MORREY'S REPRESENTATION THEOREM FOR SURFACES IN METRIC SPACES

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1. Introduction. In 1935 Morrey showed that a non-degenerate surface of finite Lebesgue area has a quasi-conformal representation on the unit circle. He made use of Schwarz' result for polyhedral surfaces and was able to use a limiting process after he had shown that the representations of the surfaces involved were sufficiently well behaved for the area to be given by the usual integral. The limiting process depended upon Tonelli's result concerning the lower semi-continuity of the Dirichlet integral.

Several years later Cesari reduced the dependence upon complex variable theory by the use of a variational technique to obtain a slightly weaker version of Schwarz' result, but he showed that for the remainder of Morrey's argument his form was adequate.

The purpose of this paper is to remove the restriction that the surfaces be in Euclidean space; the method is that of Cesari.

Morrey's theorem has proved useful in the study of certain two-dimensional problems in the calculus of variations. It is hoped that the extension of his theorem will permit corresponding extensions of that theory [3, 6, 12].

A desirable feature of quasi-conformal mappings is that the area of the surface is given by one half the Dirichlet integral. To retain this property for surfaces which are not in Euclidean space requires the definition of an appropriate integral to complement the definition of area. The definition of (Lebesgue) area used in this paper is that given in [13] which agrees with the usual definition in case the surface is in Euclidean space.

We shall make use of the ideas of [13] in two other respects. First, we need only solve our problem for surfaces in m, the space of bounded sequences [1], since the definitions are chosen so as to be invariant under an isometry and we can map other surfaces isometrically into m. Second, we shall make use of the fact that the area of a function in m depends only upon its distinct components. The last remark results from the definition of the area of a triangle. Let $r=\{r^i\}$, $s=\{s^i\}$, and $t=\{t^i\}$ be three points in m. Then the area of the triangle with these points as vertices is, by definition,

$$\frac{1}{2}\sup_{i,k}\begin{vmatrix} r^i & r^k & 1 \\ s^i & s^k & 1 \\ t^i & t^k & 1 \end{vmatrix} .$$

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2. A closure theorem for A.C.T. functions. Certain definitions applying to real-valued functions must be modified to apply to functions which range in a metric space.

DEFINITION 1. Let φ be defined on the interval $[a, b]$ with range in a metric space D. Let Φ be the interval function defined by

$$\Phi([c, d])=\delta(\varphi(c), \varphi(d)) \qquad a\leqq c\leqq d\leqq b ,$$

where $\delta(r, s)$ is the distance between r and s in D. Then φ is B.V. or A.C. according as Φ is B.V. or A.C. Define $D\varphi=D\Phi$ wherever the right hand side exists.

With this definition of bounded variation and absolute continuity of a function of one real variable in a metric space D, we extend verbatim the definitions of bounded variation and absolute continuity in the sense of Tonelli, B.V.T. and A.C.T., to apply to functions of two variables with range in D [10].

If x is continuous on an open set G contained in E_2 into D, define, where the right hand sides exist,

$$D_u x(u, v)=D\varphi(u) \qquad \text{where } \varphi(t)=x(t, v) ,$$

$$D_v x(u, v)=D\psi(v) \qquad \text{where } \psi(t)=x(u, t)$$

If x is B.V.T. then $D_u x$ and $D_v x$ exist a.e. [8].

If φ is defined on $[a, b]$ into m and is A.C. it is still possible that $\lim_{w\to t}\dfrac{\varphi(w)-\varphi(t)}{w-t}$ may not exist anywhere [5]. Hence we define a *component-wise* derivative φ' by $\varphi'=\{\varphi^{i\prime}\}$. Since φ is A.C. it follows that all of the φ^i are also and that $\varphi^{i\prime}$ and $D\varphi$ exist almost everywhere. That $D\varphi\geqq|\varphi^{i\prime}|$ for each i is evident, hence φ' is defined, and in m, almost everywhere in $[a, b]$.

THEOREM 1. *If φ is* A.C. *then $D\varphi$ exists and is equal to $\|\varphi'\|$ wherever φ' exists.*

Proof. Suppose that the theorem is true whenever φ has only a finite number of non-zero components. Let φ_n be that function whose only non-zero components are the first n, and these are the first n components of φ. Then (see the proof of Theorem 10) length $\varphi=\lim_{n\to\infty}$ length φ_n. Hence

$$\int D\varphi=\text{length } \varphi=\lim_{n\to\infty} \text{ length } \varphi_n=\lim\int D\varphi_n=\lim\int\|\varphi_n'\|=\int\|\varphi'\| .$$

Thus we may as well suppose φ has only a finite number of non-zero components. Let t be a point where φ' is defined. It suffices to show that

$$D^+\varphi(t)=\lim_{w\to t}\sup\frac{\|\varphi(w)-\varphi(t)\|}{|w-t|}\leqq\|\varphi'\|\ .$$

For some i there exists a sequence of numbers $w_n\to t$ such that

$$\lim_{n\to\infty}\frac{\|\varphi(w_n)-\varphi(t)\|}{|w_n-t|}=D^+\varphi(t)$$

and

$$|\varphi^i(w_n)-\varphi^i(t)|=\|\varphi(w_n)-\varphi(t)\|\ .$$

The existence of this sequence implies that $D^+\varphi(t)=|\varphi^{i\prime}(t)|\leqq\|\varphi'\|$.

DEFINITION 2. If x is continuous on an open set G into m, define, where the right hand sides exist,

$$x_u(u,\,v)=\varphi'(u) \qquad \text{where } \varphi(t)=x(t,\,v)\ ,$$
$$x_v(u,\,v)=\psi'(v) \qquad \text{where } \psi(t)=x(u,\,t)\ .$$

THEOREM 2. *If x is* A.C.T. *on G into m then*

$$\|x_u\|=D_ux \quad \textit{and} \quad \|x_v\|=D_vx$$

wherever the left hand sides exist.

DEFINITION 3. If x is A.C.T. on G into D and if D_ux and D_vx are in L^2, then x is a *D-mapping* [**4**]. Let

$$D(x)=\iint_G[(D_ux)^2+(D_vx)^2]\ .$$

It was shown in [13] that if x is a D-mapping on a Jordan region into a metric space, then the Lebesgue area of x, $L(x)$, is given by what corresponds to the usual integral (see § 6).

Let Π^N be the projection of m defined by

$$\Pi^N(\{a^i\})=\begin{cases}a^i & i\leqq N\ ,\\ 0 & i>N\ .\end{cases}$$

Put $\Pi^Nx={}_Nx$.

THEOREM 3. *If x_m is a sequence of* A.C.T. *functions on a bounded open set G into m, if $x_m\to x$ uniformly in each closed set H contained in G, if the norms of the partial* (*component-wise*) *derivatives $p_m=\|x_{mu}\|$,*

$q_m=\|x_{mv}\|$ *are in* L^α, $\alpha>1$, *and* $\iint_G[p_m^\alpha+q_m^\alpha]<M$ *for all* m, *then* x *is* A.C.T. *in* G, *the norms of its partials* $p=\|x_u\|$ *and* $q=\|x_v\|$ *are in* L^α *and*

$$\iint_G p^\alpha=\liminf_{m\to\infty}\iint_G p_m^\alpha\ , \qquad \iint_G q^\alpha=\liminf_{m\to\infty}\iint_G q_m^\alpha\ .$$

Proof. Let us first suppose that $x_m={}_Nx_m$ for each m and fixed N. The hypothesis, together with the closure theorem for A.C.T. real-valued functions, assures us that x^i is A.C.T. for each i. Hence ${}_Nx$ is A.C.T.

The remainder of the proof, in case $x_m={}_Nx_m$, deviates slightly from that given in [4] for real-valued functions.

Let K be a closed set contained in G whose distance from the boundary of G is $2\rho>0$. Let K_ρ be the closed set of all points whose distance from K does not exceed ρ. Let $n>2/\rho$. Define $(n;x)$ by

$(n;x)=\{(n;x,i)\}$ and

$$(n;x,i)(u,v)=n^2\int_u^{u+1/n}\int_v^{v+1/n}x^i(r,s)\,dr\,ds \qquad \text{for } (u,v)\in K\ .$$

Then $(n;x)$ has continuous first partial derivatives, $(n;x)_u=(n;x_u)$, $(n;x)_v=(n;x_v)$, and $(n;x_m)_u\to(n;x)_u$, $(n;x_m)_v\to(n;x)_v$. Furthermore, if $\|y\|$ is in L^α, where $y=\{y^i\}$ is defined on G into m, each y^i being measurable, then

$$\iint_K\|(n;y)\|^\alpha\leqq\iint_G\|y\|^\alpha\ .$$

Thus

$$\iint_K\|(n;x)_u\|^\alpha=\lim_{m\to\infty}\iint_K\|(n;x_m)_u\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|^\alpha\leqq M\ .$$

Since x is A.C.T. and x_u^i is integrable for each i, $(n;x_u^i)\to x_u^i$ a.e. in K and $\|(n;x_u)\|\to\|x_u\|$ a.e. in K. Thus $\|x_u\|$ is in L^α and

$$\iint_K\|x_u\|^\alpha\leqq\liminf_{n\to\infty}\iint_K\|(n;x_u)\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|^\alpha\leqq M\ .$$

Finally, $p^\alpha=\lim_{N\to\infty}\|{}_Nx_u\|^\alpha$ and

$$\iint_K p^\alpha=\lim_{N\to\infty}\iint_K\|{}_Nx_u\|^\alpha\leqq\lim_{N\to\infty}\liminf_{m\to\infty}\iint_G\|({}_Nx_m)_u\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|\leqq M\ .$$

Similarly

$$\iint_K q^\alpha=\liminf_{m\to\infty}\iint_G q_m^\alpha\leqq M\ .$$

3. Equicontinuity theorems. The theorems listed in this section are taken from [4], except that now the surfaces need not be in Euclidean space. The proofs carry over almost without change.

Let Q be the square $[0\leqq u,\ v\leqq 1]$, let Q^* be its boundary, and x be defined on Q into a metric space D.

THEOREM 4. [L. C. Young]. *Given two positive numbers N and ε there exists a positive number η depending only upon N and ε such that for any D-mapping x with $D(x)<N$ there exists a δ, $\eta<\delta<\varepsilon$, and a finite subdivision of Q into rectangles whose side-lengths lie between δ and 2δ and such that image of each side of such rectangles not on Q^* is a rectifiable curve whose length is less than ε. A subdivision may be obtained by means of straight lines parallel to the sides of Q.*

THEOREM 5. *Let S be a base (or open non-degenerate) surface, let S_n, $n=1, 2, \cdots$, be a sequence of surfaces such that $\|S_n, S\|\to 0$, each S_n having a D-representation x_n on Q with $D(x_n)<M$ ($\|S_n, S\|$ is the Fréchet distance between the surfaces S_n and S). Then the mappings x_n are equicontinuous in each closed set $K\subset Q^\circ$ (the interior of Q).*

THEOREM 6. *Let S be an open non-degenerate surface and S_n be a sequence of surfaces with $\|S_n, S\|\to 0$ such that each S_n has a D-representation x_n on Q with $D(x_n)<M$ and such that there exist points $w_{im}\in Q^*$, $i=1, 2, 3$, and a positive number m with $\|w_{in}-w_{jn}\|>m$, $(x_n(w_{in}), x_n(w_{jn}))>m$ for $i\neq j$. Then the mappings are equicontinuous in an open set containing Q^*. That is, for each $\varepsilon>0$ there is a $\delta>0$ such that if w, $w'\in Q$, $\|w-w'\|<\delta$,* dist $(w, Q^*)<\delta$, *and* dist $(w', Q^*)<\delta$, *then $\delta(x_n(w), x_n(w'))<\varepsilon$.*

4. Lower semi-continuity theorems. The results in this section follow from [10].

If y is a D-mapping on G into m, let

$$E_n(y)=\iint_G \sup_{i<k\leqq n}\ \{i, k, y\}$$

$$E(y)=\iint_G \sup_{i<k}\ \{i, k, y\}$$

where

$$\{i, k, y\}=\{i, y\}+\{k, y\}$$

and

$$\{p, y\}=(y_u^p)^2+(y_v^p)^2\ .$$

Let

$$F(y)=\iint_G \sup_i \{i, y\} .$$

THEOREM 7. *If x_k and x are continuous on $\overline{G}$ (the closure of G) into m and are D-mappings on G with $x_k \to x$ uniformly on $\overline{G}$, G being of finite measure, then*

$$D(x) \leqq \liminf_{k\to\infty} D(x_k) , \qquad E(x) \leqq \liminf_{k\to\infty} E(x_k) ,$$
$$F(x) \leqq \liminf_{k\to\infty} F(x_k) .$$

Proof. We shall prove that E is lower semi-continuous. The other two parts are proved in a similar manner.

The hypothesis and Theorem 2.1 [10, p. 26] show that E_n is lower semi-continuous. The theorem follows since $E_n \leqq E_{n+1}$ and $E=\lim_{n\to\infty} E_n$.

5. Quasi-conformal representations for surfaces in m. Much of this section is lifted bodily from [2]. The principal problem is to obtain a desirable representation for certain polyhedra. After this representation has been obtained, Morrey's technique yields a similar representation for other surfaces in m.

LEMMA 1. *Let $a_n \geqq 0$, b_n, and c_n be constants, $n=1, 2, \cdots, N$. If $f(t)=\max_n [a_n t^2+b_n t+c_n]$ then for some m, $f^+(0)=f'_m(0)$ where $f^+(0)=\lim_{t\to 0^+} (f(t)-f(0))/t$ and $f_m(t)=a_m t^2+b_m t+c_m$.*

Proof. That $f^+(0)$ exists is a result of the convexity of f. Now let $w_k>0$, $w_k \to 0$. Then for some m we have $f(w_k)=f_m(w_k)$ for an infinite set of k's, and in addition, $f(0)=f_m(0)$. Therefore

$$f^+(0)=\lim_{k\to\infty} \frac{f(w_k)-f(0)}{w_k}=\lim_{k\to\infty} \frac{f_m(w_k)-f_m(0)}{w_k}=f'_m(0) .$$

LEMMA 2. *Let a_n, b_n, and c_n be measurable functions on a bounded measurable set E with $a_n(x) \geqq 0$, $n=1, 2, \cdots, N$. Let a, b, and c be summable functions on E such that $a_n(x) \leqq a(x)$, $|b_n(x)| \leqq b(x)$, and $|c_n(x)| \leqq c(x)$. In addition let M be a positive constant and A and B be measurable functions on E such that $|A(x)|<2M$ and $|B(x)|<2M^2$ on E. Let*

$$f_n(x, t)=(1+A(x)t+B(x)t^2)^{-1}(a_n(x)t^2+b_n(x)t+c_n(x))$$
$$f(x, t)=\max_n f_n(x, t)$$

and

$$\varphi(t)=\int_E f(x, t)\, dx .$$

Then, for each x, there is an $r=n(x)$ such that

$$\varphi^+(0)=\int_E f_{rt}(x, 0)\, dx .$$

Proof. If we examine the proof of the theorem permitting differentiation under the integral sign [7] we see that it is sufficient to show the existence of a summable function g such that, for some $\eta>0$,

$$\left|\frac{f(x, t)-f(x, 0)}{t}\right| \leqq g(x) , \qquad 0<t<\eta .$$

If we take $\eta<(5M)^{-1}$ we may take $g(x)=2[\eta a(x)+b(x)]$.

If y is a D-mapping into m, let

$$[i, k, y]=y_u^i y_v^k-y_v^i y_u^k .$$

Then

$$L(y)=\iint \sup_{i,k}\, [i, k, y] .$$

THEOREM 8. *An open non-degenerate polyhedron $\mathscr{P}$ contained in range Π^N for some N has a representation x^* on the unit circle $\mathscr{C}$ such that x^* is a D-mapping and*

$$\max_i \{i, x^*\} = \max_{i,k}\, [i, k, x^*] \qquad \text{a.e. } in\ \mathscr{C}.$$

Proof. Let X be a representation of $\mathscr{P}$ on Q and let $C=$range $X|Q^*$. Consider the class K of all representations x of $\mathscr{P}$ which are D-mappings on $\mathscr{C}$. Since $\mathscr{P}$ is a polyhedron, K is not empty. Let $I=\inf E(x)$ for all $x \in K$. We shall see that the infimum is attained for $x=x^*$.

Let $\bar{x}_n$ be a minimizing sequence with $E(\bar{x}_n)<I+1/n$. Fix three distinct points $\overline{P}_i$ on Q^* with $Q_i=X(\overline{P}_i)$ also distinct. For each n, choose P_{in} on $\mathscr{C}^*$ so that $\bar{x}_n(P_{in})=Q_i$. Let P_i^* be three distinct points of $\mathscr{C}^*$. By means of a conformal transformation taking $\mathscr{C}$ into itself and P_{in} into P_i^*, the functions $\bar{x}_n$ are transformed into x_n where $x_n(P_i^*) =Q_i$. It is easy to verify that $E(x_n)=E(\bar{x}_n)$.

Theorems 5 and 6 assure us that the sequence $\{x_n\}$ is equicontinuous and hence a subsequence of the x_n converges uniformly to x^*. The closure theorem for D-mappings enables us to conclude that $x^* \in K$. By Theorem 7, $E(x^*)=I$.

Now let φ and ψ be Lipschitzian with constant M in $\overline{\mathscr{C}}$ and vanish on $\mathscr{C}^*$. Then [2] the transformations T and T^{-1},

$$T: \quad \alpha = u + \varepsilon\varphi(u, v) , \qquad \beta = v + \varepsilon\psi(u, v) ,$$

are both Lipschitzian if $|\varepsilon| < 1/(3M)$. Put

$$x^*[u(\alpha, \beta, \varepsilon), v(\alpha, \beta, \varepsilon)] = x(\alpha, \beta, \varepsilon) ,$$

Then $x \in K$ [10].

Now put

$$J(\varepsilon) = E(x) = \iint_{\mathscr{C}} \max_{i \neq k} \{i, k, x\}(\alpha, \beta)\, d\alpha\, d\beta .$$

A straightforward computation shows that

$$J(\varepsilon) = \iint_{\mathscr{C}} D^{-1} \max_{i \neq k} [E^*_{ik}(\alpha_v^2 + \beta_v^2) - 2F^*_{ik}(\alpha_u\alpha_v + \beta_u\beta_v) + G^*_{ik}(\alpha_u^2 + \beta_u^2)]\, du\, dv ,$$

where

$$E^*_{ik} = (x^{*i}_u)^2 + (x^{*k}_u)^2 , \qquad G^*_{ik} = (x^{*i}_v)^2 + (x^{*k}_v)^2 , \qquad F^*_{ik} = x^{*i}_u x^{*i}_v + x^{*k}_u x^{*k}_v ,$$
$$D = \frac{\partial(\alpha, \beta)}{\partial(u, v)} .$$

We apply Lemma 2 to compute

$$J^+(0) = \iint_{\mathscr{C}} \{[-(E^*_{rs} - G^*_{rs})\varphi_u - 2F^*_{rs}\varphi_v] + [(E^*_{rs} - G^*_{rs})\psi_v - 2F^*_{rs}\psi_u]\}\, du\, dv ,$$

where r and s depend upon (u, v). That $J^+(0) \geqq 0$ is evident since J assumes its minimum at $\varepsilon = 0$.

From the arbitrariness and independence of φ and ψ we obtain, first of all, that

$$\iint_{\mathscr{C}} [(E^*_{rs} - G^*_{rs})\varphi_u - 2F^*_{rs}\varphi_v]\, du\, dv \leqq 0$$

and

$$\iint_{\mathscr{C}} [(E^*_{rs} - G^*_{rs})\psi_v - 2F^*_{rs}\psi_u]\, du\, dv \geqq 0 .$$

Next we see that if we replace φ by $-\varphi$ and ψ by $-\psi$ then the equality must hold in each case.

The remainder of Cesari's proof now goes through without change, and we conclude that $E^*_{rs} = G^*_{rs}$, $F^*_{rs} = 0$ almost everywhere. It is clear that

$$[i, k, x^*] \leqq \tfrac{1}{2}\{i, k, x^*\}$$

for all i, k. Also, where the equalities above hold, if we order r and s properly we see that

$$x_u^{*r} = x_v^{*s},\ x_v^{*r} = -x_u^{*s}\ , \qquad [r, s, x^*] = \tfrac{1}{2}\{r, s, x^*\} = \{r, x^*\} = \{s, x^*\}\ .$$

Also, from the maximizing property

$$\max_{i \neq k} \{i, k, x^*\} = \{r, s, x^*\} = \{r, x^*\} + \{s, x^*\} = 2\{r, x^*\}$$
$$= 2\{s, x^*\} = 2[r, s, x^*] = 2 \max_{i,k} [i, k, x^*]\ .$$

Finally $\{r, x^*\} = \max_i \{i, x^*\}$ for otherwise $\{i, s, x^*\} > \{r, s, x^*\}$.

LEMMA 3. *Let $\mathscr{P}$ be a non-degenerate polyhedron in m. Then for some N, the projected polyhedron $\Pi^n \mathscr{P}$ is non-degenerate for all $n > N$.*

Proof. The hypothesis implies that the vertices of each triangle of $\mathscr{P}$ are distinct and not on a line. It is clear that N may be taken large enough for $\Pi^n \mathscr{P}$ to have this property for all $n > N$.

LEMMA 4. *If $\mathscr{P}$ is a non-degenerate polyhedron with representation x, and if in the countable set of functions x^i there are only a finite number of distinct functions, then $\mathscr{P}$ has a representation x^* on the unit circle $\mathscr{C}$ such that*

$$\max_i \{i, x^*\} = \max_{i,k} [i, k, x^*] \qquad \text{a.e. } in\ \mathscr{C}.$$

DEFINITION 4. A D-mapping x is *quasi-conformal* in a Jordan region R if

$$\sup_i [(x_u^i)^2 + (x_v^i)^2] = \sup_{i,k} [x_u^i x_v^k - x_v^i x_u^k] \qquad \text{a.e. in } R\ .$$

THEOREM 9. *If x_n and x are quasi-conformal mappings on R with x_n converging uniformly to x and $L(x_n) \to L(x)$, then x is quasi-conformal.*

Proof. From $\|y_u\|^2 + \|y_v\|^2 \leqq 2 \sup_i \{i, y\}$ it follows that $D(x_n) \leqq 2L(x_n)$ and hence that $D(x_n) < M$ for some M. The closure theorem for A.C.T. functions assures us that x is a D-mapping and $D(x) \leqq M$. More exactly, we have

$$L(x) \leqq \iint_R \sup_i \{i, x\} \leqq \liminf_{n\to\infty} \iint_R \sup_i \{i, x_n\} = \liminf_{n\to\infty} L(x_n) = L(x)\ .$$

Hence $\sup_{i,k}[i, k, x]=\sup_i \{i, x\}$ a.e.

THEOREM 10. *An open non-degenerate surface $\mathscr{S}$ of finite Lebesgue area has a quasi-conformal representation on $\mathscr{C}$.*

Proof. There exists a sequence of polyhedral surfaces $\mathscr{P}_n^*$ approaching $\mathscr{S}$ with $L(\mathscr{P}_n^*)\to L(\mathscr{S})$, and we may suppose that each $\mathscr{P}_n^*$ is open non-degenerate.

Using the idea of [13, § 8] we can, for each n, determine a polyhedron $\mathscr{P}_n$ with the properties

(a) The Fréchet distance between $\mathscr{P}_n$ and $\mathscr{P}_n^*$ is less than $1/n$.

(b) $L(\mathscr{P}_n^*)\geqq L(\mathscr{P}_n)>L(\mathscr{P}_n^*)-1/n$.

(c) If x_n is a representation of $\mathscr{P}_n$ then there are only a finite number of distinct functions in the collection x_n^i.

(d) The $\mathscr{P}_n$ are open non-degenerate.

Hence the sequence $\mathscr{P}_n$ approaches $\mathscr{S}$ and $L(\mathscr{S})=\lim_{n\to\infty}(\mathscr{P}_n)$.

The remainder of the proof is the same as that for a surface in Euclidean space [4].

The idea referred to is the following. If y is a representation of a polyhedron $\mathscr{P}$ then the sequence y^i is uniformly bounded and equicontinuous, thus totally bounded. Hence for each $\varepsilon>0$ there exists a finite subset y^{i_j} of the y^i with the property that $\sup|y^i-y^{i_j}|<\varepsilon$ for each i and some i_j. If $\mathscr{P}$ is open non-degenerate and $\Pi^n\mathscr{P}$ is also, then adjoin y^k, $k=1, 2, \cdots, n$ to the y^{i_j}. Now replace those components of y which are not in the subset by one which is and is within ε of it. The resulting function represents an open non-degenerate polyhedron whose Fréchet distance from $\mathscr{P}$ does not exceed ε and whose area does not exceed that of $\mathscr{P}$.

6. Isometric surfaces in m.

For later applications it is convenient to know that if x is quasi-conformal and y is isometric with x, then y is also quasi-conformal.

Let a, b, A and B be points of m.

LEMMA 5. *If $\|a\cos\theta+b\sin\theta\|=\|A\cos\theta+B\sin\theta\|$ for all θ then* $\sup_i[(a^i)^2+(b^i)^2]=\sup_i[(A^i)^2+(B^i)^2]$.

Proof. Suppose that for some p we have $(A^p)^2+(B^p)^2>0$. Then there exist real numbers $\lambda>0$ and θ such that $A^p=\lambda\cos\theta$ and $B^p=\lambda\sin\theta$. Thus

$$\begin{aligned}(A^p)^2+(B^p)^2 &= \lambda^{-2}[(A^p)^2+(B^p)^2]^2 \leqq \lambda^{-2} \sup\,[A^pA^i+B^pB^i]^2 \\ &= \sup\,[A^i\cos\theta+B^i\sin\theta]^2 = \|A\cos\theta+B\sin\theta\|^2 = \|a\cos\theta+b\sin\theta\|^2 \\ &= \sup\,|a^i\cos\theta+b^i\sin\theta|^2 \leqq \sup\,[(a^i)^2+(b^i)^2]\ .\end{aligned}$$

Similarly

$$\sup\,[(a^i)^2+(b^i)^2] \leqq \sup\,[(A^i)^2+(B^i)^2]\ .$$

COROLLARY 1. *If* $\{\theta_j; j=1, 2, \cdots\}$ *is dense in* $[0, 2\pi]$ *and if* $\|a\cos\theta_j+b\sin\theta_j\|=\|A\cos\theta_j+B\sin\theta_j\|$ *for all* j, *then* $\sup_i[(a^i)^2+(b^i)^2]=\sup_i[(A^i)^2+(B^i)^2]$.

Fix θ and let $u=r\cos\theta-s\sin\theta$, $v=r\sin\theta+s\cos\theta$. Suppose that x is A.C.T. on G into m and define y by $y(r, s)=x(u, v)$. Since x^i is A.C.T. for each i, so is y^i. Furthermore, except for a set Z of measure 0, $y^i_r=x^i_u\cos\theta+x^i_v\sin\theta$ for all i. Thus for $s_0 \notin Z$ we have

$$\begin{aligned}\text{length } y(r, s_0) &= \lim_{N\to\infty} \text{length } \Pi^N y(r, s_0) = \lim_{N\to\infty}\int D_r(\Pi^N y) \\ &= \lim_{N\to\infty}\int \sup_{i\leqq N}|x^i_u\cos\theta+x^i_v\sin\theta| \leqq \int\|x_u\cos\theta+x_v\sin\theta\| \leqq \int\|x_u\|+\int\|x_v\|\end{aligned}$$

where the first integral is taken over the intersection of dom y with the line $s=s_0$ and the other integrals are taken over the intersection of G with the line $[-u\sin\theta+v\cos\theta]=s_0$. Thus

$$\int_{s_0}\text{length } y(r, s_0) \leqq \iint_G\|x_u\|+\iint_G\|x_v\|$$

and since r and s may be interchanged in this argument, we see that y is A.C.T.

The partials of y are, of course, directional derivatives of x. We can now apply Theorem 1 to obtain, almost everywhere in G,

$$x_\theta=\{x^i_u\cos\theta+x^i_v\sin\theta\} \text{ and } D_\theta x=\|x_u\cos\theta+x_v\sin\theta\|$$

where, if $\varphi(s)=x(u+s\cos\theta, v+s\sin\theta)$, then $x_\theta=\varphi'(0)$ and $D_\theta x=D\varphi(0)$ (see Definition 1).

Now let θ_j, $j=1, 2, \cdots$, be dense in $[0, 2\pi]$. Let W be that set of measure 0 in the complement of which $x_{\theta_j}=\{x^i_u\cos\theta_j+x^i_v\sin\theta_j\}$ and $D_{\theta_j}x=\|x_u\cos\theta_j+x_v\sin\theta_j\|$.

Observe that if x and y are isometric (dom x = dom y and $\|x(p)-x(q)\|=\|y(p)-y(q)\|$ for all $p, q \in \text{dom } x$) then $D_{\theta_j}x=D_{\theta_j}y$ wherever either side exists.

THEOREM 11. *If* x *is quasi-conformal and* y *is isometric with* x, *then* y *is quasi-conformal.*

Proof. That y is a D-mapping follows directly from the definitions. By the preceding remarks and Corollary 1 we have $\sup\{i, x\} = \sup\{i, y\}$ almost everywhere. In [13] it was shown that $L(x)=L(y)$. Hence

$$L(y)=\iint \sup[i, k, y] \leqq \iint \sup\{i, y\} = \iint \sup\{i, x\} = \iint \sup[i, k, x] = L(x)$$

from which we can conclude that $\sup[i, k, y]=\sup\{i, y\}$ almost everywhere.

7. Almost conformal representations for surfaces in a metric space. If a surface is in a metric space, then there exists an isometric surface in m. The definition of 'almost-conformal' is phrased so as to be invariant under isometries. Hence the result of the last section can be applied to surfaces in metric spaces.

DEFINITION 5. Let X be continuous on a Jordan region R into a metric space D. Then X is *almost-conformal* if there exists a quasi-conformal map x on R into m which is isometric with X.

We can now repeat some familiar reasoning of [13] to obtain the following.

THEOREM 12. *An open non-degenerate surface in a metric space has an almost-conformal representation upon the unit circle.*

Proof. Let X be a representation on Q of an open non-degenerate surface $\mathscr{T}$. If p_i, $i=1, 2, \cdots$, is dense in range X then X is isometric with $x=\{X^i\}$, where $X^i(q)=\delta(p_i, X(q))$ for all $q \in Q$. By Theorem 10 there is a quasi-conformal map y on the unit circle $\mathscr{C}$ which is Fréchet equivalent to x. Define Y on $\mathscr{C}$ into D by $Y(s)=X(r)$ where $x(r)=y(s)$. If $x(r)=y(s)$ and $x(r')=y(s)$ then $X(r)=X(r')$, so Y is well defined. The map Y is a representation of $\mathscr{T}$ upon $\mathscr{C}$ which is isometric to a quasi-conformal map y. Hence Y is almost-conformal.

Let $\mathscr{S}$ be a surface in D and suppose $\mathscr{S}$ has an almost-conformal representation X on a Jordan region R. Then X is a D-mapping and $L(\mathscr{S})=\iint_R \sup_{i,k}[i, k, X]$ where X^j is defined as in the proof of Theorem 12.

Finally we observe that if X is a D-mapping then X is almost-conformal if $\sup\{i, X\} = \sup[i, k, X]$, and conversely. The direct statement is an immediate consequence of the definition. For the converse note that if $x=\{X^i\}$ then x is isometric with X and is quasi-conformal.

8. Surfaces in a Banach space. If a D-mapping has range in a Banach space B then it is possible to give a definition of quasi-conformality which is analogous to that for the case $B=m$. Then we shall see that the notions of quasi-conformal and almost-conformal are equivalent and, in case $B=E_n$, they are both equivalent to the original definition of Morrey.

Let X be defined on a Jordan region R into B. There exists a smallest (separable) subspace $B(X)\subset B$ which contains range X. A sequence $\{f_n\}$ of linear functionals of norm one over B is *admissible* with respect to X if $\sup f_i(r)=\|r\|$ for each $r\in B(X)$. The transformation T: $B(X)\to m$ defined by $T(r)=\{f_i(r)\}$ is an isometry. It was shown in [13] that such an admissible sequence always exists.

Let $\{f_i, X\}=\{i, TX\}$ and $[f_i, f_k, X]=[i, k, TX]$.

DEFINITION 6. In the notation of the preceding paragraphs, X is *quasi-conformal* if X is a D-mapping and if $\sup\{f_i, X\}=\sup[f_i, f_k, X]$ almost everywhere in R.

Theorem 11 assures us that this definition is equivalent to that given earlier for the case $B=m$.

THEOREM 13. *A necessary and sufficient condition that X be quasi-conformal is that X be almost-conformal.*

Proof. The function TX is isometric with X. If X is quasi-conformal then

$$\sup\{i, TX\}=\sup\{f_i, X\}=\sup[f_i, f_k, X]=\sup[i, k, TX]\ .$$

Thus TX is quasi-conformal in m and X is almost-conformal. If X is almost-conformal there exists a quasi-conformal function y which is isometric with X and, therefore, with TX. (The function y has the same domain as X and has range in m.) Thus TX is also quasi-conformal and

$$\sup\{f_i, X\}=\sup\{i, TX\}=\sup[i, k, TX]=\sup[f_i, f_k, X]\ .$$

Hence X is quasi-conformal.

Now suppose that B is E_n. If f is a linear functional of norm one then there exists a point p with $\|p\|=1$ such that $f(r)=p\cdot r$ for each $r\in E_n$. Since $\{f_i\}$ is admissible, $\sup p_i\cdot r=\|r\|$. If r and s are two points in E_n with $\|r\|=\|s\|$ and $r\cdot s=0$, then $(r\cdot p)^2+(s\cdot p)^2\leqq r\cdot r$ for any p with $\|p\|=1$.

If X is quasi-conformal in the sense of Morrey (almost-conformal [4]) then X is a D-mapping and $E=G$, $F=0$ almost everywhere ($E=X_u\cdot X_u$, $F=X_u\cdot X_v$, $G=X_v\cdot X_v$). Where these equations hold, $(X_u\cdot p)^2+(X_v\cdot p)^2\leqq E$ for any p on the unit sphere. Hence $\sup\{f_i, X\}\leqq E=$area of the square determined by X_u and $X_v=\sup[f_i, f_k, X]\leqq\sup\{f_i, X\}$. Thus X is quasi-conformal in the sense of this paper.

Now let X be quasi-conformal in the sense of this paper. Since E_n has the property that an absolutely continuous function on an interval into E_n does have a derivative almost everywhere, we can conclude that X_u and X_v exist almost everywhere (not only component-wise derivatives). If $\sup\{f_i, X\}=0$, then $E=F=G=0$. If X_u and X_v both exist and $\sup\{f_i, X\}>0$, it is easy to see that

$$\sup[f_i, f_k, X]=\max_{|a|=|b|=1}[(a\cdot X_u)(b\cdot X_v)-(a\cdot X_v)(b\cdot X_u)]=\sqrt{EG-F^2},$$

$$\sup\{f_i, X\}=\max_{|a|=1}[(a\cdot X_u)^2+(a\cdot X_v)^2]$$

$$=\left(\frac{E+G}{2}\right)+\sqrt{\left(\frac{E+G}{2}\right)^2-(EG-F^2)};$$

clearly these are equal only if $E=G$, $F=0$. We conclude that the definitions of almost-conformal and quasi-conformal as given in this paper are equivalent to the original definition of Morrey.

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