TWO THEOREMS OF GAUSS

DANIEL SHANKS

The purpose of this note is to show that two famous theorems of Gauss can be derived from a common source. The theorems alluded to are the following:

THEOREM 1. (The triangular-exponent identity)

$$\prod_{s=1}^{\infty} \frac{1-x^{2s}}{1-x^{2s-1}} = \sum_{s=1}^{\infty} x^{s(s-1)/2} . \tag{1}$$

THEOREM 2. (The evaluation of Gauss sums)

$$\sum_{s=0}^{m-1} e^{2\pi i s^2/m} = \begin{cases} \sqrt{m} & \text{for } m\equiv 1 \pmod 4 \\ i\sqrt{m} & \text{for } m\equiv 3 \pmod 4 . \end{cases} \tag{2}$$

Both these results will be obtained as consequences of the following identity previously stated by the author [2] without proof.

A finite identity.

THEOREM. *If* $P_0 = 1$ *and*

$$P_n = \prod_{s=1}^{n} \left(\frac{1-x^{2s}}{1-x^{2s-1}}\right)$$

for $n = 1, 2, \cdots$, *then*

$$A_n = \sum_{s=0}^{n-1} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n} x^{s(s-1)/2} = S_n \tag{3}$$

and

$$A'_n = \sum_{s=1}^{n} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n+1} x^{s(s-1)/2} = S'_n . \tag{3'}$$

Proof. We readily verify that

$$(1-x^{2n})x^{s(2n+1)} = (1-x^{2n-1})x^{s(2n-1)} + (1-x^{2s+1})x^{(s+1)(2n-1)} - (1-x^{2s})x^{s(2n-1)} ,$$

and by multiplying by $\dfrac{P_{n-1}}{P_s(1-x^{2n-1})}$ we find

Received February 5, 1958, and in revised form May 7, 1958.

$$(4) \qquad \frac{P_n}{P_s}x^{s(2n+1)} = \frac{P_{n-1}}{P_s}x^{s(2[n-1]+1)} + \alpha_{s,n} - \beta_{s,n}$$

where

$$\alpha_{s,n} = \frac{1-x^{2s+1}}{1-x^{2n-1}}\frac{P_{n-1}}{P_s}x^{(s+1)(2n-1)}$$

and

$$\beta_{s,n} = \frac{1-x^{2s}}{1-x^{2n-1}}\frac{P_{n-1}}{P_s}x^{s(2n-1)} .$$

Now

$$\beta_{s+1,n} = \alpha_{s,n} \qquad (\text{for } s = 0, 1, \cdots, n-2)$$

and since, further,

$$\beta_{0,n} = 0 \text{ and } \alpha_{n-1,n} = x^{n(2n-1)} ,$$

by summing (4) from $s = 0$ to $s = n-1$ we obtain:

$$A_n = A_{n-1} + x^{(n-1)(2n-1)} + x^{n(2n-1)} .$$

But this may be written $A_n - A_{n-1} = S_n - S_{n-1}$, and by induction

$$A_n - S_n = A_1 - S_1 = \frac{1-x^2}{1-x} - (1+x) = 0 .$$

This proves (3) and by adding $x^{n(2n+1)}$ to both sides we verify (3′).

Gauss' triangular exponent Theorem (1), now follows at once from (3).

Proof of Theorem 1. The leading term in A_n (that is $s = 0$ in the left side of eq. 3) is P_n. Since the remaining terms ($s = 1, 2, \cdots, n = 1$) are of order x^{2n+1} and higher, the power series of the function $P_n(x)$ must agree with that of $S_n(x)$ at least to terms of order x^{2n}. By induction the function P_∞ must have the power series S_∞ and this proves (1).

Proof of Theorem 2. The magnitude of

$$G = \sum_{s=0}^{m-1} e^{2\pi i s^2/m}$$

for any odd integer m is given by $|G| = \sqrt{m}$. This is easily shown, [1, p. 163], by multiplying G by its complex conjugate. The real difficulty in Theorem (2) is to show that G is positive real or positive imaginary according as $m \equiv 1$ or $m \equiv 3 \pmod 4$. But the identity (3) enables us to do this without undue computation.

First we write $x = v^2$ where $v = e^{i\theta}$. Then

$$P_n = v^n \prod_{s=1}^{n} \left(\frac{v^{2s} - v^{-2s}}{v^{2s-1} - v^{1-2s}} \right)$$

and if

$$Q_0 = 1 \,, \qquad Q_n = \prod_{s=1}^{n} \left(\frac{\sin 2s\theta}{\sin (2s-1)\theta} \right) ,$$

we may write

$$(5) \qquad S_n = \sum_{s=1}^{2n} v^{s(s-1)} = \sum_{s=1}^{n-1} v^{n+s(4n+1)} \frac{Q_n}{Q_s}$$

$$(5') \qquad S_n' = \sum_{s=1}^{2n+1} v^{s(s-1)} = \sum_{s=0}^{n} v^{n+s(4n+1)} \frac{Q_n}{Q_s} .$$

Now for any odd integer, $m = 2N+1$, if $\theta = 2\pi/m$ we have

$$v^{2N} = v^{-1}$$

and thus

$$G = \sum_{s=0}^{m-1} v^{s^2} = \sum_{s=-N}^{+N} v^{(N+s)^2} = v^{N^2} \sum_{s=-N}^{+N} v^{s(s-1)}$$

$$= v^{N^2} \left[\sum_{s=1}^{N} v^{s(s-1)} + \sum_{s=1}^{N+1} v^{s(s-1)} \right] .$$

Therefore if $m = 4n+1$ and $N = 2n$,

$$(6) \qquad G = \sum_{s=0}^{n-1} v^{(n+s)(4n+1)} \frac{Q_n}{Q_s} + \sum_{s=0}^{n} v^{(n+s)(4n+1)} \frac{Q_n}{Q_s} .$$

But $v^{4n+1} = 1$ and $Q_s =$ positive real for $s = 0, 1, \cdots, n$ so that

$$G = +\sqrt{m} \qquad\qquad m \equiv 1 \pmod 4 .$$

And, if $m = 4n+3$ and $N = 2n+1$, then

$$G = v^{N^2} \left[\sum_{s=0}^{n} v^{n+s(4n+1)} \frac{Q_n}{Q_s} + \sum_{s=0}^{n} v^{n+1+s(4n+5)} \frac{Q_{n+1}}{Q_s} \right]$$

$$= \sum_{s=0}^{n} v^{(4n+3)(n+s)} \left[v^{2n+1-2s} \frac{Q_n}{Q_s} + v^{2n+2+2s} \frac{Q_{n+1}}{Q_s} \right] .$$

But now $v^{4n+3} = 1$ and

$$Q_{n+1} = \frac{\sin [(2n+2)\theta]}{\sin [(2n+1)\theta]} Q_n = -Q_n$$

and thus

$$G = 2i \sum_{s=0}^{n} \sin [(2n + 1 - 2s)\theta] \frac{Q_n}{Q_s} \tag{7}$$

which is positive imaginary. Therefore

$$G = + i\sqrt{m} \qquad\qquad m \equiv 3 \pmod 4 .$$

References

1. E. Landau, *Aus der elementaren Zahlentheorie*, Chelsea, New York, 1946.
2. D. Shanks, *A short proof on an identity of Euler*, Proc. Amer. Math. Soc. **2** (1951), 749.

David Taylor Model Basin
Washington 7, D. C.