)$ and $\mathscr{R}(R)$ are usually topologically trivial, they are not algebraically trivial. For, in the case of the algebra $C(X)$ of example (6) where X has at least two points, it is evident that any two points of X may be separated by an element of $\mathscr{G}'$. The Stone-Weierstrass approximation theorem may be used to show that the closed complex subalgebra generated by $\mathscr{G}'$ coincides with $C(X)$.

References

1. A. A. Albert, *Absolute valued real algebras*, Ann. of Math., **48** (1947), 495-501.
2. ———, *Absolute-valued algebraic algebras*, Bull. Amer. Math., Soc. **55** (1949), 763-768.
3. ———, *A note of correction*, Bull. Amer. Math. Soc., **55** (1949), 1191.
4. R. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc., **53** (1947), 623-630.
5. R. Baer and H. Hasse, *Zusammenhang und Dimension topologischer Körperräume*, J. Reine Angew. Math., **167** (1932), 40-45.
6. N. Bourbaki, Éléments de mathématique, Livre III. *Topologie Générale*, Actualités Scientifiques et Industrielles, nos. 916, 1045, 1084, Paris, 1942-1951.
7. J. Dieudonné, *Sur les corps topologiques connexes*, C. R. Acad. Sci. Paris, **221** (1945), 396-398.
8. R. E. Edwards, *Multiplicative norms on Banach algebras*, Proc. Cambridge Philos. Soc., **47** (1951), 473-474.
9. A. Forsythe and N. H. McCoy, *On the commutativity of certain rings*, Bull. Amer. Math. Soc., **52** (1946), 523-526.
10. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sb.) N. S., **9** (1941), 3-24.
11. ———, *Zur Theorie der Charaktere der abelschen topologischen Gruppen*, Rec. Math. (Mat. Sb.) N. S., **9** (1941), 49-50.
12. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, **31**, 1948.
13. S. Kametani, *An elementary proof of the fundamental theorem of normed fields*, J. Math. Soc. Japan, **4** (1952), 96-99.
14. I. Kaplansky, *Topological rings*, Amer. J. Math., **69** (1947), 153-183.
15. ———, *Topological rings*, Bull. Amer. Math. Soc., **54** (1948), 809-826.
16. ———, *Regular Banach algebras*, J. Indian Math. Soc., **12** (1948), 57-62.
17. E. R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc., **54** (1943), 414-425.
18. ———, *The structure of normed abelian rings*, Bull. Amer. Math. Soc., **50** (1944), 447-463.
19. S. Mazur, *Sur les anneaux linéaires*, C. R. Acad. Sci. Paris, **207** (1938), 1025-1027.
20. M. Nagumo, *Einige analytische Untersuchungen in linearen, metrischen Ringen*, Jap. J. Math., **13** (1936), 61-80.
21. A. Ostrowski, *Über einige Lösungen der Funktionalgleichung $\phi(x)\phi(y)=\phi(xy)$*, Acta Math., **4** (1918), 271-284.
22. Y. Otobe, *On locally compact fields*, Jap. J. Math., **19** (1945), 189-202.
23. V. Ramaswami, *Normed algebras, isomorphism and the associative postulate*, J. Indian Math. Soc., **14** (1950), 47-64.
24. C. E. Rickart, *The singular elements of a Banach algebra*, Duke Math. J., **14** (1947), 1063-1077.
25. G. Šilov, *On the extension of maximal ideals*, C. R. (Doklady) Acad. Sci. URSS., **29** (1940), 83-84.
26. L. Tornheim, *Normed fields over the real and complex fields*, Mich. Math. J., **1** (1952), 61-68.

Queens College, New York City