ON THE LEBESGUE AREA OF A DOUBLED MAP

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If X is a metric space and A is a non-empty closed subset of X we construct a space Y by doubling X about A in such a way that X is imbedded homeomorphically in Y, the image of A is the boundary of the image of X, and X is also homeomorphic to the closure of the complement of its homeomorphic image in Y. In this way any function f on X may be doubled in a natural way to yield a function F on Y. In 17 it is shown that if X and A satisfy certain triangulability conditions, and f is continuous to Euclidean n space, E_n, with $n \geqq k \geqq 2$, then $L_k(F) \leqq 2L_k(f)$, with L_k denoting k-dimensional Lebesgue area. In 18, 21 and 22 the restrictions of 2-dimensionality are used to show that, when $k = 2$, we have in fact $L_2(F) = 2L_2(f)$.

In particular if (X, A) is a 2-dimensional manifold with boundary, then Y is a compact 2-dimensional manifold. Furthermore, if X is finitely triangulable, then X and A satisfy the required triangulability conditions and $L_2(F) = 2L_2(f)$. Thus to compute the Lebesgue area of f, we need only to know the Lebesgue area of F, whose domain is a compact 2-dimensional manifold.

Our terminology is consistent with [1]; however, some additional notations are cited below

1. NOTATIONS.

(i) 0 is the empty set,

(ii) $\{x\}$ is the set whose sole element is x.

(iii) $\sigma A = \{x \mid \text{for some } y, x \in y \in A\}$.

(iv) R is the set of real numbers.

(v) $A^{\cap} = \{x \mid x \subset A\}$.

(vi) $N(f, A, y)$ is the number of elements, possibly infinite, in the set $\{x \mid x \in A \text{ and } y = f(x)\}$.

(vii) $\operatorname{dmn} f = \{x \mid \text{for some } y, (x, y) \in f\}$.

(viii) $\operatorname{rng} f = \{y \mid \text{for some } x, (x, y) \in f\}$.

2. AGREEMENT.

(i) If X is a topological space and i is a positive integer, then $X^i = \{A \mid A \text{ is an } i\text{-cell in } A\}$.

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(ii) If for some positive integer i, A is an i-cell and f is any homeomorphism of A into E_i note that the set

$\{x \mid f(x) \in \text{bdry rng} f\}$

is independent of the homeomorphism f selected. Consequently we agree to denote this unique set by $\hat{A}$.

3. NOTATIONS.

(i) If n is a positive integer, k is an integer and $k \leqq n$, then H_n^k is *k-dimensional Hausdorff measure* over E_n.

(ii) Let X be a k-dimensional finitely triangulable topological space and let f be a continuous function on X to E_n with $k \leqq n$. Then $L_k(f)$ is the *k-dimensional Lebesgue area* of f. More precisely, $L_k(f)$ is the infimum of the set of all $t \in R$ such that for any $\varepsilon > 0$ there exists a quasi-linear function g on X to E_n such that $|g(x) - f(x)| \leqq \varepsilon$ for each $x \in X$ and $\int_{E_n} N(g, X, y) dH_n^k y < t$.

4. DEFINITION. X is a *k-dimensional manifold* if and only if X is such a connected separable metric space that for any $x \in X$ there exists A such that A is an open k-cell in X with $x \in A$.

5. DEFINITION. (X, A) is a *2-dimensional manifold with boundary* if and only if the following conditions are satisfied:

(i) X is a compact metric space and A is a closed subset of X.

(ii) $X - A$ is a 2-dimensional manifold.

(iii) If $x \in A$, there exist h and β such that β is an open subset of X, $x \in \beta$, and h is a homeomorphism of β onto $E_2 \cap \{z \mid z_2 \geqq 0\}$ such that $\text{rng}(h \mid A) = E_2 \cap \{z \mid z_2 = 0\}$.

THEOREM. *Let (X, A) be a 2-dimensional manifold with boundary. Then*

(i) *A has a finite number of components*;

(ii) *each component of A is a simple closed curve.*

Proof. A is compact, and finitely many of the open sets β described in 5 (iii) cover A. For each such β the set $A \cap \beta$ is connected. Thus A has a finite number of components.

Let x be a component of A and let $t \in x$. Then $\text{ord}_x t = 2$, [2, § 46], and x is a simple closed curve.

7. DEFINITION. Y is *obtained by doubling X about A* if and only if:

(i) X is a metric space and A is a closed, non-empty subset of A;

(ii) Y is the topological space $(\operatorname{rng} g^+ \cup \operatorname{rng} g^-) \subset (X \times R)$ where g^+ and g^- are the functions on X to $(X \times R)$ such that for each $x \in X$,

$$g^+(x) = (x, \operatorname{dist}(\{x\}, A)) ,$$
$$g^-(x) = (x, -\operatorname{dist}(\{x\}, A)) .$$

8. AGREEMENT. Throughout this paper we fix X, Y and A such that Y is obtained by doubling X about A. In addition we agree to let g^+ and g^- be the functions specified in 7 (ii).

9. THEOREM. g^+ and g^- are both homeomorphisms of X into Y such that

$$\operatorname{bdry} \operatorname{rng} g^+ = \operatorname{bdry} \operatorname{rng} g^- = \operatorname{rng}(g^+ \mid A) = \operatorname{rng}(g^- \mid A) .$$

The proof is trivial.

10. THEOREM. *If (X, A) is a 2-dimensional manifold with boundary then Y is a compact 2-dimensional manifold.*

The proof is trivial.

11. DEFINITION. The *map F is obtained by doubling the map f* if and only if f is a function on X and $F \circ g^+ = F \circ g^- = f$.

12. AGREEMENT. Throughout the remainder of this paper we fix and F such that the map F is obtained by doubling the map f.

13. THEOREM.

(i) *F is a function,* $\operatorname{dmn} F = Y$, *and* $\operatorname{rng} F = \operatorname{rng} f$.

(ii) *If f is continuous, then F is continuous.*

(iii) *If X is compact and f is light and continuous, then F is light and continuous.*

Proof. The proofs of (i) and (ii) are trivial.

Suppose X is compact and let $z \in \operatorname{rng} F$. Then

$$\{x \mid F(x) = z\} = (\operatorname{rng} g^+ \cap \{x \mid F(x) = z\}) \cup (\operatorname{rng} g^- \cap \{x \mid F(x) = z\}) .$$

Both sets on the right are closed in Y and homeomorphic to $\{x \mid f(x) = z\}$ which is 0-dimensional. Thus $\{x \mid F(x) = z\}$ is 0-dimensional.

14. DEFINITION. (P, Q) is a *finitely triangulable pair* if and only if P is a topological space, $Q \subset P$, and there exist (K, τ) and K' such that (K, τ) is a finite triangulation of P, $K' \subset K$ and $\operatorname{rng}(\tau \mid Q) = \sigma K'$.

15. THEOREM. *Let (P, Q) be a 2-dimensional manifold with boundary, such that P is finitely triangulable. Then (P, Q) is a finitely triangulable pair.*

The theorem is an immediate consequence of 6.

16. LEMMA. *Let X be k-dimensional and suppose that (X, A) is a finitely triangulable pair. Let $\operatorname{rng} f \subset E_n$ with $n \geqq k$ and let f be continuous. Let $\varepsilon > 0$.*

Let (K, τ) be a finite triangulation of X in E_q and let $K' \subset K$ such that $\operatorname{rng}(\tau \mid A) = \sigma K'$. Let u be a quasi-linear function on X to E_n such that for each $B \in K$, $(u \circ \operatorname{inv} \tau) \mid B$ is a barycentric map of B, and such that for each $x \in X$, $\mid u(x) - f(x) \mid \leqq \varepsilon$.

Then there exists a quasi-linear function h on Y to E_n such that $\mid h(y) - F(y) \mid \leqq \varepsilon$ for each $y \in Y$, and

$$\int_{E_n} N(h, Y, y) dH_n^k y \leqq 2 \int_{E_n} N(u, X, y) dH_n^k y \,.$$

Proof. We may suppose that τ is the identity map, $X \subset E_q$ and $Y \subset E_{q+1}$.

For each $B \in K$ let B^* be the set of vertices of B. Then for each $B \in K$ let φ_B^+ be the function which maps B barycentrically onto the unique Euclidean simplex in E_{q+1} spanned by the affinely independent set $\operatorname{rng}(g^+ \mid B^*)$. More precisely, if $x \in B \in K$, and γ_x is that unique function on B^* to $R \cap \{y \mid 0 \leqq y \leqq 1\}$, such that $\sum_{t \in B^*} \gamma_x(t) = 1$ and $x = \sum_{t \in B^*} \gamma_x(t) t$, then let $\varphi_B^+(x) = \sum_{t \in B^*} \gamma_x(t) g^+(t)$.

Similarly for each $B \in K$ let φ_B^- be the function which maps B barycentrically onto the unique Euclidean simplex in E_{q+1} spanned by the set $\operatorname{rng}(g^- \mid B^*)$.

Then let

$$H = \bigcup_{B \in K} (\{\operatorname{rng} \varphi_B^+\} \cup \{\operatorname{rng} \varphi_B^-\}) \,.$$

Also let λ^+ and λ^- be defined by,

$$\lambda^+ = \bigcup_{B \in K} \varphi_B^+ \,, \qquad \lambda^- = \bigcup_{B \in K} \varphi_B^- \,.$$

Then let

$$\eta = (\lambda^+ \circ \operatorname{inv} g^+) \cup (\lambda^- \circ \operatorname{inv} g^-) \,.$$

Since (X, A) is a finitely triangulable pair, η is a function and (H, η) is a finite triangulation of Y in E_{q+1}.

Next let

$$h = (u \circ \operatorname{inv} g^+) \cup (u \circ \operatorname{inv} g^-) \,.$$

Then h is a function, and $(h \circ \operatorname{inv} \eta) | B'$ is a barycentric map of B' for each $B' \in H$. Thus h is a quasi-linear map of Y into E_n. Also $|h(y) - F(y)| \leqq \varepsilon$ for each $y \in Y$.

Finally,

$$\int_{E_n} N(h, Y, y) dH_n^k y = \int_{E_n} N(h, \operatorname{rng} g^+ \cup \operatorname{rng} g^-, y) dH_n^k y$$
$$\leqq \int_{E_n} N(h, \operatorname{rng} g^+, y) dH_n^k y + \int_{E_n} N(h, \operatorname{rng} g^-, y) dH_n^k y$$
$$= 2 \int_{E_n} N(u, X, y) dH_n^k y \, .$$

17. COROLLARY. *Let X be k-dimensional and suppose that (X, A) is a finitely triangulable pair. Let $\operatorname{rng} f \subset E_n$ with $n \geqq k$, and let f be continuous. Then Y is finitely triangulable and $L_k(F) \leqq 2L_k(f)$.*

Proof. The construction of 16 guarantees that Y is finitely triangulable. Now suppose that $2L_k(f) < \delta < L_k(F)$. Let $\varepsilon > 0$. It suffices to establish a quasi-linear function h on Y to E_n such that $|h(y) - F(y)| \leqq \varepsilon$ for each $y \in Y$ and $\int_{E_n} N(h, Y, y) dH_n^k y < \delta$.

Let (K, τ) be a finite triangulation of X in E_q and let $V \subset K$ such that $\operatorname{rng}(\tau | A) = \sigma V$. By 6.24 of [1] there exists K' such that K' is a finite simplicial subdivision of K, and there exists a quasi-linear function u on X to E_n such that $|u(x) - f(x)| \leqq \varepsilon$ for each $x \in X$, $\int_{E_n} N(u, X, y) dH_n^k y < \delta/2$, and $(u \circ \operatorname{inv} \tau) | B'$ is a barycentric map of B' for each $B' \in K'$.

Now let $V' = K' \cap \bigcup_{B \in V} B^{\cap}$. Clearly (K', τ) is a finite triangulation of X in E_q, and since K' is a subdivision of K, we can state that $\operatorname{rng}(\tau | A) = \sigma V'$. Thus Lemma 16 applies to produce a quasi-linear function h on Y to E_n such that $|h(y) - F(y)| \leqq \varepsilon$ for each $y \in Y$, and

$$\int_{E_n} N(h, Y, y) dH_n^k y \leqq 2 \int_{E_n} N(u, X, y) dH_n^k y < \delta \, .$$

18. LEMMA. *Let B be a 2-cell metrized by ρ and let $V \in B^1 \cap \hat{B}^{\cap}$. Let $M = (V - \hat{V})$ and let $\varepsilon > 0$. Then there exists a function u such that:*

(i) *u is a homeomerphism of B into B.*

(ii) *$u(x) = x$ for $x \in (\hat{B} - M)$.*

(iii) *$\operatorname{rng}(u/M) \subset (B - \hat{B})$.*

(iv) *$M \cap \operatorname{rng} u = 0$.*

(v) *$\rho(x, u(x)) < \varepsilon$ for each $x \in B$.*

Proof. We may suppose that $B \subset E_2$. In fact, letting

$$\alpha = (1/2, 1) \in E_2 ,$$
$$\beta = (-1/2, 1) \in E_2 ,$$
$$\gamma = (0, 0) \in E_2 ,$$

we may assume that B is the convex hull of the set $\{\alpha, \beta, \gamma\} \subset E_2$. Furthermore we may suppose that

$$V = \{\alpha t + (1 - t)\beta \mid t \in \{z \mid 0 \leqq z \leqq 1\}\} .$$

Now let v be the function on $B - \{\gamma\}$ such that for each $x \in (B - \{\gamma\})$ we have $v(x) = [\sigma(V \cap \{tx \mid t \in R\})]_1$.

Then let w be the function on R such that $w(x) = (\varepsilon/4) - \varepsilon x^2$. For each $x \in R$.

Finally let u be the function on B such that

$$u(x) = [1 - w(v(x))]x, \text{ if } x \in (B - \{\gamma\}),$$
$$u(\gamma) = \gamma .$$

It is easy to check that u satisfies the required conditions.

19. REMARK. Let (K, τ) be a finite triangulation of a topological space P and let $\varepsilon > 0$. Then by barycentrically subdividing each element of K, we obtain K' such that (K', τ) is a finite triangulation of P, K' is a finite simplicial subdivision of K and each element of K' is less than ε in diameter.

20. DEFINITION. A subset V of E_n is *k-removable* ([1, 6.26]) if and only if V is a closed set with the following property.

If u is a continuous function on a k-dimensional finitely triangulable space T, to E_n, and

$G = \{P \mid P$ is a finitely triangulable subset of T and $\operatorname{rng}(u \mid P) \cap V = 0\}$, then $L_k(u) = \sup_{P \in G} L_k(u \mid P)$.

In the following lemma we make use of the fact that any finite subset of E_n is k removable.

21. LEMMA. *Let M be a metric space. Let K be a finite 2-dimensional cell-complex in M such that $M = \sigma K$. Suppose there exists a finite non-empty set $P \subset (K \cap M^1)$ and a function γ on P such that for each $x \in P$*

(i) $\{\gamma(x)\} = \{B \mid (B \in (K \cap M^2))$ and $(x \subset \hat{B})\}$,

and[1]

[1] Geometrically the conditions (i) and (ii) state that each 1-cell of P is a subset of the boundary of exactly one 2-cell of K, and futhermore, this 2-cell of K meets no other element of P.

(ii) $\sigma P \cap \gamma(x) = x$.

Let J be that set of all M' such that M' is a finitely triangulable subset of M and $M' \cap \sigma P = 0$. Let u be a continuous function on M to E_n with $n \geqq 2$. Then $L_2(u) = \sup_{M \in J} L_2(u \mid M')$.

Proof. Let ρ metrize M. It suffices to show that $L_2(u) \leqq \sup_{M' \in J} L_2(u \mid M')$.

The remainder of the proof is divided into 2 parts.

Part 1. Let $\varepsilon > 0$. *There exists a function φ such that*:

(i) *φ is a homeomorphism of M into M.*

(ii) *For each $x \in M$, $\rho(x, \varphi(x)) < \varepsilon$.*

(iii) $\bigcup_{B \in P} \hat{B} = \sigma P \cap \operatorname{rng} \varphi = \sigma P \cap \operatorname{rng}(\varphi \mid \sigma P)$.

Proof of Part 1. For each $x \in P$ we apply Lemma 18 to produce a function d_x is satisfying the following conditions:

(i′) d_x is a homeomorphism of $\gamma(x)$ into $\gamma(x)$.

(ii′) $d_x(t) = t$, for $t \in [\widehat{\gamma(x)} - (x - \hat{x})]$.

(iii′) $\operatorname{rng}(d_x \mid (x - \hat{x})) \subset (\gamma(x) - \widehat{\gamma(x)})$.

(iv′) $(x - \hat{x}) \cap \operatorname{rng} d_x = 0$.

(v′) For each $t \in \gamma(x)$, $\rho(d_x(t), t) < \varepsilon$.

Let Ψ be the identity map of $(M - \sigma \operatorname{rng} \gamma)$ onto itself and let

$$\varphi = \bigcup_{x \in P} d_x \cup \Psi .$$

Part 2. $L_2(u) \leqq \sup_{M' \in J} L_2(u \mid M')$.

Proof of Part 2. Let $\varepsilon > 0$ and produce a function φ satisfying the conditions (i)–(iii) of part 1.

The finite set $\operatorname{rng}(u \mid [\sigma P \cap \operatorname{rng} \varphi])$ is 2-removable. Thus if we let W be the set of all Q such that Q is a finitely triangulable subset of $\operatorname{rng} \varphi$ and

$$\operatorname{rng}(u \mid Q) \cap \operatorname{rng}(u \mid [\sigma P \cap \operatorname{rng} \varphi]) = 0$$

we can state that

$$L_2(u \circ \varphi) = L_2(u \mid \operatorname{rng} \varphi) = \sup_{Q \in W} L_2(u \mid Q) \leqq \sup_{M' \in J} L_2(u \mid M') .$$

Due to the arbitrary nature of ε we have

$$L_2(u) \leqq L_2(u \circ \varphi) \leqq \sup_{M' \in J} L_2(u \mid M') .$$

22. COROLLARY. *Let K be a finite 2-dimensional cell complex in X such that $X = \sigma K$. Suppose there exists a finite non-empty set $P \subset (K \cap X^1)$*

such that $A \subset \sigma P$, and there exists a function γ on P such that for each $x \in P$,

$$\{\gamma(x)\} = \{B \mid (B \in (K \cap X^2)) \text{ and } (x \subset \hat{B})\} ,$$

and

$$\sigma P \cap \gamma(x) = x .$$

Let $\operatorname{rng} f \subset E_n$ *with* $n \geqq 2$ *and let* f *be continuous. Then* $2L_2(f) \leqq L_2(F)$.

Proof. Let J be the set of all X' such that X' is a finitely triangulable subset of X and $X' \cap \sigma P = 0$. Let $V \in J$. Then since

$$\operatorname{rng}(g^+ \mid V) \cap \operatorname{rng}(g^- \mid V) = 0 ,$$

we infer that,

$$L_2(F \mid \operatorname{rng}(g^+ \mid V)) + L_2(F \mid \operatorname{rng}(g^- \mid V)) = L_2(F \mid (\operatorname{rng}(g^+ \mid V) \cup \operatorname{rng}(g^- \mid V)))$$
$$\leqq L_2(F) .$$

Since $F \circ g^- = F \circ g^= = f$,

$$2L_2(f \mid V) \leqq L_2(F) ,$$

and

$$2L_2(f) = 2 \sup_{X' \in J} L_2(f \mid X') \leqq L_2(F) .$$

23. COROLLARY. *Suppose that (X, A) is a 2-dimensional manifold with boundary and X is finitely triangulable. Let* $\operatorname{rng} f \subset E_n$ *with* $n \geqq 2$ *and let f be continuous. Then* $2L_2(f) = L_2(F)$.

Proof. From 15 and 17 we infer that $L_2(F) \leqq 2L_2(f)$.

Let (K, τ) be a finite triangulation of X. By appropriately subdividing each 2-cell of K we can easily produce H such that H is a finite 2-dimensional cell-complex in X, $\sigma H = X$, and such that $B \cap A \in H$ for each $B \in H$ with $B \cap A \neq 0$. Let $P = A^{\cap} \cap H \cap X^1$. Note that if $x \in P$, then the set

$$\{B \mid (B \in (H \cap X^2)) \text{ and } (x \subset \hat{B})\}$$

has precisely one element.

Thus let γ be the function on P such that for each $x \in P$

$$\gamma(x) = \sigma\{B \mid (B \in (H \cap X^2)) \text{ and } (x \subset \hat{B})\} .$$

The construction of H guarantees that $\sigma P \cap \gamma(x) = x$ for each $x \in P$. Thus 22 applies and $2L_2(f) = L_2(F)$.

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