TRANSCENDENTAL ADDITION THEOREMS FOR THE HYPERGEOMETRIC FUNCTION OF GAUSS

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1. Introduction. In this paper, integrals involving products of two Gauss functions, regarded as functions of their parameters, are evaluated it terms of other functions of the same kind. In all these integrals it is assumed that $|x|<1$. Also the integrals are taken up the entire length of the imaginary axis with loops, if necessary, to separate the increasing and decreasing sequences of poles. These formulae are:

$$(1)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\alpha+s)\Gamma(\alpha'-s)}{\Gamma(\gamma+s)\Gamma(\gamma'-s)}F(\alpha+s,\beta;\gamma+s;x)F(\alpha'-s,\beta';\gamma'-s;x)\,ds$$
$$=\frac{\Gamma(\alpha+\alpha')\Gamma(\gamma+\gamma'-\alpha-\alpha'-1)}{\Gamma(\gamma+\gamma'-1)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\alpha')}F(\alpha+\alpha',\beta+\beta';\gamma+\gamma'-1;x)\,,$$

where $\gamma+\gamma'-1\neq0,-1,-2,\cdots$; $\gamma-\alpha\neq0,-1,-2,\cdots$ and $\gamma'-\alpha'\neq0,-1,-2,\cdots$;

$$(2)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\alpha+s)\Gamma(\alpha'-s)}{\Gamma(\beta+s)\Gamma(\beta'-s)}F(\beta-\alpha,\gamma;\beta+s;x)F(\beta'-\alpha',\gamma';\beta'-s;x)\,ds$$
$$=\frac{\Gamma(\alpha+\alpha')\Gamma(\beta+\beta'-\alpha-\alpha'-1)}{\Gamma(\beta-\alpha)\Gamma(\beta'-\alpha')\Gamma(\beta+\beta'-1)}F(\beta+\beta'-\alpha-\alpha'-1,\gamma+\gamma';\beta+\beta'-1;x)\,,$$

where $\beta-\alpha\neq0,-1,-2,\cdots$; $\beta'-\alpha'\neq0,-1,-2,\cdots$ and $\beta+\beta'-1\neq0,-1,-2,\cdots$;

$$(3)\quad \frac{1}{2\pi i}\int\Gamma(\alpha+s)\Gamma(\alpha'+s)\Gamma(\beta-s)\Gamma(\beta'-s)F\left(\alpha+s,\beta-s;\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2};x\right)$$
$$\times F\left(\alpha'+s,\beta'-s;\frac{1}{2}\alpha'+\frac{1}{2}\beta'+\frac{1}{2};x\right)ds$$
$$=\Gamma(\alpha+\beta')\Gamma(\alpha'+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha'+\beta')\{\Gamma(\alpha+\alpha'+\beta+\beta'\}^{-1}$$
$$\times F\left(\alpha+\beta',\alpha'+\beta;\frac{1}{2}\alpha+\frac{1}{2}\alpha'+\frac{1}{2}\beta+\frac{1}{2}\beta'+\frac{1}{2};x\right),$$

where $\alpha+\alpha'+\beta+\beta'\neq0,-1,-2,\cdots$;

$$(4)\quad \frac{1}{2\pi i}\int\Gamma(\gamma+s)\Gamma(\gamma'-s)\Gamma(\alpha+s)\Gamma(\alpha'-s)$$
$$\times F(\alpha+s,\beta;\alpha+\gamma';x)F(\alpha'-s,\beta',\alpha'+\gamma;x)\,ds$$
$$=\Gamma(\gamma+\gamma')\Gamma(\alpha+\alpha')\Gamma(\alpha+\gamma')\Gamma(\alpha'+\gamma)\{\Gamma(\alpha+\alpha'+\gamma+\gamma')\}^{-1}$$
$$\times F(\alpha+\alpha',\beta+\beta';\alpha+\alpha'+\gamma+\gamma';x)\,,$$

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where $\alpha+\alpha'+\gamma+\gamma'\neq 0, -1, -2, \cdots$; and

$$(5)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\beta+s)\Gamma(\beta'-s)}{\Gamma(\gamma+s)\Gamma(\gamma'-s)}F(\alpha+s, \beta+s ; \gamma+s ; x)F(\alpha'-s, \beta'-s ; \gamma'-s ; x)\, ds$$
$$=\frac{\Gamma(\gamma+\gamma'-\beta-\beta'-1)\Gamma(\beta+\beta')}{\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')\Gamma(\gamma+\gamma'-1)}F(\beta+\beta', \alpha+\alpha'-1 ; \gamma+\gamma'-1 ; x)\ ,$$

where $\gamma-\beta\neq 0, -1, -2, \cdots$; $\gamma'-\beta'\neq 0, -1, -2, \cdots$ and $\gamma+\gamma'-1\neq 0, -1, -2, \cdots$.

All we need is the following two formulae [E. C. Titchmarch, Fourier integrals, p. 194]:

$$(6)\quad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\, ds$$
$$=\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)\{\Gamma(\alpha+\beta+\gamma+\delta)\}^{-1}$$

where $(-\alpha<k,\ -\beta<k, \gamma>k,\ \delta>k)$; and

$$(7)\quad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(\alpha-s)\Gamma(\gamma-s)}{\Gamma(\beta+s)\Gamma(\delta-s)}\, ds=\frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\delta-\alpha-\gamma-1)}{\Gamma(\beta-\alpha)\Gamma(\delta-\gamma)\Gamma(\beta+\delta-1)}\ ,$$

where $(-\alpha<k,\ -\beta<k, \gamma>k,\ \delta>k)$.

It may be noted that the restrictions, needed for (6) and (7), on the parameters in formulae (1)—(5), can be removed later on by the theory of analytical continuation. The proofs and two other formulae will be given in §2., while some confluent forms of addition theorems will be deduced, as a limiting case, in §3.

2. Proof. On expanding each hypergeometric function on the left hand side of (1) and changing the order of integration and summation it becomes

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta\ ; m)(\beta'\ ; n)}{m!\ n!}x^{m+n}\cdot\frac{1}{2\pi i}\int\frac{\Gamma(\alpha+m+s)\Gamma(\alpha'+n-s)}{\Gamma(\gamma+m+s)\Gamma(\gamma'+n-s)}\, ds\ .$$

From (7), it follows that the last integral is equal to

$$\frac{\Gamma(\alpha+\alpha'+m+n)\Gamma(\gamma+\gamma'-\alpha-\alpha'-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma'-\alpha')\Gamma(\gamma+\gamma'+m+n-1)}\ .$$

Thus the left hand side of (1) becomes

$$\frac{\Gamma(\alpha+\alpha')\Gamma(\gamma+\gamma'-\alpha-\alpha'-1)}{\Gamma(\gamma'-\alpha')\Gamma(\gamma-\alpha)\Gamma(\gamma+\gamma'-1)}$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta\ ; m)(\beta'\ ; n)(\alpha+\alpha'\ ; m+n)}{m!\ n!(\gamma+\gamma'-1\ ; m+n)}x^{m+n}$$

$$=\frac{\Gamma(\alpha+\alpha')\Gamma(\gamma+\gamma'-\alpha-\alpha'-1)}{\Gamma(\gamma'-\alpha')\Gamma(\gamma-\alpha)\Gamma(\gamma+\gamma'-1)}$$
$$\times\sum_{p=0}^{\infty}\frac{(\alpha+\alpha'\,;\,p)(\beta\,;\,p)}{p!(\gamma+\gamma'-1\,;\,p)}x^pF(\beta',\,-p\,;\,1-\beta-p\,;\,1)\;,$$

and from this formula (1) follows by applying Gauss's theorem. The proof of (2) is the same as the proof of (1).

To prove (3), expand each hypergeometric function on the left hand side of (3) and change the order of integration and summation ; then it becomes

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m+n}}{m!\;n!\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\,;\,m\right)\left(\frac{1}{2}\alpha'+\frac{1}{2}\beta'+\frac{1}{2}\,;\,n\right)}$$
$$\times\frac{1}{2\pi i}\int\Gamma(\alpha+m+s)\Gamma(\alpha'+n+s)\Gamma(\beta+m-s)\Gamma(\beta'+n-s)\,ds\;.$$

From (6), it follows that the last integral is equal to

$$\frac{\Gamma(\alpha+\beta+2m)\Gamma(\alpha+\beta'+m+n)\Gamma(\alpha'+\beta+m+n)\Gamma(\alpha'+\beta'+2n)}{\Gamma(\alpha+\alpha'+\beta+\beta'+2m+2n)}\;.$$

Thus the left hand side of (3) becomes

$$(2\sqrt{\pi})^{-1}\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha'+\frac{1}{2}\beta'+\frac{1}{2}\right)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+m\right)\Gamma\left(\frac{1}{2}\alpha'+\frac{1}{2}\beta'+n\right)}{m!\;n!\;\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\alpha'+\frac{1}{2}\beta+\frac{1}{2}\beta'+\frac{1}{2}+m+n\right)}$$
$$\times\frac{\Gamma(\alpha+\beta'+m+n)\Gamma(\alpha'+\beta+m+n)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\alpha'+\frac{1}{2}\beta+\frac{1}{2}\beta'+m+n\right)}x^{m+n}$$
$$=\Gamma(\alpha+\beta')\Gamma(\alpha'+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha'+\beta')\{\Gamma(\alpha+\alpha'+\beta+\beta')\}^{-1}$$
$$\times\sum_{p=0}^{\infty}\frac{(\alpha+\beta'\,;\,p)(\beta+\alpha'\,;\,p)\left(\frac{1}{2}\alpha'+\frac{1}{2}\beta'\,;\,p\right)}{p!\left(\frac{1}{2}\alpha+\frac{1}{2}\alpha'+\frac{1}{2}\beta+\frac{1}{2}\beta'\,;\,p\right)\left(\frac{1}{2}\alpha+\frac{1}{2}\alpha'+\frac{1}{2}\beta+\frac{1}{2}\beta'+\frac{1}{2}\,;\,p\right)}x^p$$
$$\times F\left(\frac{1}{2}\alpha+\frac{1}{2}\beta,\,-p\,;\,1-\frac{1}{2}\alpha'-\frac{1}{2}\beta'-p\,;\,1\right)\;;$$

and from this, formula (3) follows by applying Gauss's theorem. From the proof of (3), the following formula can be deduced :

$$(8)\quad \frac{1}{2\pi i}\int \Gamma(\alpha+s)\Gamma(\beta-s)\Gamma(\alpha'+s)\Gamma(\beta'-s)$$
$$\times F\left(\alpha+s,\ \beta-s\ ;\ \frac{1}{2}\alpha+\frac{1}{2}\beta\ ;\ x\right)F\left(\alpha'+s,\ \beta'-s\ ;\ \frac{1}{2}\alpha'+\frac{1}{2}\beta'\ ;\ x\right)ds$$
$$=\Gamma(\alpha+\beta)\Gamma(\alpha'+\beta')\Gamma(\alpha+\beta')\Gamma(\alpha'+\beta)\{\Gamma(\alpha+\alpha'+\beta+\beta')\}^{-1}$$
$$\times {}_3F_2\left[\begin{matrix}\alpha+\beta',\ \alpha'+\beta,\ \frac{1}{2}(\alpha+\alpha'+\beta+\beta'+2)\ ;\ x\\ \frac{1}{2}(\alpha+\alpha'+\beta+\beta'),\ \frac{1}{2}(\alpha+\alpha'+\beta+\beta'+1)\end{matrix}\right],$$

where $\alpha+\alpha'+\beta+\beta'\neq 0,\ -1,\ -2,\ \cdots$.

The proof of (4) is the same as the proof of (3), while formula (5) can be deduced by substituting for each hypergeometric function on the left hand side an integral of Barnes's type and changing the order of integration.

Finally, I may mention the following formula which involves a generalized hypergeometric function,

$$(9)\quad \frac{1}{2\pi i}\int \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\alpha'-s)\Gamma(\beta'-s)$$
$$\times F(\alpha+s,\ \beta+s\ ;\ \gamma\ ;\ x)F(\alpha'-s,\ \beta'-s\ ;\ \gamma'\ ;\ x)\,ds$$
$$=\Gamma(\alpha+\alpha')\Gamma(\alpha+\beta')\Gamma(\beta+\alpha')\Gamma(\beta+\beta')\{\Gamma(\alpha+\alpha'+\beta+\beta')\}^{-1}$$
$$\times {}_6F_5\left[\begin{matrix}\alpha+\alpha',\ \alpha+\beta',\ \beta+\alpha',\ \beta+\beta',\ \frac{1}{2}(\gamma+\gamma'-1),\ \frac{1}{2}(\gamma+\gamma')\ ;\ x\\ \gamma,\ \gamma',\ \gamma+\gamma'-1,\ \frac{1}{2}(\alpha+\alpha'+\beta+\beta'),\ \frac{1}{2}(\alpha+\alpha'+\beta+\beta'+1)\end{matrix}\right],$$

where $\gamma+\gamma'-1\neq 0,\ -1,\ -2,\ \cdots,\ \alpha+\alpha'+\beta+\beta'\neq 0,\ -1,\ -2,\ \cdots$ and either γ or γ' is not zero or a negative integer.

3. Confluent forms of addition theorems. In (1) take $\beta'=\beta$, write x/β for x and let $\beta\to\infty$ to get

$$(10)\quad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(\alpha+s)\Gamma(\alpha'-s)}{\Gamma(\gamma+s)\Gamma(\gamma'-s)}F(\alpha+s\ ;\ \gamma+s\ ;\ x)F(\alpha'-s\ ;\ \gamma'-s\ ;\ x)\,ds$$
$$=\frac{\Gamma(\alpha+\alpha')\Gamma(\gamma+\gamma'-\alpha-\alpha'-1)}{\Gamma(\gamma+\gamma'-1)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\alpha')}F(\alpha+\alpha'\ ;\ \gamma+\gamma'-1\ ;\ 2x),$$

where $\mathfrak{R}(k+\alpha)>0,\ \mathfrak{R}(k+\gamma)>0,\ \mathfrak{R}(\alpha'-k)>0,\ \mathfrak{R}(\gamma'-k)>0,\ \gamma+\gamma'-1\neq 0,\ -1,\ -2,\ \cdots;\ \gamma-\alpha\neq 0,\ -1,\ -2,\ \cdots$ and $\gamma'-\alpha'\neq 0,\ -1,\ -2,\ \cdots$.

In (2), take $\gamma'=\gamma$, write x/γ for x and let $\gamma\to\infty$, to get

$$(11)\quad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(\alpha+s)\Gamma(\alpha'-s)}{\Gamma(\beta+s)\Gamma(\beta'-s)}F(\beta-\alpha\ ;\ \beta+s\ ;\ x)F(\beta'-\alpha'\ ;\ \beta'-s\ ;\ x)\,ds$$

$$= \frac{\Gamma(\alpha+\alpha')\Gamma(\beta+\beta'-\alpha-\alpha'-1)}{\Gamma(\beta-\alpha)\Gamma(\beta'-\alpha')\Gamma(\beta+\beta'-1)} F(\beta+\beta'-\alpha-\alpha'-1\,;\,\beta+\beta'-1\,;\,2x)\,,$$

where $\Re(k+\alpha)>0$, $\Re(k+\beta)>0$, $\Re(\alpha'-k)>0$, $\Re(\beta'-k)>0$, $\beta-\alpha\neq 0, -1, -2, \cdots$; $\beta'-\alpha'\neq 0, -1, -2, \cdots$ and $\beta+\beta-1\neq 0, -1, -2, \cdots$.

Finally in (4), take $\beta'=\beta$; write x/β for x and let $\beta\to\infty$, to get

(12)
$$\begin{aligned}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} &\Gamma(\gamma+s)\Gamma(\gamma'-s)\Gamma(\alpha+s)\Gamma(\alpha'-s)\\ &\times F(\alpha+s\,;\,\alpha+\gamma'\,;\,x)F(\alpha'-s\,;\,\alpha'+\gamma\,;\,x)\,ds\\ &=\Gamma(\gamma+\gamma')\Gamma(\alpha+\alpha')\Gamma(\alpha+\gamma')\Gamma(\alpha'+\gamma)\{\Gamma(\alpha+\alpha'+\gamma+\gamma')\}^{-1}\\ &\times F(\alpha+\alpha'\,;\,\alpha+\alpha'+\gamma+\gamma'\,;\,2x)\,,\end{aligned}$$

where $\Re(k+\gamma)>0$, $\Re(\alpha+k)>0$, $\Re(\gamma'-k)>0$, $\Re(\alpha'-k)>0$, and $\alpha+\alpha'+\gamma+\gamma'\neq 0, -1, -2, \cdots$.

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